QCD AT SMALL x AND HEISENBERG SPIN CHAINS*

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We give a short review on the application of perturbative QCD to the description of the Regge asymptotics of hadronic scattering amplitude. Considering as an example the small x behaviour of the structure functions of deep inelastic scattering in the generalized leading logarithmic approximation, we show that the Regge asymptotics are governed in perturbative QCD by the contribution of the color-single compound states of reggeized gluons. The interaction between Reggeons is described by the effective Hamiltonian which in the multi-color limit turns out to be identical to the Hamiltonian of the completely integrable one-dimensional XXX Heisenberg magnet of noncompact spin $s = 0$. We discuss the possibility to find the spectrum of the Reggeon compound states within the Bethe Ansatz approach.

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1. Introduction

The aim of these lectures is to give a short introduction into one of the most challenging subjects in the high-energy physics which can be formulated as understanding of QCD Pomeron. The recent interest to this problem was inspired by new experimental data which indicate a steep rise of the structure function of deep inelastic electron proton scattering at small values of Bjorken variable x.

$$F_2(x,Q^2) \sim x^{-0.3}\,, \qquad \text{for } 10^{-4} < x < 10^{-3}\,.$$

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The structure function is defined as the total cross section of deep inelastic scattering $\gamma^*(Q) + p \to X$ and using the optical theorem one can relate it to the imaginary part of the elastic forward photon proton scattering

$$F(x, Q^2) = \frac{1}{s} \operatorname{Im} A_{\gamma^* p}(s, t = 0) \qquad (1.1)$$

with the center-of-mass energy $s = Q^2(1-x)/x$ and photon virtuality $-Q^2$.

The structure function has a simple interpretation in perturbative QCD. It measures the distribution density of partons inside proton which carry the fraction x of the proton momentum and have transverse size $\sim 1/Q$. At intermediate x and large Q^2, the density of partons is small and their interaction is weak since it is proportional to $\alpha_s(Q^2)$. As a result, at the intermediate x the proton can be thought of as a dilute system of quasifree partons. The situation is changed however at small x. The rise of the structure function as $x \to 0$ indicates that the density of partons increases and although the interaction between partons is still weak we are not allowed to neglect multiparton correlations anymore. Thus, at small x one has to describe the dynamics of strongly correlated system of partons in perturbative QCD. Having in mind famous example of similar systems in statistical mechanics, it is natural to expect that the description of the process using the "bare" degrees of freedom, partons, is not appropriate and one has to identify instead a new collective coordinates in terms of which the dynamics becomes much more simpler. As we will show below, the reggeized gluons or Reggeons play the role of such new coordinates at small x.

One possibility to study the small x asymptotics of the structure function is to explore the dependence of $F(x, Q^2)$ on the photon virtuality Q^2. At large Q^2 the structure function can be expanded in powers of $1/Q^2$ using the operator product expansion. In the leading $\ln Q-$approximation, $\alpha_s \ll 1$ and $\alpha_s \ln(Q^2/m^2) \sim 1$, this expansion looks like

$$F_\omega(Q^2) \equiv \int_0^1 dx\, x^{\omega-1} F(x, Q^2) = \frac{1}{Q^2} \sum_{k=1}^{\infty} C_\omega^{(k)} \left(\frac{m^2}{Q^2}\right)^{k-1+\gamma_\omega^{(k)}}, \qquad (1.2)$$

where the $k-$th term is associated with the contribution of twist$-2k$ operators, $C_\omega^{(k)}$ are dimensionless coefficients and m^2 is a mass scale of proton. The matrix elements of twist$-2k$ operators have the anomalous dimensions $\gamma_\omega^{(k)}$ which can be calculated perturbatively as

$$\gamma_\omega^{(k)} = \sum_{l \geq 1} a_l^{(k)} \left(\frac{\alpha_s}{\omega}\right)^l (1 + \mathcal{O}(\alpha_s)) \qquad (1.3)$$

with some coefficients $a_l^{(k)}$. The small x asymptotics of the structure function is translated now into $\omega \to 0$ behavior of the complex moments $F_\omega(Q^2)$. Taking the limit $\omega \to 0$ in the r.h.s. of (1.2) we find that the perturbative series (1.3) for the anomalous dimensions $\gamma_\omega^{(k)}$ becomes divergent. As a result, there is no meaning of the twist expansion of the structure function at small x and one has to find an effective way of taking into account the contributions of all terms in (1.2).

There is however another approach to the small x asymptotics of $F(x, Q^2)$. Let us notice that as $x \to 0$ the scattering amplitude in (1.1) has to be evaluated in the extreme limit of high center-of-mass energies $s = Q^2(1-x)/x \gg Q^2$, in which the famous Regge model emerges [1]. The Regge model interprets the increasing of the structure function at high energies, or equivalently at small x, by introducing the notion of the Pomerons as Regge poles of the moments (1.2) in the complex ω-plane. The contribution of the Pomerons to the structure function is given by

$$F(x, Q^2) = \sum_{\mathbb{P}} x^{-(\alpha_{\mathbb{P}}-1)}\, \beta_p^{\mathbb{P}} \beta_{\gamma^*(Q^2)}^{\mathbb{P}} \tag{1.4}$$

where summation is performed over "quantum numbers" of the Pomerons $\mathbb{P}$. Here, $\alpha_{\mathbb{P}}$ is the Pomeron intercept and $\beta_p^{\mathbb{P}}$ and $\beta_{\gamma^*}^{\mathbb{P}}$ are the so-called residue factors corresponding to proton and photon, respectively. Although the Regge model gives a successful phenomenological description of the experimental data in terms of "hard" (perturbative) and "soft" (nonperturbative) Pomerons [2] it is still unclear whether the model is consistent with QCD.

The first attempts to understand the status of the "hard" Pomerons within perturbative QCD were undertaken more than 20 years ago and they have led to the discovery of the BFKL Pomeron [3]. The BFKL Pomeron was found in the leading logarithmic approximation (LLA), $\alpha_s \ll 1$ and $\alpha_s \ln x \sim 1$, and at arbitrary small x it leads to unrestricted rise of the structure function, or equivalently of the parton densities in proton, and violates the unitary Froissart bound [1]

$$F(x, Q^2) < \text{const.} \times \ln^2 x \qquad \text{for} \quad x \to 0\,. \tag{1.5}$$

This means that at $x \to 0$ the BFKL Pomeron alone is not sufficient to describe the small$-x$ asymptotics of the structure function. One has to identify the "nonleading" Pomerons whose contribution to the structure function is suppressed in the LLA by powers of α_s with respect to that of the BFKL Pomeron but which become important for smaller values of x. In the next Sections the recent progress on this problem will be reviewed.

2. Perturbative QCD Pomeron

In perturbative QCD approach to the Pomeron, for the sake of simplicity, we replace the incoming proton by a perturbative onium state built from two heavy quarks and created via the decay of the photon with invariant mass m. Then, in deep inelastic scattering, photon and onium scatter by exchanging soft gluons in the $t-$channel and we treat their interaction using the $S-$matrix of perturbative QCD. Calculating the corresponding Feynman diagrams in powers of α_s we obtain the following general form of the perturbative expansion of the structure function of DIS at small x and fixed Q^2 [4]

$$F(x,Q^2) = \sum_{m=0}^{\infty}\Big[(\alpha_s \ln x)^m f_{m,m}(Q^2) + \alpha_s(\alpha_s \ln x)^{m-1} f_{m,m-1}(Q^2) + \ldots + \alpha_s^m f_{m,0}(Q^2)\Big]\,. \quad (2.1)$$

Here, $\alpha_s \ll 1$ and $\alpha_s \ln x$ is a large parameter at small x. The coefficient functions $f_{m,m-k}(Q^2)$ depend on the internal structure of the incoming hadron as well as the photon virtuality Q^2 and they can be calculated perturbatively for the onium state. It is clear that the number of different terms in (2.1) rapidly increases in higher orders in α_s and in order to find the structure function at small x one has to develop a "good" approximation to $F(x,Q^2)$ which, first, correctly describes the small$-x$ asymptotics of the infinite series (2.1) and, second, preserves the unitarity constraint (1.5).

