SUPERSYMMETRIC EXTENSIONS OF κ-POINCARÉ ALGEBRA

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The supersymmetric extensions of κ-Poincaré algebra are considered. All extensions of algebraic sector are classified. It is shown that, under some assmptions, no regular coproduct can be introduced.

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1. Introduction

Recently some attention has been paid to the deformed Poincaré algebra introduced by Lukierski, Nowicki and Ruegg [1] (see also [2]). It seems that it posses some atractive features and is worth to be considered in more detail. In particular one can pose the question concerning the supersymmetric extensions of deformed Poincaré algebra. This question was addressed and solved in Ref. [3]; the solution was obtained by applying the combined

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deformation and contraction procedure to the algebra $OS_p(1|4)$. The characteristic feature of the resulting structure is that the κ-Poincaré algebra *does not* form the subalgebra of the superalgebra.

The superalgebra introduced in Ref. [3] was further analysed in series of papers [4–6]. It appeared that one can change the basis in deformed superalgebra in such a way that in the new basis κ-Poincaré algebra *is* a subalgebra; however, there is always some price to be payed for that (see Section 4). This shows that the problem deserves some further studies.

In the present paper we analyse the problem of the supersymmetric extensions of the κ-Poincaré algebra in more detail. In Section 2 we classify the possible $N = 1$ extensions of the κ-Poincaré algebra making a number of rather natural assumptions. In Section 3 the coalgebra sector is analysed. A simple Ansatz is made for the coproduct of supercharges. It is then shown that, within this Ansatz, no regular (analytic) solution exists. Finally, Section 4 is devoted for short conclusions; in particular, the results obtained previously are shortly discussed within the framework studied here.

2. The algebraic sector

We are looking for deformed $N = 1$ supersymmetry algebra. The following assumptions are made:

(i) our algebra contains κ-Poincaré algebra as subalgebra;

(ii) there are two additional femionic generators Q_α, Q^+_α which obey

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= 0\,, \\ \{Q^+_\alpha, Q^+_\beta\} &= 0\,, \end{aligned} \tag{1}$$

and transform as spinors under SU(2):

$$\begin{aligned} [M_i, Q_\alpha] &= -\tfrac{1}{2}(\sigma_i Q)_\alpha\,, \\ [M_i, Q^+_\alpha] &= \tfrac{1}{2}(Q^+\sigma_i)_\alpha\,; \end{aligned} \tag{2}$$

(iii) the supercharges commute with fourmomentum

$$\begin{aligned} [Q_\alpha, P_\mu] &= 0\,, \\ [Q^+_\alpha, P_\mu] &= 0\,; \end{aligned} \tag{3}$$

(iv) the superalgebra admits involution such that κ-Poincaré subalgebra is real and Q is transformed to Q^+ ;

(v) the following commutation rules are assumed to hold;

$$[L_i, Q_\alpha] = i(\gamma_i(P))_{\alpha\beta} Q_\beta\,,$$
$$\{Q_\alpha, Q_\beta^+\} = \lambda_{\alpha\beta}(P)\,, \qquad \lambda^+ = \lambda\,, \tag{4}$$

here γ_i and λ are P-dependent matrices.

In order to find the possible forms of the superalgebra under consideration we use the following Jacobi identities

$$[L_i,[L_j,Q]] + [L_j,[Q,L_i]] + [Q,[L_i,L_j]] = 0\,,$$
$$[L_i,\{Q_\alpha,Q_\beta^+\}] - \{[L_i,Q_\alpha],Q_\beta^+\} - \{Q_\alpha,[L_i,Q_\beta^+]\} = 0\,. \tag{5}$$

