ON THE QUANTUM THEORY
OF GRAVITATING PARTICLES

P.O. MAZUR

Department of Physics and Astronomy, University of South Carolina
Columbia, SC 29208, USA

The present paper proposes a basis for new gravitational mechanics. The problem of finding the spectrum of mass-energy is reduced to a new kind of eigenvalue problem which intrinsically contains the fundamental length $l = \sqrt{Gh/c^3}$.

The similarity between physical descriptions of a "black hole atom" of the end of *20th* century and the hydrogen atom of the beginning of this century is too critical to the progress of physics that it could not remain unnoticed for too long. Indeed, the "inevitability of an electromagnetic collapse" of the hydrogen atom loomed above the heads of theoretical physicists of that long gone era in somehow similar fashion as the "inevitability of a gravitational collapse" of a quantum mechanical matter into a "black hole" does so today.

In this paper we propose to consider such a "collapse" of quantum mechanical matter as unphysical. Simply stated our proposal amounts to the statement that the matter states are stationary and only transitions between those stationary states are physical. We postulate the presence of a ground state. This requirement is compatible with the generalized gravitational correspondence principle. It should be noticed that our proposal is basically similar to the original Bohr proposal of 1913. However, the physical context is quite different in this case because we consider the "gravitational atom" described, according to Einstein, by spacetime continuum. In essence, we propose to consider the hypothesis of existence of space-time-matter 'atoms'. The problem addressed here involves all fundamental constants of Nature, G, c, and h, and contains, therefore, a length scale l to which all length scales in the problem are to be compared to.

Indeed, in the problem of the hydrogen atom solved by Bohr it was the Planck constant h which determined the scale of the atom and, therefore, its stationary states.

The similarities between phenomena of black body radiation and black hole radiance present themselves to closer analysis with the use of principles of statistical mechanics and thermodynamics [12, 11, 9, 13, 8]. In the case of radiation interacting with matter, the hypothesis of atomic nature of matter, implying the fluctuation phenomena, led to the discovery of wave-particle duality by Einstein. Once the quantum nature of phenomena of absorption and emission of radiation by matter was established it led unequivocally to the realization that atoms must have discrete energy levels. Similarly, the atomistic nature of matter-radiation together with the equivalence principle seem to imply that fluctuation-statistical properties are also intrinsic to a gravitating mass.

We know today that no approach founded on established theories of that time could have given the correct description of phenomena of black body radiation and the spectrum of hydrogen atom. Similarly, today we have no other option but to conclude that quantum mechanics fails to describe accurately gravitating particles which are classically described by the General Relativity Theory (GRT), and vice versa GRT does not seem to describe accurately quantum particles. This is so because the generalization of quantum mechanics compatible with Lorentz covariance, quantum field theory, is incompatible with the concept of fundamental length of Planck, Heisenberg, Born, Markow, Snyder, and Yukawa. The core of the problem is the concept of a point particle which leads to infinities and all kinds of incorrect mathematics [3–5, 17, 18].

Let us analyze closely some of the difficulties encountered in our description of gravitating particles. There is a clash of concepts here: on the one hand a particle with an inertial mass m_i behaves like a wave in configuration space, but on the other hand the *same* particle with a gravitational mass $m_g = m_i$ (equivalence principle) is responsible for distortion of space-time relations around it. The geometry of space-time, according to Einstein, is the mode of description of particles (corpuscules). But now, this introduction of space-time continuum is in visible contradiction to nature of phenomena on a small scale where the atomistic structure is evident. The problem of theoretical basis of physics, as we see it, is that these two modes of description are orthogonal to each other. Therefore, it seems to us that they are also in contradiction to each other. The resolution of this contradiction must result in a completely new world view which is yet to come.

The clash of concepts, as described above, leads to serious difficulties in all attempts toward quantum description of gravitating particles [9, 13,

8, 23, 25]. Indeed, the usual method of Feynman leads to an unnecessary attention paid to quantized gravitational waves or gravitons. We have convinced ourselves long time ago, after studying [6], that gravitons obscure the true physical nature of gravitating particles. True, the gravitons must be derived later on because otherwise the theory would be in contradiction with the experimental-observational evidence given by the observed properties of binary pulsars. The problem of gravitating particles as it presents itself requires that the asymmetry in our description of matter and space-time be removed. This can be done only by the postulate of general *Atomic Hypothesis*. It appears that the *Atomic Hypothesis* must be extended to the space-time-matter object.

It was only natural, after quantum mechanics was established, that the role dimensional arguments and adiabatic invariants had played in the Planck discovery of quanta and the Bohr discovery of his theory of the hydrogen atom should have become forgotten. Indeed, with his discovery of quantum of action h Planck had realized that together with the Newton constant G and the velocity of light c the three fundamental constants of Nature lead in a natural way to fundamental units of duration $\tau = \sqrt{Gh/c^5} = 1.35\,10^{-43}$ s, length $l = \sqrt{Gh/c^3} = 4.05\,10^{-33}$ cm, and mass $\mu = \sqrt{hc/G} = 5.46\,10^{-5}$ g.