To satisfy the first condition, one can neglect in (2.1) the terms containing $f_{m,m-1}, \ldots, f_{m,0}$ as suppressed by powers of α_s with respect to the leading term $f_{m,m}$. The resulting series defines $F(x,Q^2)$ in the LLA and it was resummed by BFKL [3] to all orders in α_s

$$F^{\rm LLA}(x,Q^2) = \sum_{m=0}^{\infty}(\alpha_s \ln x)^m f_{m,m}(Q^2) \sim x^{-(\alpha_s N_c/\pi)4\ln 2} \quad (2.2)$$

with N_c the number of quark colors. As was stressed before, this expression violates the unitary bound (1.5). In order to preserve the unitarity of the S-matrix of QCD and fulfill (1.5) we have to take into account an *infinite* number of nonleading terms in (2.1). This means that with the unitarity condition taken into account the series (2.1) does not have any "natural" small parameter of expansion like α_s. However, instead of searching for this parameter one may start with the LLA result (2.2) and try to identify the nonleading terms in (2.1), which should be added to (2.2) in order to restore the unitarity. At present, the following three unitarization schemes are known:

– effective reggeon field theory [5–7]

– generalized leading logarithmic approximation [8, 4, 9]

– dipole model [10].

Each of them was developed to resum special class of corrections which are expected to dominate at small x and which are closely related to identification of a new collective degrees of freedom in QCD at small x. Namely, in the first two unitarization schemes, the reggeized gluons (or Reggeons) [3] play the role of such degrees of freedom while in the last scheme one deals with the color dipoles [10]. The difference between the first two schemes is that in the generalized LLA the unitarity is preserved only in the direct channels of DIS but not in subchannels corresponding to the different groups of particles in the final state.

2.2. Generalized leading logarithmic approximation

In what follows we will analyze the structure function at small x in the generalized LLA. Once we identified the Reggeons as a new collective degrees of freedom in QCD at small x, we may try to develop a new diagram technique for calculation of the structure function [8, 4, 9]. Namely, an infinite set of standard Feynman diagrams involving "bare" gluons can be replaced by a few Reggeon diagrams describing propagation of reggeized gluons and their interaction with each other. Each Reggeon diagram appears as a result of resummation of an infinite number of Feynman diagrams with "bare" gluons.

We begin the construction of the structure function $F(x,Q^2)$ in the generalized LLA by summarizing the properties of the leading logarithmic approximation (2.2) in which one retains in (2.1) only the coefficient functions $f_{m,m}(Q^2)$. In this approximation the structure function is given by the contribution of diagrams with only two Reggeons propagating in the $t-$channel, the famous ladder diagrams [11, 3].

However, two Reggeon diagrams do not satisfy the $s-$channel unitarity condition. Once we allowed for two Reggeons to propagate in the $t-$channel, the $s-$channel unitarity requires the existence of 3-, 4-, ... Reggeon exchanges. In the generalized LLA, one restores unitarity by adding multi-Reggeon diagrams to the LLA result [4, 8]. This minimal set of diagrams is obtained from the two Reggeon diagram by iterating the number of Reggeons in the $t-$channel. For example, the first nonleading correction corresponds to the diagram with three Reggeons propagating in the $t-$channel. Continuing this procedure, we obtain that the scattering amplitude is given in the generalized LLA by the sum of the diagrams shown in Fig. 1. These diagrams have a form of generalized ladder diagrams [4, 8], in which summation is performed over all possible numbers of rungs representing the Reggeon interaction, and over all possible number of Reggeons in the $t-$channel, $n = 2, 3, \ldots$. The scattering of each two Reggeons is described by the same effective theory as we started with in the LLA.

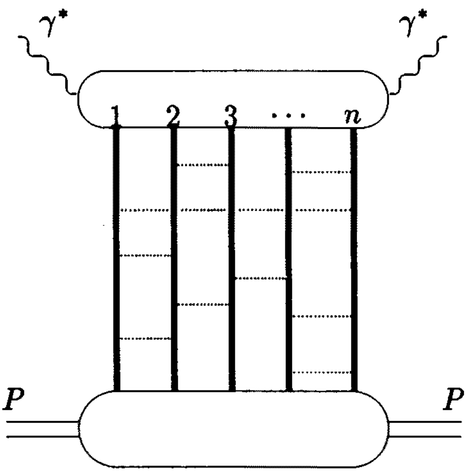

Fig. 1. Unitary Reggeon diagrams contributing to the structure function of DIS at small x in the generalized LLA in the multi-color limit, $N_c \to \infty$. For finite N_c one has to add similar diagrams with pair-wise interaction between n Reggeons.

One of peculiar features of the Reggeon scattering is that it is elastic and pair-wise. This means that the number of Reggeons is not changed and the diagrams of Fig. 1 describe the propagation in the $t-$channel of the conserved number $n = 2, 3, \ldots$ of pair-wise interacting Reggeons.

For $n = 2$ the diagram of Fig. 1 represents the leading logarithmic result for the scattering amplitude and it contributes to the coefficient functions $f_{m,m}(Q^2)$ to all orders of perturbation theory. The contribution of the diagram with $n = 3$ is suppressed by a power of α_s with respect to that for $n = 2$ and it determines the first nonleading coefficient $f^{\rm min}_{m,m-1}$ in (2.1). In general, the contribution of the diagrams of Fig. 1 to the structure function can be represented in the following form

$$F(x,Q^2) = \sum_{n=2}^{\infty} \alpha_s^{n-2}\, F^{(n)}(x,Q^2)\,, \tag{2.3}$$

where n is the conserved number of Reggeons in the $t-$channel and the functions $F^{(n)}(x,Q^2)$ have the following form

$$F^{(n)}(x,Q^2) = \sum_{m=n-2}^{\infty} (\alpha_s \ln x)^{m-n+2} f_{m,m-n+2}(Q^2) . \qquad (2.4)$$

Comparing (2.1) and (2.3) we notice that both expansions are similar and, moreover, the nonleading corrections to the structure function can be associated with the contribution of the $n-$Reggeon diagrams.

The calculation of the $n-$Reggeon diagrams requires the resummation of the whole perturbation series (2.4) and it can be effectively performed using the Bartels–Kwiecinski–Praszalowicz resummation technique [4, 9]. As we will show in the next Section, the resummed functions $F^{(n)}$ have the following Pomeron-like behaviour

$$F^{(n)}(x,Q^2) \sim x^{-(\alpha_n - 1)} , \qquad n = 2,\, 3,\, \ldots \qquad (2.5)$$

which appears as a contribution of the compound states of n reggeized gluons with vacuum quantum numbers and intercept α_n.

In the LLA, the functions $F^{(n)}$ are of the same order for different n and the contribution of the $n-$Reggeon diagrams to (2.3) is suppressed by powers of α_s with respect to the leading term, $F^{(2)}$, which describes the propagation of two reggeized gluons. At smaller values of x, the growth of the functions $F^{(n)}$ overwhelms the suppression by powers of α_s in (2.3) and one has to take into account all functions $F^{(n)}$ in the expansion (2.3). This is a scenario of how nonunitarity of the LLA result is restored in the generalized LLA.

2.2. The Bethe–Salpeter equation for the partial waves

In the Reggeon diagram of Fig. 1, the photon and onium scatter by exchanging n Reggeons in the $t-$channel. The Reggeons carry the color charge of gluons and they couple to the colorless states, $\gamma^*(Q^2)$ and p, through quark loop. The scattering amplitude corresponding to the diagram of Fig. 1 can be represented as a convolution of the probabilities to find n Reggeons inside incoming particles, Φ_{γ^*} and Φ_p, and the n Reggeon scattering amplitude, T_n. Calculating the complex moments (1.2) one can find

$$F^{(n)}(\omega,Q^2) = \int [d^2k] \int [d^2k'] \; \Phi_{\gamma^*(Q^2)}(\{k\}) \; T_n(\{k\},\{k'\};\omega) \; \Phi_p(\{k'\}) . \qquad (2.6)$$

Here, $\{k\} = (k_1,\ldots,k_n)$ and $\{k'\} = (k'_1,\ldots,k'_n)$ are 2-dimensional transverse momentum of n Reggeons emitted by photon and onium, respectively.

The functions $\Phi_{\gamma^*(Q^2)}$ and Φ_p describe the distribution of Reggeons inside incoming particles and they can be calculated perturbatively. In the limit $x \to 0$ or equivalently $\omega \to 0$, both functions depend on the transverse momenta of Reggeons [11, 3] and not on the energy s, or equivalently on ω. The function T_n describes n to n scattering of Reggeons in the t–channel, and is the main object of our consideration. In (2.6), the integration is performed over transverse momenta of n Reggeons, while integration over longitudinal components is performed inside $T_n(\omega)$,

$$[d^2k] \equiv d^2k_1 \ldots d^2k_n \delta^{(2)}(k_1 + \ldots + k_n)$$

and similar for $[d^2k']$, with a delta-function included to ensure the condition $t = 0$ for the total transferred momentum in the t–channel. The partonic distributions and the Reggeon scattering amplitude depend on the color indices of Reggeons and summation over these indices is implied in (2.6).

