UNIVERSALITY AND SCALING
IN PERTURBATIVE QCD AT SMALL x *

S. FORTE†

Theory Division, CERN
CH-1211 Genéve 23, Switzerland

R.D. BALL‡

Theory Division, CERN
Ch-1211 Genéve 23, Switzerland
and
Department of Physics and Astronomy, University of Edinburgh
Edinburgh EH9 3JZ, Scotland

We present a pedagogical review of the universal scaling properties displayed by the structure function F_2 at small x and large Q^2 as measured at HERA. We first describe the derivation of the double asymptotic scaling of F_2 from the leading-order Altarelli–Parisi equations of perturbative QCD. Universal next-to-leading order corrections to scaling are also derived. We explain why the universal scaling behaviour is spoiled when the initial distributions rise too steeply by considering the nonsinglet distribution $F_2^p - F_2^n$ as an explicit example. We then examine the stability of double scaling to the inclusion of higher order singularities, explaining how the perturbative expansion at small x can be reorganized in such a way that each order is given by the sum of a convergent series of contributions which are of arbitrarily high order in the coupling. The wave-like nature of perturbative evolution is then shown to persist throughout almost all the small x region, giving rise asymptotically to double scaling for a wide class of boundary conditions.

PACS numbers: 11.55.Jg, 12.38.Bx

---

1. Structure functions at small x

The small x limit of structure functions measured in deep-inelastic scattering is the frontier of perturbative QCD both from the experimental and the theoretical point of view. In deep-inelastic scattering experiments [1], one measures the total, fully inclusive cross section for scattering of a virtual photon with virtuality $-Q^2$ over a nucleon. The center-of-mass energy of the collision is given by the Mandelstam invariant $s = Q^2(1-x)/x$. In the large Q^2 limit the cross section is parametrized by a single form factor $F_2(x, Q^2)$ which is determined by the underlying partonic degrees of freedom, because its moments with respect to x are directly related to the nucleon matrix elements of quark and gluon operators. The scale dependence of these moments is governed by renormalization group equations which summarize the dynamical content of perturbative QCD.

The small x limit at large Q^2 thus corresponds to probing the light-cone dynamics of the nucleon (large Q^2) in the high energy limit (large $s \sim Q^2/x$). This limit stretches perturbative QCD towards its nonperturbative frontier. On the one hand, one might expect the high-energy limit of total cross sections to be governed by unitarity, and in particular by the t-channel exchange of Regge trajectories [2], which do not admit a simple perturbative interpretation. On the other hand, perturbation theory itself in this limit is somewhat problematic, hinting to its eventual breakdown. Indeed, the anomalous dimensions which govern the perturbative scaling behaviour grow without bounds, implying that Bjorken scaling is shifted to larger and larger values of Q^2. This is a manifestation of the fact that there is now another large scale in the theory besides Q^2, namely s itself: hence the renormalization group will have to be adapted in order to sum up this scale too. It is not *a priori* obvious in which kinematic region this might be possible, if at all.

Experimentally, accessing this region requires the very high center-of-mass energies that have only been attained very recently at the electron-proton collider HERA [3–5]. When the first data on F_2 at large Q^2 and small x were first presented they have provoked a considerable amount of surprise: they not only displayed the expected large violations of Bjorken scaling, growing larger as x decreases, rather, they also deviated from the Regge behaviour which is well tested by high-energy elastic scattering data, by displaying a marked rise as x decreases at fixed Q^2, whereas Regge theory would have a flat or almost flat behaviour. Yet, such a violation of Regge behaviour was predicted more than twenty years ago as a direct consequence of the leading-order renormalization group equations of perturbative QCD [6].