They give, respectively

$$[L_i,\gamma_j] - [L_j,\gamma_i] - i[\gamma_i,\gamma_j] = \frac{1}{2}\cosh\left(\frac{P_0}{\kappa}\right)\varepsilon_{ijk}\sigma_k - \frac{1}{8\kappa^2}\varepsilon_{ijk}P_kP_n\sigma_n\,, \tag{6}$$

$$[L_i,\lambda] = i\gamma_i\lambda + i\lambda\gamma_i^+\,. \tag{7}$$

Taking into account that the SU(2) subalgebra spanned by rotation generators is not deformed we can write the following general expressions for λ and γ_i :

$$\lambda = a + bP_n\sigma_n\,, \tag{8}$$

$$\gamma_i = \frac{1}{2}(f\sigma_i + gP_iP_n\sigma_n + h\varepsilon_{ikn}P_k\sigma_n + dP_i)\,; \tag{9}$$

here a, b, f, g, h, d are functions of $\vec{P}^2$ and P_0. However, it follows from our assumptions that

$$M^2 = 4\kappa^2\sinh^2\left(\frac{P_0}{2\kappa}\right) - \vec{P}^2$$

is the Casimir operator of the superalgebra under consideration. Therefore, a, b, *etc.* can be viewed as functions of P_0 and M^2 (or $\vec{P}^2$ and M^2).

Equations (6), (7) and (8), (9) give the following set of equations for scalar functions a, b, *etc.*:

$$\frac{i}{2}f' - \frac{i\kappa}{2}g\sinh\left(\frac{P_0}{\kappa}\right) + \frac{1}{2}fh + \frac{1}{2}gh\left(4\kappa^2\sinh^2\left(\frac{P_0}{2\kappa}\right) - M^2\right) = 0\,, \tag{10a}$$

$$-i\kappa h\sinh\left(\frac{P_0}{\kappa}\right) + \frac{1}{2}f^2 - \frac{i}{2}h'\left(4\kappa^2\sinh^2\left(\frac{P_0}{2\kappa}\right) - M^2\right)$$
$$+\frac{1}{2}fg\left(4\kappa^2\sinh^2\left(\frac{P_0}{2\kappa}\right) - M^2\right) = \frac{1}{2}\cosh\left(\frac{P_0}{\kappa}\right)\,, \tag{10b}$$

$$\frac{1}{2}h^2 + \frac{i}{2}h' - \frac{1}{2}fg = -\frac{1}{8\kappa^2}\,, \tag{10c}$$

$$a' = a\left(\frac{d+\bar{d}}{2}\right) + b\left(\frac{f+\bar{f}}{2}\right) + b\left(\frac{g+\bar{g}}{2}\right)\left(4\kappa^2\sinh^2\left(\frac{P_0}{2\kappa}\right) - M^2\right), \quad (10d)$$

$$b' = a\left(\frac{g+\bar{g}}{2}\right) + b\left(\frac{d+\bar{d}}{2}\right) + ib\left(\frac{h-\bar{h}}{2}\right), \quad (10e)$$

$$\kappa b\sinh\left(\frac{P_0}{\kappa}\right) = a\left(\frac{f+\bar{f}}{2}\right) - ib\left(\frac{h-\bar{h}}{2}\right)\left(4\kappa^2\sinh^2\left(\frac{P_0}{2\kappa}\right) - M^2\right), \quad (10f)$$

$$a\left(\frac{h+\bar{h}}{2}\right) + ib\left(\frac{f-\bar{f}}{2}\right) = 0; \quad (10g)$$

here prime denotes differentiation with respect to P_0 with M^2 fixed.

The above system of equations looks quite complicated. However, it posses a rather rich set of symmetries. They follow from the fact that the structure assumed is invariant under some redefinitions of supercharges. Namely we can make the following redefinitions:

(a)

$$Q \to e^{\varrho}Q, \qquad Q^+ \to e^{\bar{\varrho}}Q^+, \quad (11)$$

where $\varrho = \varrho(P_0, M^2)$ is an arbitrary complex function. The corresponding symmetry transformations read

$$a \to e^{\varrho+\bar{\varrho}}a, \qquad b \to e^{\varrho+\bar{\varrho}}b, \qquad d \to d + 2\varrho', \quad (12)$$

all other functions being unchanged;