Considering the distribution functions $\Phi_{\gamma^*(Q^2)}(\{k\})$ and $\Phi_p(\{k'\})$ as states in a two-dimensional transverse phase space for the n Reggeons, one can rewrite the scattering amplitude (2.6) as the following matrix element

$$F^{(n)}(\omega, Q^2) = \langle \Phi_{\gamma^*(Q^2)} | T_n(\omega) | \Phi_p \rangle \,, \tag{2.7}$$

where the transition operator $T_n(\omega)$ describes the elastic scattering of n Reggeons. To find the transition operator T_n we notice that diagrams of Fig. 1 have a ladder structure, which suggests a Bethe–Salpeter–like equation for $T_n(\omega)$ shown in Fig. 2.

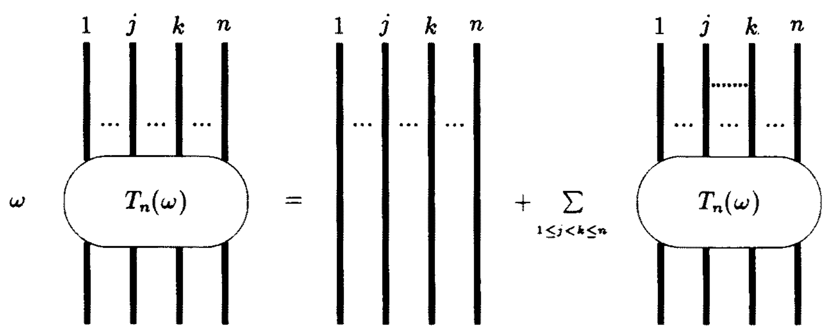

Fig. 2. The Bethe–Salpeter equation for the transition operator $T_n(\omega)$, describing $n \to n$ elastic scattering of reggeized gluons in the t–channel. Iterations of this equation reproduce the ladder diagrams of Fig. 1.

The corresponding equation has the following form [4, 9]

$$\omega T_n(\omega) = T_n^{(0)} + \mathcal{H}_n T_n(\omega) \,, \tag{2.8}$$

where $T_n^{(0)}$ corresponds to the free propagation of n Reggeons in the t-channel, while the operator $\mathcal{H}_n$ describes the pair-wise interactions of n Reggeons. The Reggeon scattering amplitude, $T_n(\{k\},\{k'\};\omega)$, depends on the transverse momenta and the color indices of n incoming and n outgoing Reggeons. The Hamiltonian $\mathcal{H}_n$ acts in (2.8) only on the incoming Reggeons, and is given by

$$\mathcal{H}_n = -\frac{\alpha_s}{2\pi} \sum_{n\geq i>j\geq 1} H_{ij}\, t_i^a t_j^a\,, \tag{2.9}$$

where the sum goes over all possible pairs (i,j) of Reggeons. Each term in this sum has a color factor, which is given by the direct product of the gauge group generators in the adjoint representation of the SU(N) group, acting in the color space of i-th and j-th Reggeons,

$$t_i^a = \underbrace{I\otimes\ldots\otimes t^a}_{i}\otimes\ldots\otimes I\,, \qquad (t^a)_{bc} = -if_{abc}\,,$$

with f_{abc} the structure constants of the SU(N). The Reggeon interaction (2.9) is described by the two-particle Hamiltonian H_{ij}, which depends only on the transverse momenta of Reggeons. If $\{k_1,\ldots,k_n\}$ and $\{k'_1,\ldots,k'_n\}$ are the transverse momenta of incoming and outgoing Reggeons, respectively, then the operator H_{ij} acts on Reggeon momenta as follows

$$\begin{aligned}&\langle k_1,\ldots,k_n|H_{ij}|k'_1,\ldots,k'_n\rangle \\ &= H(k_i,k_j|k'_i,k'_j)\delta^2(k_i+k_j-k'_i-k'_j)\prod_{l=1,l\neq i,j}^{n}\delta^2(k_l-k'_l)\,,\end{aligned} \tag{2.10}$$

where $H(k_i,k_j|k'_i,k'_j)$ is given by [3]

$$\begin{aligned}H_{ij} \equiv H(k_i,k_j|k'_i,k'_j) = \frac{1}{\pi}\Bigg\{ &\frac{k_i^2(p-k'_i)^2+(p-k_i)^2k_i'^2-(k_i-k'_i)^2p^2}{k_i^2(p-k_i)^2(k'_i-k_i)^2} \\ - \delta^2(k_i-k'_i)\int\frac{d^2k'}{(k_i-k')^2}&\left[\frac{k_i^2}{k'^2+(k_i-k')^2}+\frac{(p-k_i)^2}{(p-k')^2+(k_i-k')^2}\right]\Bigg\}\end{aligned} \tag{2.11}$$

with $p=k_j+k_i=k'_j+k'_i$. The operator $T_n^{(0)}$ is equal to the product of n Reggeon propagators, and in momentum representation it can be written as

$$\langle k_1,\ldots,k_n|T_n^{(0)}|k'_1,\ldots,k'_n\rangle = \prod_{j=1}^{n}\delta^2(k_j-k'_j)\frac{1}{k_j^2}\,. \tag{2.12}$$

Iteration of the equation (2.8) reproduces the ladder in Fig. 1, and gives the perturbative expansion of $T_n(\omega)$ as series in α_s/ω. Using the Bethe–Salpeter equation (2.8), we find the general solution for the transition operator

$$T_n(\omega) = \frac{1}{\omega - \mathcal{H}_n} T_n^{(0)} . \tag{2.13}$$

We recall that the Hamiltonian $\mathcal{H}_n$ corresponds to the pair-wise interaction (2.9) of n Reggeons and it describes the evolution of the n-Reggeon state in the t-channel.

Suppose for a moment that we know the spectrum of the Reggeon Hamiltonian

$$\mathcal{H}_n |\chi_{n,\{q\}}\rangle = E_{n,\{q\}} |\chi_{n,\{q\}}\rangle , \tag{2.14}$$

with $\{q\}$ being some set of quantum numbers, which parameterize possible solutions of the equation. Then, the eigenstates of $\mathcal{H}_n$ can be identified as compound states of n Reggeons, $|\chi_n\rangle$, and the corresponding eigenvalues, E_n, determine the energies of these states. The simplest example of such states is the BFKL Pomeron [3] which is built from $n = 2$ Reggeons. When we solve (2.14), we can expand the transition amplitude (2.13) over eigenstates of Reggeon Hamiltonian as

$$T_n(\omega) = \sum_{\{q\}} \frac{1}{\omega - E_{n,\{q\}}} |\chi_{n,\{q\}}\rangle\langle\chi_{n,\{q\}}| T_n^{(0)} , \tag{2.15}$$

where the sum over q means the summation over discrete and integration over continuous q. Combined with (2.7), this expression implies that, in accordance with the Regge model expectations, the moments of the structure function $F^{(n)}(\omega, Q^2)$ have singularities in the complex ω-plane and their positions are determined by the eigenvalues $E_{n,\{q\}}$ of the Reggeon Hamiltonian $\mathcal{H}_n$. Moreover, performing the Mellin transformation and inverting the moments as

$$F^{(n)}(x, Q^2) = \int_{\delta - i\infty}^{\delta + i\infty} \frac{d\omega}{2\pi i}\, x^\omega\, F^{(n)}(\omega, Q^2) ,$$

we substitute (2.15) into (2.7), take formally a residue at $\omega = \mathcal{H}_n$ and get the following expression for the structure function in the generalized LLA

$$F(x, Q^2) = \sum_{n=2}^{\infty} \alpha_s^{n-2} F^{(n)}(x, Q^2) ,$$
$$F^{(n)}(x, Q^2) = \langle \Phi_{\gamma^*} | x^{-\mathcal{H}_n} T_n^{(0)} | \Phi_p \rangle = \sum_{\{q\}} \beta^{\{q\}}_{n\to\gamma^*} \beta^{\{q\}}_{n\to p}\, x^{-E_{n,\{q\}}} \tag{2.16}$$