The GR Kepler problem possesses an additional adiabatic invariant whose role has not been yet exploited properly. This is the Christodoulou adiabatic invariant [12] called sometimes an area of an "event horizon". It is the opinion of the present author that the hard learnt lesson of the first quarter of this century was almost forgotten by now. It is difficult to overestimate the importance of adiabatic invariants in physics. One of the purposes of the present paper is to bring out the Christodoulou adiabatic invariant [12] from the years of neglect to the prominent role it rightfully deserves in fundamental physics [11, 8].

The principle of equivalence and Lorentz covariance determines how a small mass m moves in the space-time of a massive material body of mass M. It is a purely dimensional argument which helps to establish that the mass M is the basic space-time attribute of a material body because the characteristic extension L, or size, of such a body is given by the relation: $L = KM$, where $K = pGc^{-2}$, with p a numerical constant of order 1. This point was very clearly realized by Dirac, who posed the problem of fundamental mass spectrum for self-gravitating particles [4, 5, 7]. In the General Relativity Theory, based on the concept of a local field and space-time continuum, the fundamental solution describing gravitating particle of mass M is the Schwarzschild solution. This solution also determines the numerical constant $p = 2$. This simple observation shall be elevated to the status of *Kinematical Postulate*.

The Kinematical Postulate:

The only fundamental attribute of a material body is its space-time extension, $L = KM$, which is an attribute of space-time.

Similarly, for a given momentum P there is an associated gravitational length scale L such that $L = K/cP$. The constant K, sometimes called the Einstein gravitational constant, plays the same role in physics as the velocity of light c plays in unifying the concepts of space and time into the more general concept of space-time continuum. It unifies the concepts of space-time and matter into the more general concept of space-time-matter. *The meaning of the Postulate of Space-Time Nature of Mass-Energy* is that from all attributes of matter only one is fundamental. This attribute can be called a length L associated with a material body, or its mass M, depending on particular circumstances. All other attributes of matter will be connected to the fundamental one by dimensional constants. An example of this is the electric charge Q ($Q = \epsilon\sqrt{G}M$, ϵ is a numerical constant). So, Q is also an attribute of space-time. We shall expect that the postulate of mass-energy as an attribute of space-time must lead to new kinematics.

The clear physical meaning of this postulate could be easily seen in the context of three-dimensional gravitation [6]. There a mass M is identified by the three-dimensional analog of fundamental constant K with dimensionless geometrical object — a planar angle (an angle of rotation). I wish to comment at this point on the nature of the electric charge Q. Staruszkiewicz [16] was the first to realize that electric charge Q is an attribute of space-time, not unlike the angular momentum. In his theory he established a theoretical framework where an electric charge in proper units of $\sqrt{\hbar c}$ is compared to a hyperbolic angle (a measure of a Lorentz boost) in a similar way as in his first geometrical-kinematical theory of mass-energy [6], where a mass-energy in proper units of Planck mass-energy was compared to an angle of rotation.

The present author proposed some time ago that new kinematics be sought in order to describe quantum theory of a gravitating particle. In essence the argument can be reduced to the statement that in the same way the Planck constant h leads to new kinematics of quantum mechanics (QM), $[p, q] = h/2\pi i$, rather than new dynamics, the constant $K = 2(G/c^2)$ together with h should be a basis of new kinematics. The basic reason for this hypothesis was the observation that the Le Verrier anomaly, which was first explained by Einstein, is telling us that Nature possesses a second period. Indeed, the Mercury perihelion motion, and more visible binary pulsar periastron motion, is a signal that double periodic motions parameterized by elliptic functions in the Weierstrass (or Jacobi) form occur in the ancient Kepler problem once the fundamental constant $K = 2(G/c^2)$ is different than zero. It is important to recognize that the character of this motion is

quite different from multi-periodic motions in this respect that one of the periods is complex (purely imaginary for bound orbits). The Kepler problem in Newtonian gravitation is an example of degenerate multi-periodic motion: all periods degenerate to one. However, the GR Kepler problem is non-degenerate in a sense that there is the second complex period which tends to $i\infty$ (for bounded motions) when $K \to 0$. In the same way as the Newtonian Kepler problem is uniformized by a circle S^1, which is parameterized by the astronomers "anomaly" angle $\xi \in [0, 2\pi]$, the GR Kepler problem is uniformized by an integral lattice Λ_τ, where $\tau = \omega_2/\omega_1$, on a complex plane of ξ, *i.e.*, a complex torus T^2. The two periods of a lattice are ω_1, which is real, and ω_2 which is complex. Curiously enough, the genus- 1 elliptic curve appears in this fundamental problem. One must be prepared to say that all three classical tests of GRT are supporting this mathematical fact which should find its proper physical meaning. The change of a coordinate basis does not obscure this double periodic character of motion. It should come as no surprise that the Le Verrier anomaly is an exact four dimensional analog of the angular defect caused by a heavy body in three dimensional gravitation of Staruszkiewicz [6]. This fact was known to this author for years now but it appears not to be well known among workers in the field.

Now, we know very well the role periodic motions and adiabatic invariants have played in the Heisenberg discovery of quantum mechanics [1a]. Could it possibly be that the constant K controlling the Le Verrier's 1859 anomaly should play the fundamental role in setting up new kinematics which, somehow, is reducible to quantum mechanics in the gravitational correspondence principle limit, $K \to 0$?