This non-Regge rise takes the form of a simple universal scaling law [7] satisfied by $F_2(x, Q^2)$ at large Q^2 and small x: the structure function depends only on a variable $\sigma(x, Q^2)$. Furthermore, this dependence is univer-

sal, unlike Bjorken scaling, where asymptotically structure functions depend only on x, but in a non-universal, incalculable way: hence it actually corresponds to a double scaling law, as the universal dependence may be scaled out. The way this double asymptotic scaling behaviour has arisen out of the HERA data is shown in Fig. 1, which displays F_2 as a function of the scaling variable $\sigma(x, Q^2)$ (for all accessible values of x and Q^2), for each successive published set of data.
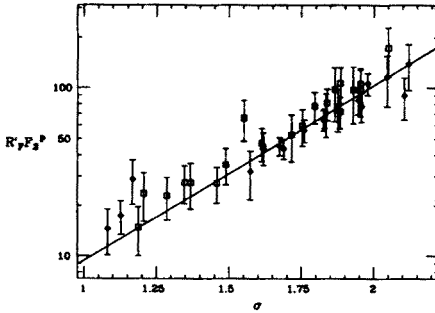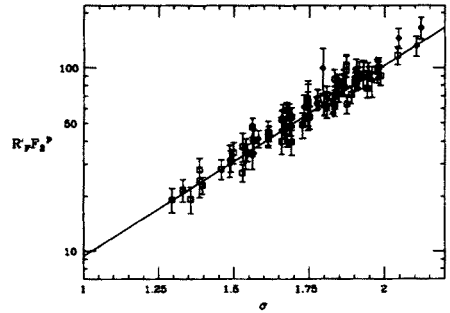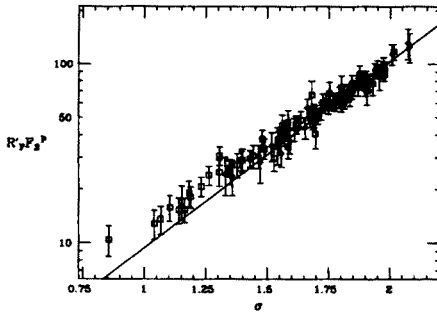
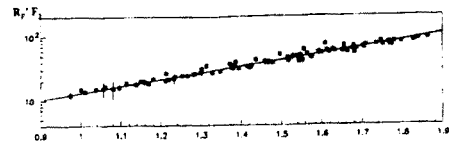

Fig. 1a

Fig. 1b

Fig. 1c

Fig. 1d

Scaling plot of the experimental data on the proton structure function $F_2(x, Q^2)$ from (a) the 1992 HERA run [3] (adapted from Ref. [7]), (b) the preliminary analysis of the 1993 HERA run [9] (adapted from Ref. [8]), (c) the 1993 run [4], (d) the preliminary analysis of the 1994 run (H1 only) (adapted from Ref. [5]). The diamonds are ZEUS data and the squares are H1 data. Only points with $\rho, \sigma > 1$ are included in the plots (b)–(d), and in (d) only those with $Q^2 > 5$ GeV^2 (see text). The straight line shown is the predicted asymptotic double scaling behaviour (with fitted normalization).

It is apparent that double asymptotic scaling is the foremost feature of F_2 at small x and large Q^2, at least in the region presently explored by the

HERA experiments. Understanding perturbative QCD in this region thus means understanding the physics of double asymptotic scaling. This entails understanding why the simple scaling prediction, which, after all, follows from a leading-order renormalization group analysis, survives the problems of perturbative instability and need for the inclusion of other large scales alluded to above.

In these lectures we will briefly summarize the current status of the current theoretical and phenomenological understanding of double scaling. In Sect. 2 we will derive the double scaling prediction from leading-order perturbative QCD and discuss the physics behind this behaviour. We will then see how double scaling may be spoiled if parton distributions at low scale are too steep, taking as an example the case of nonsinglet structure functions, where this actually happens. We will further explain how double scaling is modified (but its universality preserved) by the inclusion of next-to-leading order corrections and show that present-day data are perfectly described by such a next-to-leading order analysis. In Sect. 3 we will then discuss the behaviour of the perturbative expansion of anomalous dimensions when the effects of the other large scale which is present in the problem are included to all orders in the coupling. We will show that the perturbative expansion may be reorganized so as to sum up these effects, and that double scaling emerges then as the generic universal asymptotic behaviour. We will conclude by recalling some recent phenomenological applications of this formalism to precision tests of QCD, and summarizing the most promising future theoretical and phenomenological developments.