Here, the residue factors $\beta^{\{q\}}_{n\to\gamma^*}$ and $\beta^{\{q\}}_{n\to p}$ measure the overlapping between distribution functions of Reggeons inside photon and onium and the wave functions of the compound states of Reggeons

$$\beta^{\{q\}}_{n\to\gamma^*} = \langle \Phi_{\gamma^*} | \chi_{n,\{q\}} \rangle \,, \qquad \beta^{\{q\}}_{n\to p} = \langle \chi_{n,\{q\}} | T_n^{(0)} | \Phi_p \rangle \,. \tag{2.17}$$

Although the definitions of the residue factors look different, one can show [12] that they are closely related to each other via complex conjugation and differ only by a constant. We recall that the scalar product of states, used in (2.7), (2.16) and (2.17), implies integration over transverse momenta k_1, ..., k_n of n Reggeons as well as summation over their color indices

$$\beta^{\{q\}}_{n\to\gamma^*} = \int [d^2k] \; \Phi^{a_1\ldots a_n}_{\gamma^*}(k_1,\ldots,k_n) \chi^{a_1\ldots a_n}_{n,\{q\}}(k_1,\ldots,k_n) \,.$$

Thus, in the generalized LLA the structure function of deep inelastic scattering has the Regge behavior (2.5) and it is defined by the properties of the Reggeon Hamiltonian (2.9). The structure function (2.16) is given by a sum of $F^{(n)}(x,Q^2)$ over all possible numbers of Reggeons in the $t-$channel. With each individual $F^{(n)}(x,Q^2)$ we associate the family of the compound Reggeon states that appear as solutions of the eigenstate equation (2.14). The expression (2.16) is very similar to the predictions of the Regge model (1.4) provided that we identify the Pomerons as the compound states of the Reggeons and define their intercept as the maximal energy of the n Reggeon states

$$\alpha_n - 1 = \max_{\{q\}} E_{n,\{q\}} \,. \tag{2.18}$$

The simplest example of such states, the $n = 2$ Reggeon state, defines the structure function in the LLA and it is identical to the BFKL Pomeron with the intercept [3]

$$\alpha_2 - 1 = \frac{\alpha_s N_c}{\pi} 4 \ln 2 \,. \tag{2.19}$$

The structure function (2.16) is given by an infinite sum of terms corresponding to the diagram of Fig. 1 with fixed number n of Reggeons in the $t-$channel. The first term, $n = 2$, describes the leading logarithmic asymptotics and, taken alone, it violates the $s-$channel unitarity of the $S-$matrix. Each next term in the sum over n in (2.16) defines a nonleading contribution, which is suppressed by a power of the coupling constant with respect to that of the $(n-1)$-th term. Examining the high-energy behavior of the n-th term, $F_n(x,Q^2)$, we recognize that the contribution of compound Reggeon states with positive energy $E_n > 0$ to (2.16) grows as a power of energy, and thus violates the Froissart bound (1.5). We recall however that these nonleading terms have been defined from the very beginning in such a way that to restore unitarity of the total scattering amplitude. This means, that, although each term in the sum (2.16) violates the unitarity bound, unitarity is restored by their sum. Thus, the original problem of calculating the scattering amplitude $F(x, Q^2)$ in the generalized LLA is reduced to the problem (2.14) of the diagonalization of the Reggeon Hamiltonian $\mathcal{H}_n$ for an arbitrary number of Reggeons.

3. Properties of the Reggeon Hamiltonian

We turn now to the solution of the eigenstate problem (2.14) and (2.9) in order to find the spectrum of the compound Reggeon states. The Reggeon Hamiltonian (2.9), (2.10) and (2.11) has been found from the analysis of Feynman diagrams contributing to the hadronic scattering amplitude in the generalized LLA, and it inherits the properties of high-energy QCD in the Regge limit.

The Reggeon Hamiltonian was defined in (2.9), (2.10) and (2.11) in the two-dimensional space of transverse momenta of Reggeons. It is more convenient however to analyze equation (2.14) in two-dimensional coordinate space, the so called impact parameter space, rather than in momentum space. To this end, we perform a two-dimensional Fourier transformation, and replace in (2.6), (2.10) and (2.11) the two-dimensional transverse momenta $k_1, \dots, k_n$ and $k'_1, \dots, k'_n$ by two-dimensional impact vectors $b_1, \dots, b_n$ and $b'_1, \dots, b'_n$ which describe the transverse coordinates of Reggeons. Then, for the impact vectors $b_j = (x_j, y_j)$ we define holomorphic and antiholomorphic complex coordinates $(z_j, \bar{z}_j)$ as

$$z_j = x_j + iy_j\,, \qquad \bar{z}_j = x_j - iy_j\,, \qquad (j = 1, \dots, n)\,,$$

and analogous coordinates $(z'_j, \bar{z}'_j)$ for the impact vectors b'_j. Now one can use (2.10) and (2.11) to find the two-particle Reggeon kernel H_{ij} in the impact parameter space. It turns out that, expressed in terms of holomorphic and antiholomorphic coordinates, the kernel H_{ij} becomes holomorphically separable [17], *i.e.*,

$$H_{ik} = H(z_i, z_k) + H(\bar{z}_i, \bar{z}_k)\,. \tag{3.1}$$

Here, two operators on the r.h.s. act separately on holomorphic and antiholomorphic coordinates of Reggeons. After Fourier transformation of (2.10) and (2.11) they are given by the following *equivalent* representations

$$\begin{aligned} H(z_i, z_k) &= -P_i^{-1} \log(z_i - z_k) P_i - P_k^{-1} \log(z_i - z_k) P_k - \log(P_i P_k) - 2\gamma_{\mathrm{E}} \\ &= -2\log(z_i - z_k) - (z_i - z_k) \log(P_i P_k)(z_i - z_k)^{-1} - 2\gamma_{\mathrm{E}}\,, \end{aligned} \tag{3.2}$$

where $P_i = i\partial/\partial z_i$ and $\gamma_{\rm E}$ is the Euler constant. The same operator can be represented as [17]

$$H(z_i, z_k) = \sum_{l=0}^{\infty} \left(\frac{2l+1}{l(l+1) - \boldsymbol{L}_{ik}^2} - \frac{2}{l+1} \right), \qquad \boldsymbol{L}_{ik}^2 = -(z_i - z_k)^2 \frac{\partial^2}{\partial z_i \partial z_k}. \tag{3.3}$$

Substituting this expression into (3.1) and (2.9), we find that the Reggeon Hamiltonian $\mathcal{H}_n$ is invariant under the conformal transformations [11]

$$z_i \to \frac{az_i + b}{cz_i + d}, \qquad \bar{z}_i \to \frac{\bar{a}\bar{z}_i + \bar{b}}{\bar{c}\bar{z}_i + \bar{d}}, \tag{3.4}$$

with $ad - bc = \bar{a}\bar{d} - \bar{b}\bar{c} = 1$. Indeed, the generators of these transformations,

$$S^3 = \sum_{k=1}^{n} z_k \partial_k, \qquad S^- = -\sum_{k=1}^{n} \partial_k, \qquad S^+ = \sum_{k=1}^{n} z_k^2 \partial_k \tag{3.5}$$

and the analogous antiholomorphic generators $\bar{S}^3$, $\bar{S}^-$ and $\bar{S}^+$ form the SL(2,$\mathbb{C}$) algebra and commute with $\boldsymbol{L}_{ik}^2$ and, as a consequence, with $\mathcal{H}_n$.

Let us consider the properties of the eigenstate $\chi_{n,\{q\}}$ of the Reggeon Hamiltonian (2.14) in the impact parameter space. These states are parameterized by quantum numbers $\{q\}$, which should appear as eigenvalues of some "hidden" integrals of motion, and by a two-dimensional real vector b_0, which represents the center of mass of the compound Reggeon state. In this notation, the wave function $\chi_{n,\{q\}} = \chi_{n,\{q\}}(\{b_i\}; b_0) \equiv \chi_{n,\{q\}}(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0)$ satisfies the relations (2.14), (2.9) and (3.1), or equivalently

$$E_{n,\{q\}}\chi_{n,\{q\}}(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0) = -\frac{\alpha_s}{2\pi} \sum_{j,k=1,\, j>k}^{n} \left[H(z_j, z_k) + H(\bar{z}_j, \bar{z}_k) \right] \times t_j^a t_k^a \, \chi_{n,\{q\}}(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0), \tag{3.6}$$

which can be interpreted as a two-dimensional Schrödinger equation for a system of n pair-wise interacting particles with coordinates $\{z_i, \bar{z}_i\}$ and internal color degrees of freedom. One may try to rewrite the total Hamiltonian in (3.6) as a sum of holomorphic and antiholomorphic parts using the fact that $H(z_j, z_k)$ and $H(\bar{z}_j, \bar{z}_k)$ commute. However, the resulting two terms do not commute with each other due to nontrivial color factor in (3.6), and as a consequence holomorphic and antiholomorphic sectors become coupled to each other via color degrees of freedom. It is this interaction which makes difficult the solution of the Schrödinger equation (3.6).