Later it became quite clear to the present author that the second period in Nature is closely related to the Christodoulou [12] adiabatic invariant $(A/16\pi) = M_{ir}^2$, where A is the area of the Schwarzschild sphere[1]. Indeed, one is compelled to consider integrals of pdq one-forms over two homology cycles of a complex torus T^2 which is inherent in the GR Kepler problem. The Bohr-Sommerfeld-Einstein semiclassical quantization conditions amount to the statement that integrals of all pdq's over the *real homology cycles of real* tori in phase space are natural numbers in Planck constant units (modulo some half-integers).

What is the meaning of the other homology cycle and the corresponding purely imaginary adiabatic invariant? This is precisely here where the pioneering work of Christodoulou [12] finds its quantum mechanical context. Consider an integral $I_2 = \int p_0 dx^0$ over the homology cycle of a complex

---

[1] We consider for simplicity the static case, *i.e.*, when the angular momentum of a gravitating particle is vanishing. In the Christodoulou formula we take $G = c = 1$.

torus with a period ω_2 (which is purely imaginary for bounded motions). Since $p_0 = E$ is a constant of motion, then the imaginary part of the adiabatic invariant I_2 is equal to $EIm\Delta x^0$. We propose here the requirement that this adiabatic invariant satisfies the same Bohr-Sommerfeld-Einstein quantization condition as before. This is new, and quite a surprising, basic physical condition which contains all constants of Nature in it. The Planck mass must appear in this condition, as well as the mass of the central heavy body. It appears that this condition is a mass-energy quantization condition (quantization of E) which contains both the Planck constant h and the Newton constant G. It must be stressed again that this condition is different from the usual one where an energy E is compared to a frequency ν, $E = nh\nu$ (n a natural number).

The important lesson we have learnt from Heisenberg [1a] is that the adiabatic invariant $I = \int pdq$, evaluated in the phase space with singly periodic motions, leads, via Ritz combination principle and quantum hypothesis in the form of replacement of differential relations by the difference ones, to quantum mechanics. The question we have asked is:

Would not the Christodoulou adiabatic invariant and the double-periodic character of motion of gravitating particles necessarily lead to kinematics of the new gravitational mechanics? The hope has arisen that such a parallel development could possibly lead to our understanding of gravitation and spacetime at the deeper level. The present paper grew out of such considerations. In the following we will present our quite simple and rather basic observations plus some modest results. We will consider the GR Kepler problem in the light of our *Kinematical Postulate*.

In the simplest, no "back reaction" approximation the motion of a small mass m particle in the Schwarzschild field of a massive material body of mass M is described by the geodesic equation which follows from a variational principle. General Relativity Mechanics in the Hamiltonian form has an intimate connection to the wave propagation phenomena in nonhomogeneous media. Naturally, the argument due to Schrödinger [1b] applied to the motion in the Schwarzschild field leads to the first relativistic Schrödinger wave equation describing scalar wave propagation in the gravitational field of a massive body. For simplicity we will consider radiation field only. In this way the third fundamental constant of Nature, h, came into consideration, and, therefore, the fundamental length l, also. The usual method of second quantization when applied to the problem of radiation in the gravitational field of a particle leads to the so-called "black hole radiance" [9], but it does not take into account the presence of fundamental length l in the problem. This leads to all kind of problems which suggest that both General Relativity Theory and Quantum Mechanics fail to describe phenomena correctly in the domain where both must be applied [25, 23].

In particular, "black hole radiance" comes out thermal [9, 13]. It is clear that the arguments advanced up to now miss the obvious point that both theories applied to the problem utilize the concept of space-time continuum. In particular, the arguments proposing modification of quantum mechanics in such a way that transitions from pure quantum states to von Neumann mixed states are allowed should be considered unphysical. The difficulties encountered with the proposal of taking properly into account an "infinite blue-shift" of quanta and their "infinite back-reaction" appear to be insurmountable in the present quantum field theory scheme. Clearly, this is not the resolution of the problem of gravitating particles as it presents itself.

We must go back more than one and half century back in time and realize that the fundamental ideas of Hamilton's "*Optics of Nonhomogeneous Media*", which underlie Mechanics in Hamiltonian form, were based on the concept of the continuum. These ideas and formal analogies between mechanics of Hamilton and wave propagation in nonhomogeneous media, which appear natural, had later led Schrödinger to establish his wave mechanics [1b]. However, today we know that it is the atomistic nature of media which is responsible for dispersion and wave propagation in nonhomogeneous dispersive media.

We shall propose that the description of fundamental properties of matter and space-time free from contradictions must entail somehow the atomistic nature of space-time-matter entity. No wonder that the space-time continuum survived quantum revolution as it is clear that phenomena on the scale of 10^{20} in fundamental length scale were sought to be described adequately. Indeed, the phenomena are described adequately even at the scale of 10^{17} (CERN and Fermilab experiments), and perhaps even at the scale of 10^{11} (Dehmelt experiments [21]), in natural units of Planck length l.