2. Double asymptotic scaling at leading and next-to-leading order

The perturbative evolution of the structure function $F_2(x, Q^2)$ is determined by first decomposing it into parton distributions:

$$x^{-1}F_2(x, Q^2) \equiv \sum_{i=1}^{n_f} e_i^2 C_i \otimes (q_i + \bar{q}_i) + C_g \otimes g, \qquad (2.1)$$

where n_f is the number of active flavors, e_i is the electric charge of the quark distribution $q_i(x; Q^2)$, $\otimes$ denotes the convolution with respect to x, i.e.

$$[f \otimes g](x) \equiv \int_x^1 \frac{dy}{y} f\left(\frac{x}{y}\right) g(y),$$

and the coefficient functions at leading order are simply

$$C_i(x, Q^2) = \delta(1 - x); \qquad C_g(x, Q^2) = 0, \qquad (2.2)$$

while at higher orders they depend on the specific choice of renormalization prescription (factorization scheme): in most of the subsequent treatment we will choose a scheme (parton scheme) in which Eq. (2.2) remains true to all orders. The evolution of parton distributions is in turn determined by the Altarelli–Parisi equations [10]

$$\frac{d}{dt}\begin{pmatrix} g \\ q_{\mathrm{S}} \end{pmatrix} = \frac{\alpha_s(t)}{2\pi}\begin{pmatrix} P_{gg} & P_{gq} \\ P_{qg} & P_{qq}^{\mathrm{S}} \end{pmatrix} \otimes \begin{pmatrix} g \\ q_{\mathrm{S}} \end{pmatrix},$$

$$\frac{d}{dt}q_{\mathrm{NS}} = \frac{\alpha_s(t)}{2\pi}P_{qq}^{\mathrm{NS}} \otimes q_{\mathrm{NS}}, \qquad\qquad (2.3)$$

where $t \equiv \ln(Q^2/\Lambda^2)$ and the singlet and nonsinglet quark distributions are respectively given by

$$q_{\mathrm{S}}(x,t) = \sum_{i=1}^{n_f}\left(q_i(x;Q^2) + \bar{q}_i(x;Q^2)\right),$$

$$q_{\mathrm{NS}}(x,t) = \sum_{i=1}^{n_f}\left(\frac{e_i^2}{\langle e^2\rangle} - 1\right)\left(q_i(x;Q^2) + \bar{q}_i(x;Q^2)\right), \qquad (2.4)$$

with $\langle e^2\rangle = (1/n_f)\sum_{i=1}^{n} e_i^2$. In terms of these distributions the decomposition (2.1) becomes, in a parton scheme (2.2), simply

$$F_2(x,Q^2) = \langle e^2\rangle x \left(q_{\mathrm{S}}(x,t) + q_{\mathrm{NS}}(x,t)\right). \qquad (2.5)$$

The splitting functions depend on t through the strong coupling $\alpha_s(t)$, and at n-th perturbative order are given by

$$\frac{\alpha_s(t)}{2\pi}P(x,t) = \sum_{i=1}^{n} P^{(i)}\left(\frac{\alpha_s(t)}{2\pi}\right)^i + O\left(\alpha_s^{n+1}\right). \qquad (2.6)$$

The Altarelli–Parisi equations Eq. (2.3) are simply the inverse Mellin transforms of the renormalization group equations satisfied by the nucleon matrix elements of quark and gluon operators, namely

$$\frac{d}{dt}\begin{pmatrix} g(N,t) \\ q_{\mathrm{S}}(N,t) \end{pmatrix} = \frac{\alpha_s(t)}{2\pi}\begin{pmatrix} \gamma_{gg}\left(N,\alpha_s(t)\right) & \gamma_{gq}\left(N,\alpha_s(t)\right) \\ \gamma_{qg}\left(N,\alpha_s(t)\right) & \gamma_{qq}^{\mathrm{S}}\left(N,\alpha_s(t)\right) \end{pmatrix}\begin{pmatrix} g(N,t) \\ q_{\mathrm{S}}(N,t) \end{pmatrix},$$

$$\frac{d}{dt}q_{\mathrm{NS}}(N,t) = \frac{\alpha_s(t)}{2\pi}\gamma_{qq}^{\mathrm{NS}}\left(N,\alpha_s(t)\right)q_{\mathrm{NS}}(N,t). \qquad (2.7)$$

