WEYL–WIGNER–MOYAL FORMALISM. I. OPERATOR ORDERING

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General approach to the problem of the operator ordering for the flat phase space is given. It is shown how the operator ordering is determined by some natural axioms. The Weyl, symmetric and the Born–Jordan orderings are considered. The general form of the momentum operator in curvilinear coordinates is found.

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1. Introduction

The Weyl–Wigner–Moyal formalism provides us with a powerful link between the classical and quantum physics [1–8]. The formalism has found its application in many branches of quantum theory, as for example in statistical physics, quantum collision theory, quantum optics, semiclassical approximation *etc.* (See Refs [6] and [7] and the references cited therein).

Then, rather surprisingly, the Weyl–Wigner–Moyal formalism plays an important role in self-dual gravity [9–11] and quantum groups [12]. Briefly speaking this formalism describes the connection between the functions on the phase space and the quantum objects. In most cases the phase space is assumed to be Euclidean. However, some effort has been made to carry over the Weyl–Wigner–Moyal formalism to the phase space of non-Euclidean topology [13–16]. One can expect that it is possible to generalize this formalism to any curved phase space. Such a generalization would provide us with the natural method of quantization in general relativity.

The present paper is the first one in the series of our works devoted to the Weyl–Wigner–Moyal formalism in curved phase spaces. Here we are concerned with the problem of operator ordering. This problem has been considered by many authors [1–8], [17] and [18].

Our intention is to derive the rule of the operator ordering from some simple and "natural" assumptions. We hope that such an approach enable us to establish the correspondence between the classical observables and the operators in Hilbert space in the case of a curved phase space.

In Section 2 the assumptions are proposed which determine the operator ordering for a monomial $p^m q^n$, $m, n \in N$ (N stands for the set of natural numbers). It is shown that this ordering is defined by a sequence of complex numbers $g(s, s, s)$, $s \geq 0$, satisfying some additional conditions (2.14) and (2.21). We have not found the general solution of those conditions.

However, as it is shown in Section 3, if one makes the assumption defining the rule of extension of the prescription given in Section 2 to any (analytic) function on phase space then the Fourier representation method enables us to find $g(s, s, s)$ for any $s \geq 0$. Namely, the sequence $(g(s, s, s))_{s \in N}$ is defined according to the formula (3.17) by a complex valued function $f = f(x)$, $x \in R$ satisfying (3.15) and (3.16).

This agrees with the considerations of Refs [4–7, 17] and [18] but our approach is different. (Especially the considerations of Ref. [18] are of a great interest and in some aspect are similar to our treatment). At the end of Section 3 we deal with the Weyl, symmetric and the Born–Jordan orderings in some details.

Section 4 is devoted to the momentum operator in curvilinear coordinates. Using our formalism we find the general form of the operator. This generalizes the results of other authors [19] and [20]. In the next paper we deal with the generalized Moyal bracket and related topics.

2. Operator ordering

In this section the problem of the operator ordering in quantum mechanics is considered. First we make some assumptions which seem to be natural and then we find the general formulas for the operator ordering which follow from those assumptions. For simplicity we mainly deal with the flat 2-dimensional phase space $\Gamma_2 = R^1 \times R^1$ but it is a straightforward matter to generalize our results to the case of a flat 2n-dimensional phase space $\Gamma_{2n} = R^n \times R^n$. Moreover, we believe that the considerations presented here can be quickly carried over to the case of any curved phase space.

The problem of operator ordering can be stated as follows. Given the flat 2-dimensional phase space $\Gamma_2 = R^1 \times R^1$ with the symplectic 2-form $\omega = dq \wedge dp$ one looks for the rule which assigns to any classical observable $A = A(p, q)$ the selfadjoint operator $\hat{A}$ in Hilbert space. The map $A(p, q) \longmapsto \hat{A}$ we denote by W_g and call the *generalized Weyl application* [1–8]. We write $W_g(A(p, q)) = \hat{A}$ or simply $W_g(A) = \hat{A}$.