The equation (3.6) has a form of $(2+1)$-dimensional Schrödinger equation for the system of n particles with additional color degrees of freedom. One should notice that this equation has been solved in the special case of $n = 2$ Reggeon states [3] and it remained unclear whether it could be solved for an arbitrary n. Recently, a significant progress has been achieved [3, 14] after it was realized that in the multi-color limit, $N_c \to \infty$, the Schrödinger equation (3.6) has an interesting interpretation in terms of exactly solvable lattice models [15].

3.1. Multi-color limit

Each Reggeon carries a color charge t_j^a, and the total charge of n Reggeons is equal to their sum $\sum_{j=1}^n t_j^a$. Since the compound Reggeon states propagate in the $t-$channel between two hadrons, they carry zero color charge, unchanged by the Reggeon interaction,

$$[\mathcal{H}_n, \sum_{j=1}^n t_j^a] = 0\,, \qquad \sum_{j=1}^n t_j^a \, |\chi_{n,\{q\}}\rangle = 0\,. \tag{3.7}$$

An essential simplification occurs in (3.6) in multi-color limit, $N \to \infty$ and $\alpha_s N$ is fixed, [17, 16]. In this limit, only planar diagrams of Fig. 1 survive, which have the form of a cylinder attached by both edges to the hadronic states [16]. Reggeons propagate along the sides of the cylinder and it makes them possible to interact only with two nearest Reggeons. Using the double line representation for Reggeon color charge and applying the standard rules of large N counting, one finds that the color structure in (3.6) can be simplified as

$$t_1^a t_2^a \to -N\,, \qquad \text{for } n = 2\,; \qquad\qquad t_i^a t_j^a \to -\frac{N}{2}\delta_{i,j+1}\,, \qquad \text{for } n \geq 3\,,$$

where $i, j = 1, \ldots, n$ and the Reggeons with $i = 1$ and $i = n+1$ are considering as coinciding. Then, in the multi-color limit we find that, first, the color factors become trivial in (3.6) and as a consequence holomorphic and antiholomorphic sectors become decoupled and, second, inside each sector in (3.6) the pair-wise Reggeon interaction is replaced by a nearest-neighbour interaction with periodic boundary conditions. Thus, in the large-N limit, the two-dimensional Schrödinger equation (3.6) is replaced by a system of two one-dimensional Schrödinger equations [17],

$$\begin{aligned} H_n \varphi_{n,\{q\}}(\{z_i\}; z_0) &= \varepsilon_{n,\{q\}} \varphi_{n,\{q\}}(\{z_i\}; z_0)\,, \\ \bar{H}_n \bar{\varphi}_{n,\{q\}}(\{\bar{z}_i\}; \bar{z}_0) &= \bar{\varepsilon}_{n,\{q\}} \bar{\varphi}_{n,\{q\}}(\{\bar{z}_i\}; \bar{z}_0)\,. \end{aligned} \tag{3.8}$$

The Hamiltonians H_n and $\bar{H}_n$ are defined as

$$H_n = \sum_{k=1}^{n} H(z_k, z_{k+1}), \qquad \bar{H}_n = \sum_{k=1}^{n} H(\bar{z}_k, \bar{z}_{k+1}), \tag{3.9}$$

with two-particle Hamiltonians given by (3.2) or (3.3) and $z_{n+1} \equiv z_1$. Once we know the solution of (3.8), the eigenstates (2.14) of the Reggeon Hamiltonian in the multi-color limit can be found as

$$\chi_{n,\{q\}}(\{z_i, \bar{z}_i\}; z_0, \bar{z}_0) = \varphi_{n,\{q\}}(\{z_i\}; z_0)\, \overline{\varphi}_{n,\{q\}}(\{\bar{z}_i\}; \bar{z}_0), \tag{3.10}$$

and the corresponding eigenvalues are given by

$$E_{n,\{q\}} = \frac{\alpha_s N}{4\pi}\left(\varepsilon_{n,\{q\}} + \overline{\varepsilon}_{n,\{q\}}\right). \tag{3.11}$$

We conclude that starting with the calculation of the structure function of DIS at small x in $(3+1)$-dimensional multi-color QCD in the generalized LLA we came to the solution of the system of $(1+1)$-dimensional Schrödinger equations (3.8).

4. Multi-color QCD at small x as XXX Heisenberg magnet

For fixed number of Reggeons, n, each of the Schrödinger equations in (3.8) describes the system of n one-dimensional particles on a line interacting with their neighbours via Hamiltonian (3.3). It is of no surprise now that this quantum-mechanical system can be exactly solved for $n = 2$ particles leading to the BFKL Pomeron, but it is not obvious that the same could be done for an arbitrary number of Reggeons. The famous example of the one-dimensional exactly solvable multiparticle system is the XXX Heisenberg chain of n interacting $s = 1/2$ spins with the Hamiltonian [15]

$$H_n^{\mathrm{XXX}_{1/2}} = -\sum_{m=1}^{n}\left(\vec{S}_m \vec{S}_{m+1} - \tfrac{1}{4}\right), \tag{4.1}$$

where $\vec{S}_m$ are spin $s = 1/2$ operators defined in the $m-$th site. It turns out [18] that this simple model admits nontrivial generalizations to the XXX magnets for an arbitrary complex value of the spin s. The unique feature of these models is that they are completely integrable, that is, that they contain additional integrals of motion whose number is equal to the number of degrees of freedom. The reason why we might be interested in considering complex spin s is that the one-dimensional lattice models (3.8), (3.9) and

(3.3), which describe the Regge behavior of multi-color QCD, turn out to be identical to the XXX Heisenberg magnet in the special case of spin $s = 0$.

To explain this correspondence let us consider the one-dimensional lattice with periodic boundary conditions and with the number of sites n equal to the number of Reggeons. Each site is parameterized by the holomorphic coordinates z_k ($k = 1, \dots, n$) and the spin s operators are introduced in all sites as the following differential operators

$$S_k^+ = z_k^2 \partial_k - 2s z_k \,, \qquad S_k^- = -\partial_k \,, \qquad S_k^3 = z_k \partial_k - s \,, \quad (4.2)$$

with $S^\pm = S^1 \pm i S^2$. The total spin of the lattice $\vec{S} = \sum_{k=1}^n \vec{S}_k$ coincides for $s = 0$ with the generators of conformal transformations, Eq. (3.5). The definition of the integrable XXX spin chain is based on the existence of a fundamental operator $R_{km}(\lambda)$, which acts on the holomorphic coordinates in sites k and m, depends on an arbitrary complex parameter λ and which satisfies the Yang–Baxter equation [20–23]

$$R_{km}(\lambda-\mu) R_{kl}(\lambda-\rho) R_{ml}(\mu-\rho) = R_{ml}(\mu-\rho) R_{kl}(\lambda-\rho) R_{km}(\lambda-\mu) \,, \quad (4.3)$$

with λ, μ and ρ being arbitrary complex spectral parameters and k, m and l three different sites on the lattice. The solution of this equation for an arbitrary complex s is defined up to an arbitrary c-number function $f(\lambda)$ and it is given by [18, 19]

$$R_{km}(\lambda) = f(\lambda) \frac{\Gamma(i\lambda - 2s) \Gamma(i\lambda + 2s + 1)}{\Gamma(i\lambda - J_{km}) \Gamma(i\lambda + J_{km} + 1)} \,, \quad (4.4)$$

where the operator J_{km} acts on the holomorphic coordinates in the sites k and m and satisfies the equation

$$J_{km}(1 + J_{km}) = (\vec{S}_k + \vec{S}_m)^2 = 2 \vec{S}_k \vec{S}_m + 2s(s+1) \,. \quad (4.5)$$

Then, the Hamiltonian of exactly solvable XXX magnet of spin s is defined as [18]

$$H_n^{\mathrm{XXX}_s} = \sum_{m=1}^n H_{m,m+1} \,,$$
$$H_{m,m+1} = -i \frac{d}{d\lambda} \ln R_{m,m+1}(\lambda) \bigg|_{\lambda=0} \,. \quad (4.6)$$

Let us show first that for $s = 1/2$ this definition leads to the spin-1/2 XXX Heisenberg magnet (4.1). Using the well-known rules for the sum of two $s = 1/2$ spins, we find from (4.5) that the operator J_{km} has two eigenvalues: 0 and 1

$$J_{km} = 0 \cdot \Pi_0 + 1 \cdot \Pi_1$$

with Π_0 and Π_1 being the projectors onto corresponding subspace

$$\Pi_0 = -\vec{S}_k\vec{S}_m + 1/4\,, \qquad \Pi_1 = \vec{S}_k\vec{S}_m + 3/4\,, \qquad 1 = \Pi_0 + \Pi_1\,.$$

Applying this decomposition to the R-operator in (4.4) we find

$$R_{km}(\lambda) = \Pi_0 \frac{i\lambda + 1}{i\lambda - 1}$$

where we fixed the ambiguity in the definition (4.4) by choosing $f(\lambda) = 1$. Substituting the last two expressions into (4.6) we recover the Hamiltonian (4.1) of the XXX magnet of spin $s = 1/2$.