Quantization as introduced by Schrödinger [1b] and based on Einstein's and de Broglie's physical insight has something to do with "vibrations" and/or wave phenomena. In essence, quantization entails introducing integers in the same way as counting number of nodes/zeros of some Ψ-functions satisfying some wave equations. Today we know that the formal analogy between the mechanics of Hamilton and wave propagation in nonhomogeneous media, which led to wave mechanics, must be modified accordingly once we realize that dispersive nonhomogeneous media appear as such due to their atomistic/molecular nature.

We shall observe that a particle motion/wave propagation in curved space-time continuum is not unlike wave propagation in nonhomogeneous media. It appears to the present author that such a propagation must occur effectively as a process of simple fundamental interactions of a "particle"

with a "molecular medium" [2]. There would be not too much to this physical analogy if it were not for our *Kinematical Postulate*. On the other hand, we have already argued that a gravitating particle of mass M has a length $L = KM$ associated with it. The fundamental postulate of wave mechanics is that a material particle of mass M behaves like a wave under some conditions. Now, when the constant K is assumed to be vanishing, *i.e.*, $K = G/c^2 \to 0$, there is no way to argue that a given mass-energy must be quantized (this is the limit of point particles, which is described by quantum field theory on space-time continuum). It is clearly the case that when $K \neq 0$ one can associate a purely dimensionless number γ with a mass M: $\gamma = Ll^{-1} = KMl^{-1}$.

We shall state the basic *heuristic principle* which leads to quantization of mass-energy of a gravitating particle:

Postulate of Quantization of Mass-Energy of a Gravitating Particle

Quantum states of a particle are characterized by the condition that the length L corresponding to a given mass-energy ΔM associated with transitions between two such states must correspond to a standing wave of wavelength $\lambda = 2l$, i.e., $L = n(\lambda/2) = nl$, where n is an integer.

Of course, the factor of 2 is just a convention. This condition is similar to the de Broglie reformulation of the Bohr [3] quantization conditions determining energy levels of the hydrogen atom alluded to above. We shall demand that $\gamma = n$: $L = nl = K\Delta M$. From this relation we obtain the heuristic mass-energy quantization condition: $\Delta M = \mu n$, where μ is the fundamental mass scale.

The principle of Lorentz covariance, which must be valid, tells us that the arguments applied to a particle at rest must be extended to particles in motion. It may help to think about the familiar principle of Lorentz covariance as valid in the correspondence principle limit. This principle must be generalized accordingly so it may accommodate our *Kinematical Postulate*. This seems to suggest that all components of four-momentum must satisfy some kind of periodicity condition, *i.e.*, all components of the four-momentum must be defined modulo some constant(s) only. The mass-energy is defined modulo the Planck mass-energy [4].

---

[2] It may help to think about it in a sense of some kind of a random walk — not unlike Brownian motion first explained theoretically by Einstein and Smoluchowski.

[3] We need to elaborate here on the Bohr quantum condition for stationary orbits and the resulting stability criterion in the context of our GR Kepler problem. We postulate that there are no gravitational radiation or other energy loss when the GR two-body system satisfies new quantization condition.

[4] One may ask where is the place for an electron whose mass is of the order of

We are led to view the four-momentum vector as defined modulo some lattice vector. It seems to the present author that the concept of fundamental cell in the four-momentum space must be introduced. The situation is not unlike the one encountered in the case of a harmonic crystal were the pseudomomentum is defined modulo the inverse lattice vectors.

It appears to the present author that the situation is indeed very unusual in this respect that the fundamental periodicity in four-momentum of natural phenomena was not uncovered earlier [15]. The real implication of that work [15] is that all physical observables depending on four-momentum are defined modulo some lattice vector in four-momentum space. It seems that we should also consider a space-time lattice dual to that one in the four-momentum space. However, this only means that the underlying mathematical structure may somehow involve difference equations in "space-time coordinates". This author suggests that the *heuristic principle* should be considered as a guiding principle toward the goal of finding new equations. Only in this way we can understand the periodicity of observables in four-momentum. Such fundamental periodicity in four-momentum space must follow from new wave equations. We shall see that these wave equations are difference equations in many variables.

It is clear to the present author that applying the concepts of existing theories at the intersection of the area of their validity physicists can find new avenues toward new physics[5]. This happens to be the case of quantum mechanics applied to the much simpler three-dimensional gravitation [6, 10, 15, 19, 24]. The "spinning particle solution" [10] in three-dimensional gravitation is obtained from flat space-time by an application of "improper

---

10^{-22} of the Planck mass. This question is also valid for masses of all particles discovered until now and those which will be discovered in the future. It appears that empirically the energy has a continuous spectrum. On the other hand the processes of energy exchange must be periodic on fundamental energy scale. It is the enormous number of "primitive elements" in any domain of space-time which is responsible for continuous spectrum of energy. The similar situation is encountered in any solid material body. This is why thermodynamics works. We expect that the "band spectrum" of energy levels must arise in a consistent theory which incorporates all three fundamental constants of Nature. An example showing this point explicitly will be presented in the following paper.