Consider the family of classical observables of the form

$$A_{m,n} \stackrel{\text{def}}{=} p^m q^n ; \quad m, n \in N . \tag{2.1}$$

We make the following assumptions

(i) The operator $\hat{A}_{m,n} = W_g(A_{m,n})$ is selfadjont for every $m, n \in N$ and it has the form of a polynomial with respect to the operators $\hat{p}$ and $\hat{q}$. Moreover, each term of $\hat{A}_{m,n}$ is of the same unit as the classical observable $A_{m,n}$. In particular

$$W_g(1) = \hat{1}, \tag{2.2}$$

where 1 is a constant dimensionless function and $\hat{1}$ is an identity operator. Let us make a new function $\tilde{A}_{m,n}(p,q)$ from the operator $\hat{A}_{m,n}$ by the change:

$$\hat{p} \to p , \quad \hat{q} \to q , \quad \hat{1} \to 1 .$$

(ii) For every $m, n \in N$

$$\lim_{\hbar \to 0} \tilde{A}_{m,n}(p,q) = A_{m,n}(p,q) . \tag{2.3}$$

(iii) For every $m, n \in N$, $m \geq 1$,

$$\hat{A}_{m-1,n} = m^{-1} \frac{\partial \hat{A}_{m,n}}{\partial \hat{p}} ; \tag{2.4}$$

for every $m, n \in N$, $n \geq 1$,

$$\hat{A}_{m,n-1} = n^{-1} \frac{\partial \hat{A}_{m,n}}{\partial \hat{q}} , \tag{2.5}$$

where

$$\frac{\partial \hat{A}_{m,n}}{\partial \hat{p}} \stackrel{\text{def}}{=} \frac{1}{i\hbar}[\hat{q}, \hat{A}_{m,n}] ,$$
$$\frac{\partial \hat{A}_{m,n}}{\partial \hat{q}} \stackrel{\text{def}}{=} -\frac{1}{i\hbar}[\hat{p}, \hat{A}_{m,n}] . \tag{2.6}$$

(Compare with [1]).

Remarks

a) The operators $\hat{p}$ and $\hat{q}$ are the momentum and position operators, respectively. In the general case of the flat phase space $\Gamma_{2n} = R^n \times R^n$ one deals with n momentum operators $\hat{p}_k$ and n position operators

$\hat{q}_k$, $k = 1, \ldots, n$. They are selfadjoint operators satisfying the well known commutation relations

$$\begin{aligned} [\hat{q}_k, \hat{q}_l] &= 0\,, \\ [\hat{p}_k, \hat{p}_l] &= 0\,, \\ [\hat{q}_k, \hat{p}_l] &= i\hbar\delta_{kl}\,, \qquad k, l = 1, \ldots, n\,. \end{aligned} \tag{2.7}$$

b) Condition *(ii)* means that $A_{m,n}(p,q)$ is the classical limit of the operator $\hat{A}_{m,n}$.

We now consider the consequences of the assumptions *(i)*, *(ii)* and *(iii)*.

From *(i)* and the fact that we have at our disposal one universal parameter *i.e.* Planck's constant $\hbar$ which has the same dimension as pq or $\hat{p}\hat{q}$ (see (2.7)) it follows that

$$\hat{A}_{m,n} = \sum_{s=0}^{\min(m,n)} g(m,n,s)\hbar^s \hat{p}^{m-s}\hat{q}^{n-s}, \tag{2.8}$$

where $g(m,n,s)$ are some complex numbers that should be chosen in such a manner that the operator $\hat{A}_{m,n}$ defined by (2.8) be selfadjoint *i.e.*

$$\hat{A}^{+}_{m,n} = \hat{A}_{m,n}\,, \tag{2.9}$$

where, as usually, $\hat{A}^{+}_{m,n}$ denotes the operator adjoined to $\hat{A}_{m,n}$. Substituting (2.8) into (2.9) and applying the formula (which can be immediately proved by the induction)