Let us consider now the case $s = 0$. Using the explicit expressions (4.2) for the spin operators for $s = 0$ and choosing $f(\lambda) = \lambda$ in (4.4) we obtain from (4.6) the two-particle holomorphic Hamiltonian as

$$\begin{aligned} H_{k,k+1} &= -\psi(-J_{k,k+1}) - \psi(1 + J_{k,k+1}) + 2\psi(1)\,, \\ J_{k,k+1}(1 + J_{k,k+1}) &= -(z_k - z_{k+1})^2\partial_k\partial_{k+1} \end{aligned} \tag{4.7}$$

with $\psi(x) = (d\Gamma(x))/dx$. Comparing (4.7) and (3.3) we find that both expressions for the two-particle Hamiltonians coincide after we identify $J_{k,k+1}(1 + J_{k,k+1}) = \boldsymbol{L}^2_{k,k+1}$ and perform summation over l in (3.3). This means that the holomorphic Reggeon Hamiltonian (3.9) is identical to the Hamiltonian (4.7) of the XXX Heisenberg magnet for spin $s = 0$

$$H_n^{\rm Reggeon} \equiv H_n^{{\rm XXX},s=0} \tag{4.8}$$

which immediately implies that the system of the Schrödinger equations (3.8) describing the multi-color QCD Pomerons in the generalized LLA is completely integrable [13, 14].

Moreover, it follows from (4.8), that the n Reggeon compound states share all their properties with the eigestates of the XXX Heisenberg magnet for spin $s = 0$. In particular, changing a sign of the Hamiltonian (4.7) one could obtain that the intercept of the n-Reggeon states, (2.18), is a ground state energy of the XXX magnet. The latter can be found by using the Bethe Ansatz technique [20–23]. This program was initiated in [14, 12] where the generalized Bethe Ansatz was developed for diagonalization of the Reggeon Hamiltonian.

5. Bethe Ansatz for QCD Pomerons

The fact that the system of Schrödinger equations (3.8) is completely integrable implies that there exists a family of "hidden" holomorphic and antiholomorphic conserved charges, $\{q\}$ and $\{\bar{q}\}$, which commute with the Reggeon Hamiltonian (3.9) and among themselves. Their explicit form can be found using the quantum inverse scattering method as [13, 14]

$$q_m = \sum_{n\geq i_1>i_2>\cdots>i_m\geq 1} i^m z_{i_1i_2}z_{i_2i_3}\ldots z_{i_mi_1}\partial_{i_1}\partial_{i_2}\ldots\partial_{i_m} \tag{5.1}$$

with $z_{ij}\equiv z_i - z_j$ and the expression for $\bar{q}_m$ is similar. The appearance of these operators is closely related to the invariance of the Reggeon Hamiltonian (3.9) under conformal SL(2,$\mathbb{C}$) transformations (3.4). Indeed, we recognize q_2 and $\bar{q}_2$ as quadratic Casimir operators of the SL(2,$\mathbb{C}$) group while the remaining conserved charges $\{q_m, \bar{q}_m\}$, $m = 3,\ldots,k$ can be interpreted as higher Casimir operators. The Reggeon compound states belong to the principal series representation of the SL(2,$\mathbb{C}$) group and under the conformal transformations (3.4) they are transformed as quasiprimary fields with conformal weights $(h,\bar{h})$ [24]

$$\begin{aligned}\chi_{n,\{q\}}(\{z_i,\bar{z}_i\};z_0,\bar{z}_0) &\to \chi'_{n,\{q\}}(\{z'_i,\bar{z}'_i\};z'_0,\bar{z}'_0)\\ &= (cz_0+d)^{2h}(\bar{c}\bar{z}_0+\bar{d})^{2\bar{h}}\chi_{n,\{q\}}(\{z_i,\bar{z}_i\};z_0,\bar{z}_0)\,.\end{aligned}$$

The n Reggeon states diagonalize the operators $\{q,\bar{q}\}$ and the eigenvalues of the conserved charges q_2, q_3, ..., q_n play a role of their additional quantum numbers. In particular, the eigenvalues of the quadratic Casimir operators are related to the conformal weights of the n Reggeon state as

$$q_2 = -h(h-1)\,, \qquad \bar{q}_2 = -\bar{h}(\bar{h}-1)\,, \qquad \bar{q}_2 = q_2^*\,.$$

where the possible values of h and $\bar{h} = 1 - h^*$ can be parameterized by integer n and real ν

$$h = \frac{1+n}{2} + i\nu\,, \qquad n\in\mathbb{Z}\,, \quad \nu\in\mathbb{R}\,. \tag{5.2}$$

As to remaining charges, $q_3, \ldots, q_n$, their possible values also become quantized [12]. The explicit form of the corresponding quantization conditions is more complicated and it was obtained in [25].

To find the explicit form of the eigenstates and eigenvalues of the n Reggeon compound states corresponding to a given set of quantum numbers $\{q,\bar{q}\}$ we apply the generalized Bethe Ansatz developed in [14, 12]. The

Bethe Ansatz for Reggeon states in multi-color QCD is based on the solution of the Baxter equation

$$\Lambda(\lambda)Q(\lambda) = (\lambda + i)^n Q(\lambda + i) + (\lambda - i)^n Q(\lambda - i)\,. \tag{5.3}$$

Here, $Q(\lambda)$ is a real function of the spectral parameter λ, $\Lambda(\lambda)$ is the eigenvalue of the so-called auxiliary transfer matrix for the XXX Heisenberg magnet of spin $s = 0$

$$\Lambda(\lambda) = 2\lambda^n + q_2\lambda^{n-2} + \ldots + q_n \tag{5.4}$$

and n is the number of reggeized gluons or, equivalently, the number of sites of the one-dimensional spin chain. For fixed n it is convenient to introduce the function

$$\widetilde{Q}(\lambda) = \lambda^n Q(\lambda) \tag{5.5}$$

and rewrite the Baxter equation (5.3) as

$$\Delta\,\widetilde{Q}(\lambda) = \left(-\frac{h(h-1)}{\lambda^2} + \frac{q_3}{\lambda^3} + \ldots + \frac{q_n}{\lambda^n}\right)\widetilde{Q}(\lambda)\,, \tag{5.6}$$

where Δ is a second-order finite difference operator

$$\Delta\,\widetilde{Q}(\lambda) = \widetilde{Q}(\lambda + i) + \widetilde{Q}(\lambda - i) - 2\widetilde{Q}(\lambda)\,.$$

Once we know the function $\widetilde{Q}(\lambda)$, the energy E_n of the $n-$Reggeon compound state can be evaluated using the relation [14, 12]

$$E_n = \frac{\alpha_s N_c}{2\pi}\,\mathrm{Re}\,\varepsilon_n(h, q_3, \ldots, q_n)\,, \tag{5.7}$$

where the holomorphic energy ε_n is defined as

$$\varepsilon_n(h, q_3, \ldots, q_n) = i\frac{d}{d\lambda}\log\frac{\widetilde{Q}(\lambda - i)}{\widetilde{Q}(\lambda + i)}\bigg|_{\lambda=0}\,. \tag{5.8}$$

The expression for the wave function of the n Reggeon states in terms of the function $\widetilde{Q}$ can be found in [14, 12].