(b)

$$Q \to e^{\alpha\hat{P}_n\sigma_n}Q, \qquad Q^+ \to Q^+e^{\bar{\alpha}\hat{P}_n\sigma_n}, \quad (13)$$

with arbitrary complex $\alpha = \alpha(P_0, M^2)$; the relevant symmetry reads

$$\begin{aligned}
a &\to a\cosh(\alpha+\bar{\alpha}) + b|\vec{P}|\sinh(\alpha+\bar{\alpha}),\\
b &\to b\cosh(\alpha+\bar{\alpha}) + \frac{a}{|\vec{P}|}\sinh(\alpha+\bar{\alpha}),\\
f &\to f\cosh 2\alpha + ih|\vec{P}|\sinh 2\alpha + \frac{\kappa}{|\vec{P}|}\sinh\left(\frac{P_0}{\kappa}\right)\sinh 2\alpha,\\
g &\to g - \frac{2f}{|\vec{P}|^2}\sinh^2\alpha - \frac{ih}{|\vec{P}|}\sinh 2\alpha - \frac{\kappa}{|\vec{P}|^3}\sinh 2\alpha\sinh\left(\frac{P_0}{\kappa}\right) + \frac{2\alpha'}{|\vec{P}|},\\
h &\to h\cosh 2\alpha - \frac{if}{|\vec{P}|}\sinh 2\alpha - \frac{2i\kappa}{|\vec{P}|^2}\sinh^2\alpha\sinh\left(\frac{P_0}{\kappa}\right),\\
d &\to d.
\end{aligned} \quad (14)$$

Now we can analyse the set of equations (10). Transformation (14) allows us (with some exception — see below) to put

$$h = \frac{i}{2\kappa}, \quad (15)$$

which determines α up to the discrete number of possibilities. Analogously, equation (12) can be used to set

$$d = 0\,, \tag{16}$$

which determines ϱ up to complex additive constant.

Now, equations (10a)–(10c) give

$$f = \pm e^{-\frac{P_0}{2\kappa}}\,, \tag{17a}$$
$$h = \frac{i}{2\kappa}\,, \tag{17b}$$
$$g = 0\,. \tag{17c}$$

Equation (10g) is identically fulfiled while (10e) gives

$$b = Ce^{-\frac{P_0}{2\kappa}}\,. \tag{18}$$

Using the freedom in the choice of additive constant in ϱ we set $C = \pm 1$. Therefore we get finally

$$g = 0\,, \tag{19a}$$
$$d = 0\,, \tag{19b}$$
$$h = \frac{i}{2\kappa}\,, \tag{19c}$$
$$f = \pm e^{-\frac{P_0}{2\kappa}} \equiv \varepsilon_f e^{-\frac{P_0}{2\kappa}}\,, \tag{19d}$$
$$b = \pm e^{-\frac{P_0}{2\kappa}} \equiv \varepsilon_b e^{-\frac{P_0}{2\kappa}}\,. \tag{19e}$$

It remains to determine a. From the equations (10d), (10f) we get

$$a = \varepsilon_b \varepsilon_f \left(\kappa \sinh\left(\frac{P_0}{\kappa}\right) - \frac{\vec{P}^2}{2\kappa} \right)\,. \tag{19f}$$

In order to reveal the meaning of the above solutions let us take $\kappa \to \infty$ which yields :

$$[L_i, Q] = \frac{i}{2}\varepsilon_f \sigma_i Q\,, \tag{20a}$$
$$\{Q, Q^+\} = \varepsilon_b \varepsilon_f \left(P_0 + \varepsilon_f \vec{P}\vec{\sigma} \right)\,. \tag{20b}$$

We can get rid off the factor ε_b by noting that the κ-Poincaré algebra admits an automorphism $P_\mu \to \varepsilon_b P_\mu$ which allows to exclude ε_b. The sign factor ε_f determines the choice of $D^{(\frac{1}{2}0)}$ or $D^{(0\frac{1}{2})}$ representation for Q.

It remains to consider the case when the transformation (14) does not allow us to put $h = \frac{i}{2\kappa}$.