$$\begin{aligned} [\hat{q}^k, \hat{p}^j] &= \sum_{r=1}^{\min(j,k)} b(j,k,r)\hbar^r \hat{p}^{j-r}\hat{q}^{k-r}, \\ b(j,k,r) &\stackrel{\text{def}}{=} i^r \binom{j}{r}\binom{k}{r} r!\,, \quad r \geq 1\,, \end{aligned} \tag{2.10}$$

one gets

$$\sum_{r=0}^{\min(m,n)} g^*(m,n,s)\hbar^s \sum_{r=0}^{\min(m-s,n-s)} b(m-s,n-s,r)\hbar^r \hat{p}^{m-s-r}\hat{q}^{n-s-r} = \sum_{s=0}^{\min(m,n)} g(m,n,s)\hbar^s \hat{p}^{m-s}\hat{q}^{n-s}, \tag{2.11}$$

where $b(j,k,0) \stackrel{\text{def}}{=} 1$ and the star "*" stands for the complex conjugation.

We conclude that

$$g(m,n,l)=\sum_{s=0}^{l} g^*(m,n,s)b(m-s,n-s,l-s) \tag{2.12}$$

or in another form

$$\mathrm{Im}[g(m,n,l)]=\frac{1}{2i}\sum_{s=0}^{l-1} g^*(m,n,s)b(m-s,n-s,l-s)\,. \tag{2.13}$$

As we will see in a moment the conditions (2.13) can be remarkably simplified if the remaining assumptions *(ii)* and *(iii)* are employed. Using (2.8) one quickly finds that *(ii)* leads to the following condition

$$g(m,n,0)=1\,, \tag{2.14}$$

for every $m,n\in N$.

Then we consider *(iii)*. From (2.4), (2.6) and (2.8) one gets

$$\sum_{s=0}^{\min(m-1,n)} g(m-1,n,s)\hbar^s\hat{p}^{m-1-s}\hat{q}^{n-s}= m^{-1}\sum_{s=0}^{\min(m,n)}(m-s)g(m,n,s)\hbar^s\hat{p}^{m-s-1}\hat{q}^{n-s}\,, \tag{2.15}$$

for every $m,n\in N,\quad m\geq 1$.

Hence

$$g(m-1,n,s)=\frac{m-s}{m}g(m,n,s)\,, \tag{2.16}$$

for every $m,n,s\in N,\quad m\geq s+1,\quad n\geq s$.

Analogously (2.5), (2.6) and (2.8) yield

$$g(m,n-1,s)=\frac{n-s}{n}g(m,n,s)\,, \tag{2.17}$$

for every $m,n,s\in N\,,\ m\geq s\,,\ \ n\geq s+1\,$.

The relations (2.16) and (2.17) show that given $s\in N$ the coefficient $g(m,n,s)$ for any $m,n\in N,\ \ m\geq s\,,\ \ n\geq s\,$, is defined by $g(s,s,s)$.

Indeed, from (2.16) and (2.17) we obtain for any $m>s\,,\ \ n>s$

$$\begin{aligned} g(m,n,s)&=g((m-s)+s,(n-s)+s,s)\\ &=\frac{(s+1)\ldots m}{1\ldots(m-s)}\,\frac{(s+1)\ldots n}{1\ldots(n-s)}g(s,s,s)\\ &=\binom{m}{s}\binom{n}{s}g(s,s,s)\,. \end{aligned} \tag{2.18}$$

Then one easily finds the general formula

$$g(m,n,s) = \binom{m}{s}\binom{n}{s} g(s,s,s)\,, \tag{2.19}$$

for every $m, n, s \in N\,,\quad m \geq s\,,\quad n \geq s$. Concluding, the assumption *(ii)* leads to (2.14) and *(iii)* yields (2.19).

Remark

From the formula (2.19) we infer that assuming $g(0,0,0) = 1$ one obtains (2.14).