The Baxter equation (5.6) has the following properties [12]. We notice that for $q_n = 0$ it is effectively reduced to a similar equation for the states with $n-1$ reggeized gluons. The corresponding solution, $Q(\lambda)$, gives rise to the degenerate unnormalizable n Reggeon states with the energy

$$\varepsilon_n(h, q_3, \ldots, q_{n-1}, 0) = \varepsilon_{n-1}(h, q_3, \ldots, q_{n-1})\,. \tag{5.9}$$

These states should be excluded from the spectrum of the n Reggeon Hamiltonian and solving the Baxter equation for n reggeized gluons we have to satisfy the condition

$$q_n \neq 0\,. \tag{5.10}$$

As a function of the quantum numbers, the holomorphic energy obeys the relations

$$\varepsilon_n(h, q_3, \ldots, q_n) = \varepsilon_n(1-h, q_3, \ldots, q_n) = \varepsilon_n(h, -q_3, \ldots, (-)^n q_n)\,, \tag{5.11}$$

which follow from the symmetry of the Baxter equation (5.6) under the replacement $h \to 1-h$ or $\lambda \to -\lambda$ and $q_m \to (-)^m q_m$. This relation means that the spectrum of the Reggeon Hamiltonian is degenerate with respect to quantum numbers h, q_3, ..., q_n. Then, assuming that the ground state of the XXX Heisenberg magnet of spin $s=0$ is not degenerate we can identify the quantum numbers corresponding to the maximal value of the Reggeon energy as [12]

$$h\Big|_{\rm max} = \tfrac{1}{2} \tag{5.12}$$

and for the states with only even number of Reggeons, $n = 2m$,

$$q_3\Big|_{\rm max} = q_5\Big|_{\rm max} = \ldots = q_{2m-1}\Big|_{\rm max} = 0\,.$$

For the states with odd number of Reggeons the latter condition is not consistent with (5.10).

We notice that the conformal weight h enters as a parameter into the Baxter equation (5.3) and, in general, one is interesting to find the solutions of (5.3) only for its special values (5.2). In [14, 12] the following way of solving the Baxter equation was proposed. One first solves (5.3) for integer positive values of the conformal weight h and then analytically continues the result, $Q(\lambda)$, to all possible values (5.2) including the most physically interesting value (5.12). The Baxter equation (5.3) has two linear independent solutions and in order to select only one of them we have to impose the additional condition on the function $Q(\lambda)$ for $h = \mathbb{Z}_+$

$$Q(\lambda) \stackrel{\lambda\to\infty}{\sim} \lambda^{h-n}\,. \tag{5.13}$$

For integer positive conformal weight, $h \geq k$, the solution, $Q(\lambda)$, of the Baxter equation (5.3) under the additional condition (5.13) is given by a polynomial of degree $h-n$ in the spectral parameter λ, which can be expressed in terms of its roots λ_1, ..., λ_{h-n} as follows [14, 12]

$$Q(\lambda) = \prod_{i=1}^{h-n} (\lambda - \lambda_i)\,. \tag{5.14}$$

Substituting (5.14) into (5.6) and putting $\lambda = \lambda_i$ we obtain that the roots satisfy the famous Bethe equation for the XXX spin chain

$$\left(\frac{\lambda_m + i}{\lambda_m - i}\right)^n = \prod_{\substack{j=1 \\ j\neq m}}^{h-n} \frac{\lambda_m - \lambda_j - i}{\lambda_m - \lambda_j + i}, \qquad m = 1, \ldots, h. \tag{5.15}$$

Unfortunately, there is no regular way of solving the Bethe equation (5.15) for an arbitrary number n of Reggeons. At the same time, for $n = 2$ Reggeon states the explicit solution of the Baxter equation (5.3) was found in [14, 12]

$$Q_{n=2}(\lambda) = i^h h(1-h)\; {}_3F_2 \left(\begin{matrix} 1+h, 2-h, 1-i\lambda \\ 2, 2 \end{matrix} ; 1 \right). \tag{5.16}$$

where ${}_3F_2$ is the generalized hypergeometric function. This expression admits an interesting interpretation [12, 25] in terms of classical orthogonal polynomials and conformal field theories. Substituting the solution (5.16) into (5.8) we obtain the holomorphic energy of the $n = 2$ Reggeon states as [12]

$$\varepsilon_2(h) = -4\left[\psi(h) - \psi(1)\right], \tag{5.17}$$

where ψ-function was defined in (4.7). We substitute $\varepsilon_2(h)$ into (5.7) and analytically continue the result from integer h to all possible complex values (5.2)

$$E_2(h) = -2\frac{\alpha_s N_c}{\pi} \operatorname{Re}\left[\psi\left(\frac{1+|n|}{2} + i\nu\right) - \psi(1)\right].$$

This relation coincides with the well-known expression [3] for the energy of the $n = 2$ Reggeon compound state, the BFKL Pomeron. The maximum value of the energy

$$E_2^{\max} = \frac{\alpha_s N_c}{\pi} 4 \ln 2$$

is achieved at $h = 1/2$ and it is in agreement with (2.19), (2.18) and (5.12).

It is of most interest to find the solution of the Baxter equation (5.3) for higher $n \geq 3$ Reggeon states. Different approaches have been proposed [12, 26] but explicit expression similar to that (5.16) for the BFKL Pomeron was not found yet.

It is not difficult however to solve the Baxter equation (5.3) numerically for $n = 3, 4 \ldots$ Reggeon states and for lowest values of the conformal weight h and then find the quantized charges $\{q_m\}$ and the energy ε_n. The results of numerical solution of the Baxter equation for $n = 3$ and $n = 4$ presented in [12] indicate the remarkable regularity in the distribution of the quantized values of charges $\{q_m\}$ and energy ε_n and they strongly suggest that the analytical solution of the Baxter equation for high Reggeon states should exist. Recently, such kind of solution was found in [25] for an arbitrary number of Reggeons, n, in the special limit of large values of the conformal weight of the Reggeon states, $h \gg 1$.

The method developed in [25] is based on the observation that after rescaling of the spectral parameter $\lambda \to \lambda h$ in the Baxter equation (5.6), the operator Δ can be replaced in the naive limit $h \to \infty$ by a second-order derivative. Then, the Baxter equation (5.6) takes a form of the one-dimensional Schrödinger equation for a particle in external potential, in which the inverse conformal weight $1/h$ of the Reggeon states plays a role of the Planck constant. This fact allows us to apply the well-known quasi-classical expansion and obtain the solution of the Baxter equation as well as quantized values of the charges and energy of n Reggeon states in the form of asymptotic series in $1/h$. For large integer h the obtained analytical expressions completely agree with the results of numerical solutions while for small h the asymptotic expansions for the energy of n Reggeon states becomes divergent and it should be replaced by the asymptotic approximation. As first nontrivial application of these results, the intercept of the $n = 3$ Reggeon state, the so-called perturbative Odderon [27], was obtained as [25]

$$\alpha_3 - 1 \leq \frac{\alpha_s N_c}{\pi} 2.41 \,. \tag{5.18}$$

This relation estimates the Odderon intercept from above and it implies, in particular, that it is smaller than the intercept (2.19) of the BFKL Pomeron. The expression (5.18) is also in agreement with the lower bound for the Odderon intercept proposed in [28].

6. Analytical properties of the holomorphic energy

In previous sections we studied the behaviour of the structure function of DIS at small x, but we did not explore yet its dependence on the photon virtuality Q^2. Let us consider the asymptotics of the obtained expressions (2.16) in the double scaling limit of small x and large Q^2/m^2 with m being onium mass. In this limit one should be able to reproduce the operator product expansion (1.2). Moreover, as we will show below, there is a close relation between the analytical properties of the holomorphic energy $\varepsilon_n(h, \{q\})$ of the n-Reggeon states in the complex h-plane and the anomalous dimensions $\gamma_\omega^{(k)}$ of the higher twist operators in the generalized LLA.