[5] A. Staruszkiewicz has recognized it long time ago and consistently has been exploring the domain of 'infrared physics' where the charges reside. Also Gerard 't Hooft has been pursuing this path for more than a decade now. His attempts at formulating a new theory of Planckian scale physics should satisfy, what we called, the generalized correspondence principle: $K \to 0$ should lead to quantum mechanics (QM) and $\hbar \to 0$ to General Relativity (GR).

coordinate transformation" [15] in the same way one introduces the electromagnetic potential of Aharonov and Bohm [14]. The present author has applied quantum mechanics to a test particle in the field of a "spinning line source" in four-dimensional gravitation [15]. The presence of a "line source" is essential for dimensional reasons again; and this was one of the reasons it was considered in the first place. We should keep the dimensionality of physical constants intact and this is the dictum one must obey. In order to best present the argument we shall write this amazingly simple metric below:

$$ds^2 = -(cdt - Ad\phi)^2 + dx^2 + dy^2 + dz^2 \,, \tag{1}$$

where, in one interpretation, $A = 4GJc^{-3}$, with J an angular momentum per unit length of the line source. A acquires new physical meaning which is far beyond the original circumstances which led to it. It is clear that A has the physical dimension of length. Therefore it can always be written as some pure number times the Planck length l. Quantum mechanics in this space is the quantum mechanics of the Aharonov–Bohm effect [14, 15]. This basic observation [15] was first stated explicitly as early as in 1986. The "spinning string" metric [10, 15] served a purpose of an agent through which the fundamental length was introduced into physics. One may say that the three-dimensional gravitation [6] is a perfect theoretical laboratory [6] in which the basic concepts of quantum mechanics and gravitation lead inevitably to the theory in which the fundamental length scale must be introduced in a consistent way. We have argued earlier that quantum mechanics of a test particle in the space-time described by equation (1) is the quantum mechanics of the Aharonov–Bohm effect. For simplicity we consider here a massless particle scattering in the gravitational field (1) of a "line source" with vanishing mass per unit length in order to preserve the perfect analogy to the Aharonov–Bohm effect [14]. We have calculated the scattering cross-section for such a two-body problem and found the general formula which reduces in the special case discussed here to the formula

$$\frac{d\sigma}{dzd\theta} = \frac{hc}{4\pi^2 E} \frac{\sin^2 \frac{2\pi AE}{h}}{\sin^2 \frac{\pi-\theta}{2}} \,. \tag{2}$$

The scattering cross-section is a periodic function of the product AE, where E is an energy of a particle scattered by a line source [15]. The historical role of the famous Dirac magnetic monopole quantization condition [2] has been to help to determine the natural scale of a magnetic monopole

---

[6] Gerard 't Hooft and his collaborators have also recognized the fundamental importance of this fact as their published work demonstrates clearly [10, 19, 23, 24].

charge, and, therefore, a magnetic flux, also. Today we know that the unit of magnetic flux is hc/e [14], and this is so because there exists quantum of an electric charge e. Of course, this does not mean that the magnetic flux could exist in Nature only in quanta of hc/e. In the similar way, the presence of a "fundamental length" A in the problem of quantum mechanics of a test particle in the field of a line source implies that the energy E of a particle is defined modulo $h/2A$ [15]. This argument is, to this author's knowledge, the first known and clearly realized example of fundamental periodicity in the four-momentum space. This does not mean that energy could be exchanged only in quanta of $h/2A$. One can argue that this simple argument establishes the hypothetical presence of inverse lattice to the "space-time lattice". The basic argument originated in three-dimensional gravitation [6] but for dimensional reasons ended up with line sources in four-dimensional gravitation. One comment is in order here; the constant A determines also a scale of acceleration $a = c^2/A$ [7].

The equation for a selfgravitating particle

The present author has discovered not long ago that gravitating particles in four-dimensional gravitation satisfy, in the simplest situation of spinless particles, a new kind of s-wave Schrödinger type wave equation. This equation has the fundamental Planck length l incorporated in naturally. It is, indeed, the difference equation of the second order and it describes the processes of emission and absorption of radiation by a gravitating particle. In fact, it describes the quantum mechanics of the GR Kepler problem without spin degrees of freedom taken into account. We present this equation here without too long discussion of its origin. Suffice to say that it cannot be derived from the Einstein theory of gravitation and quantum mechanics only. An additional postulate/assumption, related to the *heuristic principle* stated above, was made in deriving it. Here we present the equation which describes the emission and absorption of quanta by the self-gravitating particle in the s-wave only:

$$x\left[\Psi(x + il) + \Psi(x - il)\right] = (x + 2KE)\Psi(x), \qquad (3)$$

where $x = r - 2KM$, $l^2 = Gh$, (we take $c = 1$ here.) E is the energy of the emitted quantum of radiation and M is the mass of a gravitating

---

[7] It is known that quantum mechanics in accelerated frames is related to the second (imaginary) period of Nature. We suggest that the reader convince herself/himself of this by considering quantum mechanics of a pendulum (planar rotator) in two frames: inside the Einstein elevator and outside of it.

particle. The mass M' after emission of quantum of radiation of energy E is $M' = M - E$. We shall show that equation (3) implies the following spectrum of mass-energy for a gravitating particle: $M' = M - nE_1$, where $E_1 = \mu \sin(\pi/3)$, and n is a natural number.