Consider the conditions (2.12) for $m = n = l$. Using also (2.19) one gets

$$\begin{aligned} g(m,m,m) &= \sum_{s=0}^{m} g^*(m,m,s) b(m-s, m-s, m-s) \\ &= \sum_{s=0}^{m} \left[\binom{m}{s}\right]^2 g^*(s,s,s) b(m-s, m-s, m-s)\,. \end{aligned} \tag{2.20}$$

Finally, substituting into (2.20) $b(m-s, m-s, m-s)$ as defined by (2.10) and remembering that $b(j,k,0) = 1$ we obtain for each $m \in N$

$$g(m,m,m) = \sum_{s=0}^{m} i^{\,m-s}(m-s)! \left[\binom{m}{s}\right]^2 g^*(s,s,s)\,, \tag{2.21}$$

or, equivalently

$$Im[g(m,m,m)] = \tfrac{1}{2} \sum_{s=0}^{m-1} i^{\,m-s-1}(m-s)! \left[\binom{m}{s}\right]^2 g^*(s,s,s)\,. \tag{2.22}$$

Now we prove an important proposition.

Proposition 2.1 *The relations (2.21) (or, equivalently (2.22)) and (2.19) imply (2.12) (or, equivalently (2.13)).*

Proof

$$g(m,n,l) \overset{\text{by}(2.19)}{=} \binom{m}{l}\binom{n}{l} g(l,l,l)$$

$$\overset{\text{by}(2.21)}{=} \sum_{s=0}^{l} \binom{m}{l}\binom{n}{l} i^{\,l-s}(l-s)! \left[\binom{l}{s}\right]^2 g^*(s,s,s)$$

$$= \sum_{s=0}^{l} \binom{m}{s}\binom{n}{s} g^*(s,s,s) i^{\,l-s} \frac{(m-s)!(n-s)!}{(m-l)!(n-l)!(l-s)!}$$

$$\overset{\text{by}(2.19)}{=} \sum_{s=0}^{l} g^*(m,n,s) i^{\,l-s} \binom{m-s}{l-s}\binom{n-s}{l-s}(l-s)!$$

$$\overset{\text{by}(2.10)}{=} \sum_{s=0}^{l} g^*(m,n,s) b(m-s,n-s,l-s)\,.$$

(In the last step we also use the formula $b(j,k,0) = 1$.) This completes the proof. ■

Gathering all that one concludes that *the operator ordering satisfying the axioms (i), (ii) and (iii) is defined by the sequence of complex numbers* $g(s,s,s)$, $s \in N$, *such that* $g(0,0,0) = 1$ *and (2.21) (or, equivalently (2.22)) holds. The remaining coefficients* $g(m,n,s)$ *are determined by the formula (2.19).*

We have not succeeded in finding the general solution of (2.21). However, as it is shown in the next section, making some additional reasonable assumption one can find the general form of $g(m,n,s)$.

3. Generalized Weyl application and the Fourier transformation

The assumption we now make is in fact a definition of the generalized Weyl application W_g. Namely we assume
(iv) The generalized Weyl application W_g *is defined on a class of complex functions on phase space by the* C*-linear extension of the mapping* $A_{m,n} \to \hat{A}_{m,n}$.

Consider a complex function $F = F(p,q)$ that has the Fourier representation

$$F = F(p,q) = \frac{1}{(2\pi)^2} \int_{R^2} \tilde{F}(\lambda,\mu) \exp[i(\lambda p + \mu q)] d\lambda d\mu\,, \tag{3.1}$$

where $\tilde{F} = \tilde{F}(\lambda,\mu)$ is the Fourier transform of F

$$\tilde{F} = \tilde{F}(\lambda,\mu) = \int_{R^2} F(p,q) \exp[-i(\lambda p + \mu q)] dp dq\,. \tag{3.2}$$

From the assumption *(iv)* it follows that

$$\hat{F} \stackrel{\rm def}{=} W_g(F(p,q)) = \frac{1}{(2\pi)^2} \int\limits_{R^2} \tilde{F}(\lambda,\mu) W_g\left(\exp[i(\lambda p + \mu q)]\right) d\lambda d\mu \,. \quad (3.3)$$

Define

$$\hat{F}_0(\lambda,\mu) \stackrel{\rm def}{=} W_g\left(\exp[i(\lambda p + \mu q)]\right) . \quad (3.4)$$