Let us consider the moments (1.2) of the structure function of DIS in the generalized LLA. Using the relation (2.16), one can find the contribution of the n-Reggeon states to $F_\omega(Q^2)$ as

$$F^{(n)}_\omega(Q^2)=\sum_{\{q\}}\frac{1}{\omega-E_{n,\{q\}}}\beta^{\{q\}}_{n\to\gamma^*(Q^2)}\beta^{\{q\}}_{n\to h(m^2)}\,. \qquad (6.1)$$

Here the summation is performed over all quantum numbers h, q_3, ..., q_n corresponding to the n Reggeon compound state with the energy $E_{n,\{q\}}$. The residue factors have been defined in (2.17) and in the case of the forward scattering, $t=0$, they depend only on the invariant masses of scattering particles, Q^2 and m^2. In the generalized LLA, one may calculate the residue factors for perturbative states of virtual photon, $\gamma^*(Q^2)$, and onium, $h(m^2)$, in the Born approximation and neglect α_s corrections. As a result, $\beta^{\{q\}}_{n\to\gamma^*}$ and $\beta^{\{q\}}_{n\to h}$ do not have anomalous dimension and their scaling dimensions are equal to the sum of the scaling dimensions of the n Reggeon state, $h+\bar h=1+2i\nu$, and the scaling dimensions of photon and onium states

$$\beta^{\{q\}}_{n\to\gamma^*(Q^2)}=C^{\{q\}}_{n\to\gamma^*}\,Q^{-1+2i\nu}\,,\qquad \beta^{\{q\}}_{n\to h(m^2)}=C^{\{q\}}_{n\to h}\,m^{-1-2i\nu}\,,$$

where the dimensionless coefficients depend on the quantum numbers of the Reggeon states and scattering particles. Substituting these relations into (6.1) we obtain

$$F^{(n)}_\omega(Q^2)=\frac{1}{Q^2}\sum_{q_3,\dots,q_n}\int_{-\infty}^{\infty}d\nu$$
$$\times\sum_{m\ge 0}\frac{C^{\{q\}}_{n\to\gamma^*}C^{\{q\}}_{n\to h}}{\omega-\frac{\alpha_sN}{4\pi}\left[\varepsilon_n(\frac{1+m}{2}+i\nu;\{q\})+\varepsilon_n(\frac{1+m}{2}-i\nu;\{q\})\right]}\left(\frac{m}{Q}\right)^{-1-2i\nu} \qquad (6.2)$$

Here, we extracted the sum over quantized values (5.2) of the conformal weight, that is summation over discrete m and integration over continuous ν, from the sum over all quantum numbers in (6.1).

Let us consider (6.2) in the limit $Q^2\gg m^2$ and try to expand $F^{(n)}_\omega(Q^2)$ in powers of $1/Q^2$ according to (1.2). For $Q^2\gg m^2$ one can enclose the integration contour over ν into the lower half-plane, $\operatorname{Im}\nu<0$, and calculate the integral over ν in (6.2) by taking the residue at the values of ν which satisfy the relation

$$\frac{4\pi\omega}{\alpha_sN}=\varepsilon_n\left(\frac{1+m}{2}+i\nu;\{q\}\right)+\varepsilon_n\left(\frac{1+m}{2}-i\nu;\{q\}\right). \qquad (6.4)$$

Solving this equation one can find the values of $i\nu$ which determine the power of m/Q in the $1/Q$−expansion of the structure function (1.2), or equivalently define the scaling dimensions of the composite operators entering into the OPE. The comparison of (6.2) with (1.2) requires that the solutions of (6.3) should have the following form for $\mathrm{Im}\,\nu < 0$:

$$i\nu = \tfrac{1}{2} - k - \gamma_\omega^{(k)}(\alpha_s)\,, \qquad k = 1\,,2\,,\ldots\,, \tag{6.4}$$

where $\gamma_\omega^{(k)}(\alpha_s)$ is the anomalous dimension of twist $2k$ operators in the generalized LLA. Let us now take into account that in order for the contribution of the pole in ν to (6.2) to be nonvanishing, the residue factors, or equivalently the coefficients $C^{\{q\}}_{n\to\gamma^*}$ and $C^{\{q\}}_{n\to h}$, should be different from zero. This condition imposes selection rules on the quantum numbers h, q_3, ..., q_n of the Reggeon states. In particular, for the residue factors to be scalar, the conformal spin of the Reggeon state, $h - \bar{h} = m$, should be equal to the conformal spin of the photon and onium states [11]: $m = 0$ or $m = 2$.

Let us substitute (6.4) into (6.3) and take the limit $\alpha_s \to 0$ in the both sides of (6.3). The α_s dependence enters into the r.h.s. of (6.3) only through the anomalous dimension which has a perturbative expansion (1.3). For small α_s we may invert the functional dependence in (1.3) and express α_s/ω as a power series in $\gamma_\omega^{(k)}$. Then, the l.h.s. of (6.3) becomes a Laurant series in $\gamma_\omega^{(k)}$ with the first term proportional to $\sim 1/\gamma_\omega^{(k)}$. Comparing the $\gamma_\omega^{(k)}$–dependence of the both sides of (6.3) we find that the holomorphic energy $\varepsilon_n = \varepsilon_n(h;\{q\})$ has simple poles at the origin and at the integer negative values of the conformal spin

$$\varepsilon_n(h;\{q\}) \stackrel{h\to -l}{\sim} \frac{A_{l,\{q\}}}{h+l}\,, \qquad l = 0,\,1,\,2\,,\ldots\,. \tag{6.5}$$

Let us verify this relation in the simplest case of the $n = 2$ Reggeon states, in which the holomorphic energy is given by (5.17). We find that $\varepsilon_2(h)$ is analytical function of h in the half-plane $\mathrm{Re}\,h \geq 1/2$ with the asymptotics at infinity

$$\varepsilon_2(h) \stackrel{h\to\infty}{\sim} -4\ln h\,, \tag{6.6}$$

while for $\mathrm{Re}\,h < 1/2$ it has poles at the origin $h = 0$ and at negative integer h

$$\varepsilon_2(h) \stackrel{h\to -l}{\sim} \frac{4}{h+l}\,, \qquad l = 0, 1, 2, \ldots\,. \tag{6.7}$$

Thus, the relation (6.5) holds for the energy of the $n = 2$ Reggeon states (5.17) provided that $A_l = 4$.

For the higher Reggeon states the holomorphic energy ε_n can be obtained in the form of the asymptotic series in $1/h$ whereas the analytical expression for ε_n similar to (5.17) is not available yet. In the large h limit ε_n has the asymptotic behaviour

$$\varepsilon_n(h;\{q\}) \overset{h\to\infty}{\sim} -2n\ln h\,. \tag{6.8}$$

Then, using this behaviour and calculating the discontinuity of the energy at the negative h one can write the subtracted dispersion relation for the function $\varepsilon_n(h;\{q\})$ in the complex h-plane which leads to

$$\varepsilon_n(h;\{q\}) = \sum_{l=0}^{\infty} \frac{A_{l,\{q\}}}{h+l} + C \tag{6.9}$$

with C some infinite $h-$independent subtraction constant. For the $n=2$ Reggeon states the constant C can be found using the condition $\varepsilon_2(1)=0$. Being combined together, the relations (6.9) and (6.8) allow us to obtain the asymptotics of the coefficients: $A_{l,\{q\}}\sim 2n$ as $l\to\infty$. Substituting (6.9) into (6.3) and taking into account the relation (6.4) we can express the anomalous dimensions $\gamma_\omega^{(k)}$ in terms of the coefficients $A_{l,\{q\}}$ as follows

$$\gamma_\omega^{(k)} = -\frac{\alpha_s N_c}{4\pi\omega} A_{k-1-m/2,\{q\}} + \mathcal{O}(\alpha_s^2)\,. \tag{6.10}$$

This expression describes the scaling dimensions of the operators entering into the $1/Q^2$ expansion of the contribution of the n-Reggeon states into the structure function. For $n=2$ Reggeon state we have $A_l=4$ and (6.10) coincides with the well-known result [11].

7. Summary

It is still remains a challenge for QCD to understand the mechanism responsible for the rise of the structure function of DIS at small x. In this regime one has to deal with the system of strongly correlated partons in proton, for which "good" standard methods like operator product expansion are not applicable.

It is widely believed that in the Regge limit and, in particular in the small$-x$ limit, QCD should be replaced by a two-dimensional effective theory (dual models, QCD string *etc.)*, in which Reggeons play a role of a new collective degrees of freedom. This theory should inherit all symmetries of QCD and one may try to identify them by studying the small x asymptotics of the structure function in DIS.

Indeed, analysing the small x limit of the structure function of DIS in the generalized LLA we found that multi-color QCD turns out to be equivalent to the exactly solvable XXX Heisenberg magnet for noncompact spin $s = 0$. This relation still looks mysterious and one should understand better its origin starting from QCD lagrangian. From practical point of view it allows us to apply the powerful methods of exactly solvable models for calculation the spectrum of Pomerons in perturbative QCD. The work in this direction is at the very beginning and one should expect a lot of surprises to come soon.

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