Let us write out the relevant equation:

$$\frac{i}{2\kappa}x = \frac{h}{2}\left(x^2+1\right) - \frac{if}{2|\vec{P}|}\left(x^2-1\right) - \frac{i\kappa}{2|\vec{P}|^2}\sinh\left(\frac{P_0}{\kappa}\right)\left(x^2-2x+1\right) , \quad (21)$$

where $x \equiv e^{2\alpha}$. If the above equation has double root at $x = 0$ the transformation under consideration cannot be performed. However, this can happen only provided

$$\frac{|\vec{P}|}{2\kappa} = \kappa \sinh\left(\frac{P_0}{\kappa}\right) , \quad (22)$$

which cannot be fulfiled generically.

Let us write our final solution (we choose $\varepsilon_f = 1$ in what follows)

$$[L_i, Q] = \frac{i}{2}e^{-\frac{P_0}{2\kappa}}\sigma_i Q + \frac{i}{2\kappa}\varepsilon_{ikn}P_n\sigma_n Q , \quad (23a)$$

$$\{Q, Q^+\} = \left(\kappa \sinh\left(\frac{P_0}{\kappa}\right) - \frac{\vec{P}^2}{2\kappa}\right) + e^{-\frac{P_0}{2\kappa}}P_n\sigma_n . \quad (23b)$$

Let us note that the second rule can be written in factorized form

$$\{Q, Q^+\} = \frac{1}{2\kappa}\left(2\kappa\cosh\left(\frac{P_0}{2\kappa}\right) - \vec{P}\vec{\sigma}\right)\left(2\kappa\sinh\left(\frac{P_0}{2\kappa}\right) + \vec{P}\vec{\sigma}\right) . \quad (24)$$

All other solutions can be obtained from the one given by equations (19) by applying the transformations (11) and (13) given above.

3. The coalgebra sector

We shall make the following rather mild assumptions concerning the coproducts:

(i) the coproduct for P_μ has the same form as for κ-Poincaré algebra.
(ii) the coproduct for Q_α can be written as :

$$\triangle Q_\alpha = F_{\alpha\beta}(P, \widetilde{P})Q_\beta + \sigma_{\alpha\beta}(P, \widetilde{P})\widetilde{Q}_\beta , \quad (25)$$

where for simplicity, we denote the second factor in tensor product by $\widetilde{\ }$.

The basic constraint for F and G is provided by the condition

$$\{\triangle Q, \triangle Q^{+}\} = a(\triangle P) + b(\triangle P)\triangle P_n \sigma_n \tag{26}$$

and reads

$$a(FF^{+}) + \widetilde{a}(GG^{+}) + bp_i(F\sigma_i F^{+}) + \widetilde{b}\widetilde{p}_i(G\sigma_i G^{+}) = a(\triangle p) + b(\triangle p)\triangle P_i \sigma_i\,; \tag{27}$$

here $a \equiv a(p)$, $b = b(p)$, $\widetilde{a} = a(\widetilde{P})$, $\widetilde{b} = b(\widetilde{P})$.

Let us consider the above condition for a and b given by equations (19). We get then

$$\begin{aligned}
&F\left(2\kappa \cosh\left(\frac{P_0}{2\kappa}\right) - \vec{P}\cdot\vec{\sigma}\right)\left(2\kappa \sinh\left(\frac{P_0}{2\kappa}\right) + \vec{P}\cdot\vec{\sigma}\right)F^{+} \\
&\quad + G\left(2\kappa \cosh\left(\frac{\widetilde{P}_0}{2\kappa}\right) - \vec{\widetilde{P}}\cdot\vec{\sigma}\right)\left(2\kappa \sinh\left(\frac{\widetilde{P}_0}{2\kappa}\right) + \vec{\widetilde{P}}\cdot\vec{\sigma}\right)G^{+} \\
&\quad = \left[e^{-\frac{P_0}{2\kappa}}\left(2\kappa \cosh\left(\frac{\widetilde{P}_0}{2\kappa}\right) - \vec{\widetilde{P}}\cdot\vec{\sigma}\right) + e^{\frac{\widetilde{P}_0}{2\kappa}}\left(2\kappa \sinh\left(\frac{P_0}{2\kappa}\right) - \vec{P}\cdot\vec{\sigma}\right)\right] \\
&\quad \times \left[e^{-\frac{P_0}{2\kappa}}\left(2\kappa \sinh\left(\frac{\widetilde{P}_0}{2\kappa}\right) + \vec{\widetilde{P}}\cdot\vec{\sigma}\right) + e^{\frac{\widetilde{P}_0}{2\kappa}}\left(2\kappa \sinh\left(\frac{P_0}{2\kappa}\right) + \vec{P}\cdot\vec{\sigma}\right)\right].
\end{aligned} \tag{28}$$