The operator matrix elements are related to the corresponding parton distributions by Mellin transform, *i.e.* the matrix element of the (leading twist)

spin $N + 1$ operator is the N-th moment of the corresponding parton distribution

$$p(N, t) = \mathcal{M}[p(x, t)] \equiv \int_0^1 dx \, x^N p(x, t) \,, \qquad (2.8)$$

where $p(x, t)$ is any linear combination of parton densities. Likewise, anomalous dimensions are found by taking moments of the splitting functions

$$\gamma(N, \alpha_s(t)) = \mathcal{M}[P(x, t)] \equiv \int_0^1 dx \, x^N P(x, t) \,. \qquad (2.9)$$

Notice that only operators for odd values of $N > 0$ in Eqs (2.8), (2.9) actually exist, because even (odd) spin operators are charge-conjugation even (odd), while F_2 and the parton distributions which contribute to it are charge-conjugation even; the anomalous dimensions for all other (complex, in general) values of N can only be defined by analytic continuation. This continuation is provided by the Altarelli–Parisi formalism, where the primary quantities, namely the parton densities and splitting functions, can be used to define the values of matrix elements and anomalous dimensions for all N through (2.8) and (2.9), provided only that they are known for all values of $0 < x < 1$. Of course, these are also precisely the quantities which can be extracted directly from the structure functions measured experimentally, albeit in practice only over a limited range of x.

The Altarelli–Parisi equations determine the parton distributions at (x', t') in terms of their values for all $x > x'$ and $t < t'$, hence they actually describe evolution in the whole (x, t) plane, even though the renormalization group equations Eq. (2.7) for each value of N only specify an evolution law with respect to t. The evolution with respect to x, while causal, is however nonlocal. The basic idea behind double scaling is the realization that at small x the Altarelli–Parisi equations actually reduce to local evolution equations in both variables, x and t, which can then be treated symmetrically.

This can be shown by constructing a systematic approximation [7] to the evolution equations Eq. (2.3) at small x and large Q^2. As x gets smaller, one would expect the behaviour of parton distributions to be dominated by that of their Mellin transforms for small N, and thus by the small N behaviour of the anomalous dimensions. Since the anomalous dimensions are singular at small N one would specifically expect their right most singularity to provide the dominant behaviour.

A simple way of showing [6] that this is indeed the case is to solve the Altarelli–Parisi equations by Mellin transformation, *i.e.* solve the renormalization group equations Eq. (2.7) for the eigenvectors of the anomalous

dimension matrix, which are linear combinations $p(x,t)$ of the parton densities. Using the leading order (LO) form of the anomalous dimensions (which is t independent) and of $\alpha_s = (4\pi/\beta_0 t)$ with $\beta_0 = 11 - (2/3)n_f$, this procedure gives generically

$$p(N,t) = p(N,t_0) \exp\left[\frac{2}{\beta_0}\zeta\gamma^{(1)}(N)\right], \tag{2.10}$$

where

$$\zeta \equiv \ln\left(\frac{t}{t_0}\right) = \ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right). \tag{2.11}$$

The x space solution is then the inverse Mellin transform

$$p(x,t) = \int\limits_{-i\infty}^{i\infty} dN \, \exp(\xi N) p(N,t), \tag{2.12}$$

where

$$\xi \equiv \ln\left(\frac{x_0}{x}\right) \tag{2.13}$$

and the integration runs over a contour located to the right of all singularities of $\gamma(N)$ and $p(N,t_0)$. Assume for the moment that any singularities of the initial condition $p(N,t_0)$ are to the left of those of $\gamma(N)$ (which are always poles on the real axis at non-positive integer values of N). Then at small x, *i.e.* as ξ grows the integral may be evaluated by the saddle point method: the saddle point condition is of the form

$$\xi_s + \frac{2}{\beta_0}\frac{d\gamma^{(1)}_{\mathrm{sing}}}{dN}\zeta = 0, \tag{2.14}$$