Previously, difference equations of a type introduced here were discussed in the context of quantization of motion of material membranes and "vacuum bubbles" [20] [8]. The equation (3) is a homogeneous difference equation of the second order with linear coefficients. From the theory of linear difference equations we know that the equation (3) has solutions which are acceptable, *i.e.*, normalizable in the domain: $x \in [0, \infty]$. The solutions of (3) are given in terms of the Gauss hypergeometric function $F(\alpha, \beta, \gamma; z)$. No regular solutions with positive or vanishing E exist for negative x. In this sense only, the region inside classical Schwarzschild sphere has no physical meaning at all. The "black hole" interpretation of quantum solutions to equation (3) is untenable. This is quite similar to the situation we encounter in the Schrödinger equation for the hydrogen atom. The reduced radial Schrödinger equation is defined on the whole complex r-plane but the real negative values of the radial variable r are physically excluded because physically acceptable solutions for bound states are divergent for negative r. The condition of regularity of $\Psi(x)$ at $x = 0$ and quick decay at $x = \infty$ leads to quantization condition for E. The spectrum of energy of emitted quanta is linear in n, where n is a natural number:

$$E_n = E_1 n, \qquad (4)$$

where $E_1 = \mu \sin(\pi/3)$ and μ is the Planck mass-energy.

This seems to be an example of a general rule that even though the differential equation limit of vanishing Planck length l of the difference equation (3) is of the confluent hypergeometric type, the solution to the difference equation (3) is given in terms of the Gauss hypergeometric series. These wave functions are quite unusual transcendental functions with qualitatively different behavior from that of the continuous limit wave functions. One should not take the limit $l \to 0$ too easily because interesting physics might be lost in this process. This point of view is quite opposite to the method of, say, lattice gauge theories.

---

[8] The authors of [20, 27] where the first, to this author knowledge, to go beyond Dirac's 1962 papers [4, 5] as far as quantization of Dirac's Hamiltonian is concerned. Dirac has wondered how to take a square root of a nonquadratic Hamiltonian, but he did not proceed beyond the semiclassical approximation. With few exceptions [7], confirming the rule, the basic message of Dirac's proposal, as the questions after Dirac's lecture at Jabłonna Conference [4, 5] readily show, was not understood at all.

The equation (3) is of the general type of hypergeometric difference equation. The method of solving the hypergeometric difference equation is pretty standard. It is based on the application of the Laplace transformation [22]. One obtains an integral representation for solutions of the hypergeometric difference equation. The contour integrals in a complex plane of the Laplace transform variable t are characterized by pairs of points chosen from among four points: $t = 0$, $t = \infty$, $t = \rho_1$, and $t = \rho_2$. This leads to six solutions together with their analytic continuations. These solutions are analogous to 24 Kummer solutions of the Gauss hypergeometric differential equation [22]. ρ_1 and ρ_2 are the roots of the characteristic equation associated with the difference equation (3): $\rho^2 - \rho + 1 = 0$. Here we present, for the sake of completeness, the following solution of (3) corresponding to the discrete spectrum (4) only:

$$\Psi_n(x) = y \exp\left(-\frac{\pi y}{3}\right) F(1 + iy, 1 - n, 2; \kappa), \qquad (5)$$

where $y = x/l$, $n \geq 1$, and $\kappa = 1 - (\rho_1/\rho_2) = 1 - \exp(2\pi i/3)$. It should be noted that $n = 0$ is allowed in the spectrum [9]. The wave function for $n = 0$ is $\Psi_0(x) = \exp(-\pi y/3)$.

The arguments in favor of the inverse lattice in the four-momentum space presented above lead to the fundamental difference wave equation, which for a spinless (scalar) gravitating particle, is the difference analog of the first relativistic Schrödinger equation:

$$\left([P_0]_q{}^2 - [P_1]_q{}^2 - [P_2]_q{}^2 - [P_3]_q{}^2 - [M]_q{}^2\right)\Xi = 0, \qquad (6)$$

where

$$[x]_q = (q - q^{-1})^{-1}(q^x - q^{-x}), \qquad (7)$$

$$P_\mu = i\partial_\mu. \qquad (8)$$

We put $\hbar = h/2\pi = 1$, $c = 1$. Then we have $G = l^2$, and $q = \exp(i(l/2))$. In the vanishing gravitational constant G limit $q \to 1$ this equation becomes what is known as the Klein–Gordon relativistic equation. We could easily notice that the only quite apparent reference to the phenomenon of gravitation in (6) is in the presence of the Newton constant G in the parameter $q = \exp((i/2)\sqrt{G})$. It seems that the decomposition of the

---

[9] I am grateful to Professor A. Staruszkiewicz for his penetrating observations on the subject of this paper and I thank him for reminding me about the need to exercise caution.

Ξ-function into "spherical harmonics" should lead to a "radial" equation similar to equation (3). More work is required to prove this.