We shall now show that the above equation does not posses the solutions F, G which are everywhere regular. To show this let us note that the covariance under the rotation subalgebra spanned by M_i's imply the following structure for F and G :

$$\begin{aligned}
F &= A + (BP_i + C\widetilde{P}_i + D\varepsilon_{ijk}P_j\widetilde{P}_k)\sigma_i\,, \\
\sigma &= E + (HP_i + K\widetilde{P}_i + R\varepsilon_{ijk}P_j\widetilde{P}_k)\sigma_i\,;
\end{aligned} \tag{29}$$

here A, B etc. are functions of P_0 and M^2 to be determined by equation (28). Let us now specify P and $\widetilde{P}$ as follows:

$$P_\mu = (P_0, 0, 0, P_3), \qquad \widetilde{P}_\mu = (\widetilde{P}_0, 0, 0, \widetilde{P}_3)\,.$$

Then all matrices appearing in equation (28) become diagonal. Let us now choose

$$P_0 > 0, \qquad \widetilde{P}_0 > 0, \qquad P_3 = 2\kappa \cosh\left(\frac{P_0}{2\kappa}\right), \qquad \widetilde{P}_3 = 2\kappa \cosh\left(\frac{\widetilde{P}_0}{2\kappa}\right).$$

It is then easy to see that, if F and G are nonsingular, the $(1,1)$-element on the left-hand side vanishes while the corresponding element on the right-hand side does not.

Still we cannot conclude that there exists no regular coproduct for our algebra. This is because we could in principle choose another solutions to equations (19) and again look for regular coproduct (25). However, we can argue that no such choice exists. As we have shown all solutions to equations (10) are related to each other by the transformations (12), (14) (once the sign factors ε_f, ε_b have been choosen). Therefore, any solution can be obtained from our particular solution (19) by applying first, say, transformation (14) and then transformation (12).

Let us first consider the transformations (14). Equations (14) imply that $a^2 - b^2|\vec{p}|^2$ is an invariant. However, we have

$$a^2 - b^2|\vec{p}|^2 = (a + b\vec{p}\cdot\vec{\sigma})(a - b\vec{p}\cdot\vec{\sigma})\,. \tag{30}$$

The left-hand side can be now calculated using our particular solution (19). It vanishes, if

$$|\vec{p}| = 2\kappa\cosh\left(\frac{P_0}{2\kappa}\right)\,, \tag{31}$$

i.e.

$$M^2 = -4\kappa^2\,. \tag{32}$$

The matrices on the right-hand side of equation (27) commute and can be diagonalized simultaneously. For $\vec{p}\neq 0$ they are nonzero. Thus we conclude that for the momentum configuration (31) both have one eigenvalue zero and these eigenvalues stand on different places. As both matrices are related by transformation $\vec{p}\to-\vec{p}$ we conclude that for any a, b obtained from the particular solution (19) by transformation (14) the matrix $a + b\vec{p}\cdot\vec{\sigma}$ has the following property : if we fix the direction $\vec{n}$ of $\vec{p}$ then one of its eigenvalues vanishes for $\vec{p} = 2\kappa\cosh\left(\frac{P_0}{2\kappa}\right)\vec{n}$ while the second — for $\vec{p} = -2\kappa\cosh\left(\frac{P_0}{2\kappa}\right)\vec{n}$.