Applying *(iv)* we get

$$\begin{aligned} \hat{F}_0(\lambda,\mu) &= W_g\left(\sum_{k,l=0}^{\infty} \frac{(i\lambda)^k}{k!}\frac{(i\mu)^l}{l!} p^k q^l\right) \\ &= \sum_{k,l=0}^{\infty} \frac{(i\lambda)^k}{k!}\frac{(i\mu)^l}{l!} W_g(p^k q^l) = \sum_{k,l=0}^{\infty} \frac{(i\lambda)^k}{k!}\frac{(i\mu)^l}{l!} \hat{A}_{k,l} \,, \end{aligned} \quad (3.5)$$

where $\hat{A}_{k,l}$ is defined by (2.8). Substituting (2.8) into (3.5) and using also (2.19) one finds

$$\begin{aligned} \hat{F}_0(\lambda,\mu) &= \sum_{k,l=0}^{\infty} \left\{ \frac{(i\lambda)^k}{k!}\frac{(i\mu)^l}{l!} \sum_{s=0}^{\min(k,l)} g(k,l,s)\,\hbar^s \hat{p}^{k-s}\hat{q}^{l-s} \right\} \\ &= \sum_{m,n,s=0}^{\infty} \frac{(i\lambda)^{m+s}}{(m+s)!}\frac{(i\mu)^{n+s}}{(n+s)!}\, g(m+s,n+s,s)\,\hbar^s \hat{p}^m \hat{q}^n \\ &= \sum_{m,n,s=0}^{\infty} (-1)^s \frac{(\lambda\mu\hbar)^s}{(s!)^2} \binom{m+s}{s} \binom{n+s}{s} \\ &\times \frac{s!}{(m+s)!}\frac{s!}{(n+s)!} g(s,s,s)(i\lambda)^m (i\mu)^n \hat{p}^m \hat{q}^n \\ &= \left(\sum_{s=0}^{\infty} (-1)^s \frac{(\lambda\mu\hbar)^s}{(s!)^2} g(s,s,s)\right) \left(\sum_{m=0}^{\infty} \frac{(i\lambda)^m}{m!}\hat{p}^m\right) \left(\sum_{n=0}^{\infty} \frac{(i\mu)^n}{n!}\hat{q}^n\right) \\ &= f(\lambda\mu\hbar)\exp(i\lambda\hat{p})\exp(i\mu\hat{q}) \,, \end{aligned} \quad (3.6)$$

where

$$f(\lambda\mu\hbar) \stackrel{\rm def}{=} \sum_{s=0}^{\infty} (-1)^s \frac{(\lambda\mu\hbar)^s}{(s!)^2} g(s,s,s) \,. \quad (3.7)$$

Employing the Baker-Campbell-Hausdorff formula

$$\begin{aligned}\exp[i(\lambda\hat{p}+\mu\hat{q})] &= \exp(i\lambda\hat{p})\exp(i\mu\hat{q})\exp\left[-\frac{1}{2}[i\lambda\hat{p},i\mu\hat{q}]\right]\\ &= \exp(i\lambda\hat{p})\exp(i\mu\hat{q})\exp\left(-\frac{i\lambda\mu\hbar}{2}\right)\\ &= \exp(i\mu\hat{q})\exp(i\lambda\hat{p})\exp\left(\frac{i\lambda\mu\hbar}{2}\right)\end{aligned} \tag{3.8}$$

one has

$$\begin{aligned}\hat{F}_0(\lambda,\mu) &= f(\lambda\mu\hbar)\exp(i\lambda\hat{p})\exp(i\mu\hat{q})\\ &= f(\lambda\mu\hbar)\exp\left(\frac{i\lambda\mu\hbar}{2}\right)\exp[i(\lambda\hat{p}+\mu\hat{q})].\end{aligned} \tag{3.9}$$