The difference equation analog of the relativistic Schrödinger equation has the mathematical properties we seek. Similarly we can write down a difference equation describing an object with a spin. It should be clear that for the elementary plane wave solution of (6), $\Xi(x^\mu) = \exp(-iP_\mu x^\mu)$, we obtain the following dispersion relation:

$$\sin^2 \frac{P_0 l}{2} = \sin^2 \frac{P_1 l}{2} + \sin^2 \frac{P_2 l}{2} + \sin^2 \frac{P_3 l}{2} + \sin^2 \frac{M l}{2}. \tag{9}$$

The energy-momentum vector P_μ is defined modulo the inverse lattice, exactly like in the case of pseudomomentum of phonons in harmonic crystals. This property suggests the presence of *Umklapp* processes in interactions of fundamental modes of "space-time vibrations". We know that *Umklapp* processes [26] are very important in solid state because they protect the system from developing infinite heat and electric conductivities. This is to say that their presence stops the development of instability. It is not inconceivable that similarly *Umklapp* processes in interactions of "space-time vibrations" could be responsible for absence of instabilities like the development of singularities [10]. In particular, for the special case of vanishing spatial components of four momentum $P_1 = P_2 = P_3 = 0$, modulo $2\pi/l$, we have from (9)

$$P_0 = M + \frac{2\pi}{l} n. \tag{10}$$

Clearly, equation (10) is essentially equivalent to the equation (4) when $M = 0$. They differ only by a numerical constant which must be fixed by comparison of those two equations.

There are many questions we shall ask about the whole framework presented here. Among those is the question of "many particle" ("many mode") states *etc.* Our method calls for understanding first the two-body problem of gravitating particles, *i.e.*, the GR Kepler problem, before approaching the general problem of many bodies. We shall argue that the more fundamental and more natural description of a self-gravitating material particle is achieved in terms of the Ξ-function satisfying the difference equation rather than in terms of curved space-time continuum produced by a massive particle. The difference equation strongly suggests the presence of some "space-time lattice" in some representation of the equation. Clearly,

---

[10] We have in mind here the space-time curvature singularities. *Umklapp* processes [26] in solid state are processes of collision of *phonons* where the pseudomomentum is conserved modulo the inverse lattice vectors. These processes reduce the resulting total pseudomomentum to the first Brillouin zone.

there are more than one representations of the operator equation (6). The presence of a "lattice" [11] rather than the continuum is what distinguishes clearly our attempt at the theory from the General Relativity Theory. The questions of principle arise:

In what limit and how our theory converges to the Einsteinian description of macroscopic reality? What is the meaning of a particle in our theory? It seems that the concept of a particle must be replaced by the concept of a "primitive element" which is basically the concept of the mode of vibration of the fundamental "space-time lattice". It appears to this author that the analog of transport phenomena, collective excitations, statistics, and the "elastic" properties of continuum space-time limit of the "space-time lattice", as described by the General Relativity Theory, must be deduced first from the equations of the type presented in this paper before the present theory could be accepted.

This paper is the first in the program of establishing, what we prefer to call, the new gravitational mechanics. The mathematical framework which is necessary for establishing the new gravitational mechanics must be developed as the next step in the program. One comment seems to be needed here. The mathematics of the algebraic formulation of new gravitational mechanics, as we see it now, seems to be related to the algebraic theory of generalized Hopf algebras. One good reason for that is that the difference equations and special functions related to them are intimately related to the representation theory of Hopf algebras. This seems to be the case of our equations (3) and (6). Another reason is that the algebraic concept of a coproduct, inherent in the mathematical definition of a Hopf algebra, must be considered a necessary ingredient in description of "fusion" of two or more "modes of vibration" into one. The composition rule for four-momenta of elementary systems ("particles") must take into account the presence of an analog of the Brillouin zone in four-momentum space. The concept of a "point particle" is replaced here by the concept of a "primitive element of matter" or "space-time quantum". However, these formal similarities are suggested only by the 'prototype equations', which are not the last word in the development of the new gravitational mechanics. We need the formal framework which should limit the number of possibilities for the theory of gravitating particles in accord with the *Atomic Hypothesis*.

---

[11] which should be not taken too literally; only equations have their meaning which extends far beyond simple physical pictures.

of gravitating particles long ago.   Special thanks must go to Professors A. Casher, Y. Aharonov, and S. Nussinov for many interesting discussions which helped me to clarify the approach presented here.

Note added in proof. I have learnt meanwhile that Hájíček *et al.* [28] have addressed the physical problem first considered in [20] and they have clarified the formal aspects of it greatly.

Appendix

The empirical evidence shows a great disparity between the natural scale of mass-energy and the spectrum of masses of observable particles. Moreover, the processes of absorption and emission of radiation, and observable particle interactions, do not seem to show any signal of 'energy quantization' (see [15] for an early attempt to compare the experimental data and 'energy quantization' property). The theory presented in this paper proposes to introduce new kinematics which in essence means that the phenomena of absorption and emission are periodic in four-momentum, or in other words it says that we should treat four-momentum as a modular variable defined modulo an integer multiplicity of some constant unit of four-momentum given in terms of a fundamental (Planck) length. It is clear that for the four-momentum scale probed presently the modular character of four-momentum is not quite evident yet. However, this does not mean that it is not apparent in phenomena when inspected closely. *The evidence* for the modular character of four-momentum *may exist in quite indirect form.*

In the processes like 'gravitational collapse', according to GR Theory, one would expect the presence of arbitrary high four-momenta in the collision of matter quanta and unlimited energy-momentum density, leading ultimately to a gravitational singularity. The presence of relativistic *Umklapp* processes, which is implied by the modular character of four-momentum, would mean that the processes like 'gravitational collapse' of matter do not occur in Nature. The physical role of *Umklapp* processes is to arrest the growing instability in exactly the same way as in solids where they save a day by producing a finite heat conductivity (without *Umklapp* processes the heat conductivity in solids would be infinite).