Let us again consider equation (27) for

$$P_\mu = \left(P_0, 0, 0, 2\kappa\cosh\left(\frac{P_0}{2\kappa}\right)\right), \qquad \widetilde{P}_\mu = \left(\widetilde{P}_0, 0, 0, 2\kappa\cosh\left(\frac{\widetilde{P}_0}{2\kappa}\right)\right).$$

The left-hand side of equation (27) has eigenvalue zero provided (new) F and G are nonsingular. On the other hand, using again the invariance of $a^2 - b^2|\vec{p}|^2$ under transformations (14) we easily check that, with P_μ, $\widetilde{P}_\mu$ chosen as above,

$$a^2(\triangle p) - b^2(\triangle p)(\triangle\vec{p})^2 > 0\,, \tag{33}$$

so the right-hand side of equation (27) cannot have zero eigenvalue.

There remains to consider the effect of transformation (12). As the transformation (14) does not change the coefficient d, and $d = 0$ for the particular solution (19) we have now

$$d = \frac{\partial \varrho(P_0, M^2)}{\partial P_0}\,.$$

However, d is regular by assumption, so ϱ must be a sum of regular function of P_0 and M^2 plus a (possibly singular) function of M^2. The regular part cannot transform singular coproduct into a regular one.

Therefore, we can restrict ourselves to the functions $\varrho = \varrho(M^2)$. Under the transformation (11) $a \to e^{\varrho+\bar{\varrho}}a$, $b \to e^{\varrho+\bar{\varrho}}b$; ϱ can cure the singularity of F and G if either

(i) $e^{\varrho+\bar{\varrho}}$ kills zero eigenvalue on the left-hand side of equation (27) or
(ii) it provides the zero eigenvalue on the right-hand side.

As far as *(i)* is concerned, ϱ must have singularity at $M^2 = -4\kappa^2$. In consequence, this singularity appears for both $P_3 = \pm 2\kappa \cosh\left(\frac{P_0}{2\kappa}\right)$; but any of two eigenvalues of $a + b\vec{p}\cdot\vec{\sigma}$ vanishes only for one of these values so that

$$e^{\varrho+\bar{\varrho}}(a + b\vec{p}\cdot\vec{\sigma}) \equiv a' + b'\vec{p}\vec{\sigma}$$

is necessarily somewhere singular.

In order to cosider *(ii)* let us take again

$$P_3 = 2\kappa \cosh\left(\frac{P_0}{2\kappa}\right), \qquad \widetilde{P}_3 = 2\kappa \cosh\left(\frac{\widetilde{P}_0}{2\kappa}\right)$$

and vary P_0, $\widetilde{P}_0$ (P_0, $\widetilde{P}_0 > 0$). Then $\triangle M^2 = M^2(\triangle p)$ varies in the interval $(-\infty, -8\kappa^2)$. Therefore, $e^{\varrho+\bar{\varrho}}$ should vanish for M^2 taking values in this interval; consequently a' and b' vanish for such M^2. This contradicts the assumption that a' and b' are regular (analytic). We conclude that no regular coproduct of the form (25) exists.

4. Conclusions

We have analysed the possible extensions of κ-Poincaré algebra to $N = 1$ superalgebra. Some rather mild assumptions have been made concerning the algebraic as well as coalgebraic sectors.

We have found all possible algebraic structures admitted by the assumptions we made. It appears that, in principle, they are all related by redefinition of supercharges. To put our results in proper framework let us compare them with those obtained in Ref. [5, 6]. It was shown there that the deformed superalgebra of Lukierski, Nowicki and Sobczyk [3] can be put in the form in which the bosonic operators form κ-Poincaré superalgebra.

However the price to be paid for that is that Q and $\bar{Q}$ cannot be interpreted as conjugated to each other (if we assume that κ-Poincaré superalgebra is invariant under conjugation). This can be further cured by performing an additional transformation which, however, is necessarily singular at $M^2 = -4\kappa^2$. Therefore, the coproduct resulting from such transformation will be singular at that point. This again supports our conclusion that it is not possible to construct regular coproduct (except $\kappa = \infty$), at least within the Ansatz we have used.

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