Inserting (3.9) into (3.3) we obtain

$$\begin{aligned}\hat{F} \stackrel{\text{def}}{=} W_g(F(p,q)) &= \frac{1}{(2\pi)^2}\int\limits_{R^2}\tilde{F}(\lambda,\mu)f(\lambda\mu\hbar)\exp(i\lambda\hat{p})\exp(i\mu\hat{q})d\lambda d\mu\\ &= \frac{1}{(2\pi)^2}\int\limits_{R^2}\tilde{F}(\lambda,\mu)f(\lambda\mu\hbar)\exp\left(\frac{i\lambda\mu\hbar}{2}\right)\exp[i(\lambda\hat{p}+\mu\hat{q})]d\lambda d\mu.\end{aligned} \tag{3.10}$$

The formula (3.10) has been considered in several works, [5, 17] and [18]. Here we have found it from some simple and natural axioms.

From (3.7) and (2.14) we get

$$f(0)=1. \tag{3.11}$$

Consider a real function (an observable) $A = A(p,q)$. Then

$$A = A^* \Longleftrightarrow \tilde{A}^*(\lambda,\mu) = \tilde{A}(-\lambda,-\mu). \tag{3.12}$$

From our axioms *(i)* and *(iv)* it follows that $\hat{A} \stackrel{\text{def}}{=} W_g(A(p,q))$ is the self-adjoint operator *i.e.*,

$$\hat{A}^+ = \hat{A}. \tag{3.13}$$

Consequently, (3.10), (3.12) and (3.13) lead to the condition on the function $f = f(\lambda\mu\hbar)$

$$\left(f(\lambda\mu\hbar)\exp\left(\frac{i\lambda\mu\hbar}{2}\right)\right)^* = f(\lambda\mu\hbar)\exp\left(\frac{i\lambda\mu\hbar}{2}\right). \tag{3.14}$$

Equivalently, (3.14) can be written as follows

$$f(\lambda\mu\hbar) = \alpha(\lambda\mu\hbar)\exp\left(\frac{-i\lambda\mu\hbar}{2}\right), \tag{3.15}$$

where $\alpha = \alpha(\lambda\mu\hbar)$ is a real function, and by (3.11)

$$\alpha(0) = 1. \tag{3.16}$$

Finally, using (3.7) one can find $g(s,s,s)$ to be

$$g(s,s,s) = (-1)^s s!\, f^{(s)}(0). \tag{3.17}$$

(Compare with [18]).

Now we consider some well known examples of the operator ordering (compare with [17] and [18])

a) **Weyl's ordering.**
Here $\alpha(\lambda\mu\hbar) = 1$. Then one gets

$$g(s,s,s) = \left(\frac{i}{2}\right)^s s! \tag{3.18}$$

and by (2.19) and (2.10)

$$g(m,n,s) = \left(\frac{i}{2}\right)^s \binom{m}{s}\binom{n}{s} s! = \frac{b(m,n,s)}{2^s}. \tag{3.19}$$

b) **Symmetric ordering**
In this case $\alpha(\lambda\mu\hbar) = \cos\left(\frac{\lambda\mu\hbar}{2}\right)$. Then we quickly find

$$g(0,0,0) = 1 \text{ and } g(s,s,s) = \frac{i^s}{2}s! \text{ for } s \geq 1, \tag{3.20}$$

and consequently

$$g(m,n,0) = 1$$

and

$$g(m,n,s) = \frac{i^s}{2}\binom{m}{s}\binom{n}{s} s! = \frac{b(m,n,s)}{2} \text{ for } s \geq 1. \tag{3.21}$$

c) **Ordering of Born and Jordan**

Now $\alpha(\lambda\mu\hbar) = \frac{\sin(\frac{\lambda\mu\hbar}{2})}{\frac{\lambda\mu\hbar}{2}}$. Thus

$$g(s,s,s) = \frac{i^s}{s+1} s! \tag{3.22}$$

and

$$g(m,n,s) = \frac{i^s}{s+1}\binom{m}{s}\binom{n}{s} s! = \frac{b(m,n,s)}{s+1}\,. \tag{3.23}$$