In a similar way the energy-momentum concentration caused by 'gravitational collapse' would be dealt with by distributing the excess four-momentum in the collision of matter quanta to the whole 'space-time lattice'. This is the meaning of the statement that on dimensional grounds the upper limit on the energy density of matter is the Planck density $\hbar c/l^4$.

REFERENCES

[1] (a) W. Heisenberg, *Z. Phys.* **33**, 879 (1925), reprinted in English in *Sources of Quantum Mechanics*, edited with a Historical Introduction by B. L. Van Der Waerden, Dover Publications Inc., New York 1968; (b) E. Schrödinger, *Ann. Phys.* (Germany) **4**, vol. **79**, (1926), reprinted in *Collected Papers on Wave Mechanics*, Chelsea Publishing Company, New York, N.Y. 1978.

[2] P.A.M. Dirac, *Proc. R. Soc. London* **A133**, 60 (1931).

[3] P.A.M. Dirac, *Proc. R. Soc. Edinburgh* **A59**, 122 (1938-39).

[4] P.A.M. Dirac, (1962), *Motion of an Extended Particle in the Gravitational Field*, in *Relativistic Theories of Gravitation*, Proceedings of a Conference held in Warsaw and Jabłonna, July 1962, ed. by L. Infeld, P.W.N. Publishers, 1964, Warsaw, pp 163-171, discussion pp 171-175.

[5] P.A.M. Dirac, *Proc. R. Soc. London* **A268**, 57 (1962); P.A.M. Dirac, *Proc. R. Soc. London* **A270**, 354 (1962).

[6] A. Staruszkiewicz, *Acta Phys. Pol.* **24**, 735 (1963).

[7] A. Staruszkiewicz, (1966/67), Syracuse University lecture; private communication to the author, unpublished.

[8] P.O. Mazur, *Gen. Relativ. Gravitation* **19**, 1173 (1987).

[9] S.W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975).

[10] S. Deser, R. Jackiw, G. 't Hooft, *Ann. Phys.* (NY) **152**, 220 (1984).

[11] J.D. Bekenstein, *Lett. Nuovo Cimento* **11**, 467 (1974).

[12] D. Christodoulou, *Phys. Rev. Lett.* **25**, 1596 (1970); D. Christodoulou, PhD Thesis, Princeton University, 1970.

[13] J.D. Bekenstein, *Gravitation, the Quantum, and Statistical Physics*, in *To Fulfill a Vision: Jerusalem Einstein Centennial Symposium on Gauge Theories and Unification of Physical Forces*, Addison–Wesley Publishing Co., New York 1981, pp 42-59; discussion pp 60-62.

[14] Y. Aharonov, D. Bohm, *Phys. Rev.* **115**, 485 (1959).

[15] P.O. Mazur, *Phys. Rev. Lett.* **57**, 929 (1986); P.O. Mazur, *Phys. Rev. Lett.* **59**, 2380(C) (1987).

[16] A. Staruszkiewicz, *Ann. Phys.* (NY) **190**, 354 (1989); A. Staruszkiewicz, *Quantum Mechanics of the Electric Charge*, contribution to the Yakir Aharonov's Festschrift *Quantum Coherence and Reality*, ed. by J.S. Anandan and J.L. Safko, World Scientific, Singapore 1994, page 90.

[17] P.A.M. Dirac, *Int. J. Theor. Phys.* **21**, 603 (1984).

[18] P.A.M. Dirac, *Int. J. Theor. Phys.* **23**, 677 (1984).

[19] G. 't Hooft, *Commun. Math. Phys.* **117**, 685 (1988).

[20] V.A. Berezin, N.G. Kozimirov, V.A. Kuzmin, I.I. Tkachev, *Phys. Lett.* **B212**, 415 (1988).

[21] H. Dehmelt, *Rev. Mod. Phys.* **62**, 525 (1990).

[22] P.M. Batchelor, *An Introduction to Linear Difference Equations*, The Harvard University Press, Cambridge 1927.

[23] G. 't Hooft, *Nucl. Phys.* **B335**, 138 (1990).

[24] G. 't Hooft, *Classical Quantum Gravity* **10**, 1653 (1993).

[25] Y. Aharonov, A. Casher, S. Nussinov, *Phys. Lett.* **B191**, 51 (1987).

[26] R.E. Peierls, *Quantum Theory of Solids*, Oxford University Press 1955.

[27] V.A. Berezin, *Quantum Black Holes and Hawking's Radiation*, IHES/P/91/76 Bures-sur-Yvette report 1991, unpublished; V.A. Berezin, *Phys. Lett.* **B241**, 194 (1990).

[28] P. Hájíček, B.S. Kay, K.V. Kuchař, *Phys. Rev.* **D46**, 5439 (1992).