In order to generalize our formalism on the case of the $2n$-dimensional flat phase space $\Gamma_{2n} = R^n \times R^{2n}$ endowed with the symplectic form $\omega = dq_1 \wedge dp_1 + \cdots + dq_n \wedge dp_n$ we make the following assumption:

(v) The generalized Weyl application W_g for $\Gamma_{2n} = R^n \times R^{2n}$ is defined by the C-linear extension of the following formula

$$W_g(p_1^{j_1} q_1^{k_1} \ldots p_n^{j_n} q_n^{k_n}) = W_g(p_1^{j_1} q_1^{k_1}) \ldots W_g(p_n^{j_n} q_n^{k_n})\,. \tag{3.24}$$

From this assumption one implies that if $F = F(p_1,\ldots,p_n,q_1\ldots q_n)$ is a function on Γ_{2n} then

$$\begin{aligned}\hat{F} &\stackrel{\text{def}}{=} W_g(F(p_1,\ldots,p_n,q_1\ldots q_n))\\ &= \frac{1}{(2\pi)^{2n}} \int\limits_{R^{2n}} \tilde{F}(\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_n) f(\lambda_1\mu_1\hbar)\cdots f(\lambda_n\mu_n\hbar)\\ &\quad\times \exp(i\vec{\lambda}\cdot\hat{\vec{p}})\exp(i\vec{\mu}\cdot\hat{\vec{q}})d\lambda_1\ldots d\lambda_n d\mu_1\ldots d\mu_n\,,\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}\tilde{F} = \tilde{F}(\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_n) &= \int\limits_{R^{2n}} F(p_1,\ldots,p_n,q_1\ldots q_n)\\ &\quad\times \exp[-i(\vec{\lambda}\cdot\vec{p} + \vec{\mu}\cdot\vec{q})]dp_1\ldots dp_n dq_1\ldots dp_n\end{aligned} \tag{3.26}$$

and $\vec{\lambda}\cdot\vec{p} \stackrel{\text{def}}{=} \lambda_1 p_1 + \ldots + \lambda_n p_n$, $\vec{\lambda}\cdot\hat{\vec{p}} \stackrel{\text{def}}{=} \lambda_1\hat{p}_1 + \ldots + \lambda_n\hat{p}_n, \ldots$ *etc.*

In the next section we show how the formalism presented here enables us to define the generalized momentum operator.

4. The momentum operator in curvilinear coordinates

Consider $2n$-dimensional flat phase space $\Gamma_{2n} = R^n \times R^n$. Let Q^j, $j = 1,\ldots, n$, be some curvilinear coordinates and denote by P_j, $j = 1,\ldots,n$, canonically conjugate momenta. Thus we have (here we use the symbol q^j instead of q_j)

$$Q^j = Q^j(q^k)\,, \quad P_j = \frac{\partial q^k}{\partial Q^j} p_k\,, \quad j = 1, \ldots, n\,, \tag{4.1}$$

where the Einstein summation convention is assumed. Then from (3.25) one gets

$$\hat{P}_j = W_g(P_j) = \frac{1}{(2\pi)^{2n}} \int\limits_{R^{2n}} \tilde{P}_j(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n) f(\lambda_1 \mu_1 \hbar) \ldots f(\lambda_n \mu_n \hbar) \times \exp(i\vec{\lambda} \cdot \hat{\vec{p}}) \exp(i\vec{\mu} \cdot \hat{\vec{q}}) d\lambda_1 \ldots d\lambda_n d\mu_1 \ldots d\mu_n\,, \tag{4.2}$$

where

$$\begin{aligned} \tilde{P}_j =& \tilde{P}_j(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n) \\ =& \int\limits_{R^{2n}} \frac{\partial q^k}{\partial Q^j} p_k \exp[-i(\vec{\lambda} \cdot \vec{p} + \vec{\mu} \cdot \vec{q})] dp_1 \ldots dp_n dq^1 \ldots dq^n \\ =& (2\pi)^n i \delta(\lambda_1) \ldots \delta'(\lambda_k) \ldots \delta(\lambda_n) \\ & \times \int\limits_{R^n} \frac{\partial q^k}{\partial Q^j} \exp(-i\vec{\mu} \cdot \vec{q}) dq^1 \ldots dq^n \end{aligned} \tag{4.3}$$

(Summation over k).

Substituting (4.3) into (4.2) and employing (3.11) and (3.8) we obtain

$$\hat{P}_j = \frac{i}{(2\pi)^n} \sum_{k=1}^{n} \int\limits_{R^{n+1}} \left\{ \left[\int\limits_{R^n} \frac{\partial q^k}{\partial Q^j} \exp(-i\vec{\mu} \cdot \vec{q}) dq^1 \ldots dq^n \right] \exp(i\vec{\mu} \cdot \hat{\vec{q}}) \times \delta'(\lambda_k) f(\lambda_k \mu_k \hbar) \exp(i\lambda_k \mu_k \hbar) \exp(i\lambda_k \hat{p}_k) \right\} d\lambda_k d\mu_1 \ldots d\mu_n\,. \tag{4.4}$$

Then simple calculations show that in the Schrödinger representation (4.4) leads to

$$\begin{aligned} \hat{P}_j &= \frac{\partial q^k}{\partial Q^j} \hat{p}_k - (i + f'(0))\, \hbar \, \frac{\partial}{\partial q^k} \left(\frac{\partial q^k}{\partial Q^j} \right) \\ &= \frac{\partial q^k}{\partial Q^j} \hat{p}_k - (i + f'(0))\, \hbar \, \frac{\partial Q^l}{\partial q^k} \frac{\partial^2 q^k}{\partial Q^l \partial Q^j}\,, \end{aligned} \tag{4.5}$$

where $\hat{p}_k = -i\hbar \dfrac{\partial}{\partial q^k}$.

Let ds^2 denotes the metric on the configuration space *i.e.*,

$$ds^2 = dq^1 \otimes dq^1 + \ldots + dq^n \otimes dq^n = \gamma_{jk} dQ^j \otimes dQ^k. \tag{4.6}$$

Then it is well known that the components Γ^l_{jm} of the connection with respect to the coordinate system $Q^1, \ldots, Q^n$ read [21]

$$\Gamma^l_{jm} = \frac{\partial Q^l}{\partial q^k} \frac{\partial^2 q^k}{\partial Q^m \partial Q^j}. \tag{4.7}$$

(Remember that the connection components with respect to the coordinate system $q^1, \ldots, q^n$ vanish). Comparing (4.7) with (4.5) and using (3.17) and the well known formula [21]

$$\Gamma^l_{jl} = \frac{\partial \ln \sqrt{\gamma}}{\partial Q^j}, \quad \gamma \stackrel{\text{def}}{=} \det(\gamma_{jk}), \tag{4.8}$$

we get

$$\begin{aligned} \hat{P}_j &= \frac{\partial q^k}{\partial Q^j} \hat{p}_k - (i - g(1,1,1))\hbar \Gamma^l_{jl} \\ &= \frac{\partial q^k}{\partial Q^j} \hat{p}_k - (i - g(1,1,1))\hbar \frac{\partial \ln \sqrt{\gamma}}{\partial Q^j}. \end{aligned} \tag{4.9}$$

This is our general formula for the momentum operator with respect to the curvilinear coordinates Q^j.

In particular assuming

$$g(1,1,1) = \frac{i}{2} \tag{4.10}$$

one obtains for (4.9) the form considered in Refs [19, 20]

$$\hat{P}_j = \frac{\partial q^k}{\partial Q^j} \hat{p}_k - i\hbar \frac{\partial \sqrt[4]{\gamma}}{\partial Q^j}. \tag{4.11}$$

Notice that the relation (4.10) is satisfied if and only if

$$W_g(pq) = \tfrac{1}{2}(\hat{p}\hat{q} + \hat{q}\hat{p}). \tag{4.12}$$

From (3.18), (3.20) and (3.22) one infers that (4.10) holds true for the Weyl, symmetric and Born–Jordan orderings.

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