where $\gamma^{(1)}_{\mathrm{sing}}$ is the leading singularity of the one loop anomalous dimension. Higher order terms in the expansion around the singularity then give subleading corrections both to the location of the saddle point and to the integral over it. If the initial condition $q_{\mathrm{NS}}(N,t_0)$ has a singularity (typically a branch point) to the right of that of the anomalous dimension this will also contribute to the integral and indeed may dominate the contribution from the saddle point as $\xi \to \infty$. However the saddle point always gives the dominant contribution at large Q^2 *i.e.* large ζ, and in any case the dominant contribution to the evolution is always given at large ξ by the leading singularity $\gamma^{(1)}_{\mathrm{sing}}$.

We may thus obtain the leading small x behaviour of the Altarelli-Parisi equations by expanding the matrix of anomalous dimensions around

its right most singularity, determining the corresponding splitting functions by inverse Mellin transformation, and then solving the resulting simplified Altarelli–Parisi equations. The singularity is located at $N = 0$ in the singlet case, and at $N = -1$ in the nonsinglet. This means that all other things being equal $q_{NS}(x, t)$ will display the same qualitative behaviour as $x q_S(x, t)$, and $x g(x, t)$, *i.e.* at small x the nonsinglet will be down by a power of x compared to the singlet.

Expanding the singlet anomalous dimension about its leading singularity, at one loop we find

$$\gamma_S^{(1)}(N) = \frac{1}{N} \begin{pmatrix} 2C_A & 2C_F \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{11}{6}C_A - \frac{2}{3}T_R n_f & -\frac{3}{2}C_F \\ \frac{4}{3}T_R n_f & 0 \end{pmatrix} + O(N),$$

(2.15)

while doing the same thing for the nonsinglet

$$\gamma_{NS}^{(1)}(N) = \frac{C_F}{N+1} + \tfrac{1}{2}C_F + O(N+1),$$ (2.16)

where $C_A = 3$, $C_F = 4/3$, $T_R = 1/2$ for QCD with three colors. The corresponding splitting functions are simply found by noting that

$$\mathcal{M}\left[\frac{1}{x}\right] = \frac{1}{N}, \qquad \mathcal{M}[\delta(1-x)] = 1, \qquad \mathcal{M}[1] = \frac{1}{N+1}.$$ (2.17)

Positive powers of N correspond to logarithmic derivatives of $\delta(1-x)$ which, when summed up, produce the nonlocal x propagation kernel of Eq. (2.3). If instead the expansion is truncated, the evolution equations are local. Specifically, using the splitting functions obtained according to Eq. (2.17) from the anomalous dimensions Eq. (2.15) in the evolution equations Eq. (2.3), and then differentiating with respect to ξ, we get

$$\frac{\partial}{\partial\xi\partial\zeta}\begin{pmatrix} G(\xi,\zeta) \\ Q(\xi,\zeta) \end{pmatrix} =$$
$$\frac{2}{\beta_0}\left[\begin{pmatrix} \frac{11}{6}C_A - \frac{2}{3}T_R n_f & -\frac{3}{2}C_F \\ \frac{4}{3}T_R n_f & 0 \end{pmatrix}\frac{\partial}{\partial\xi} + \begin{pmatrix} 2C_A & 2C_F \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} G(\xi,\zeta) \\ Q(\xi,\zeta) \end{pmatrix},$$

(2.18)

$$\frac{\partial}{\partial\xi\partial\zeta}q(\xi,\zeta) = \frac{C_F}{\beta_0}\left[\frac{\partial}{\partial\xi} + 2\right]q(\xi,\zeta),$$

where we have for convenience defined

$$G(\xi,\zeta) \equiv x g(x,t), \quad Q(\xi,\zeta) \equiv x q_S(x,t), \quad q(\xi,\zeta) \equiv q_{NS}(x,t).$$ (2.19)

The Altarelli–Parisi equations thus simply reduce to two-dimensional partial differential equations with constant coefficients, thereby proving the local nature of the evolution.

The singlet evolution equations Eq. (2.18) can be solved by diagonalizing the matrix of anomalous dimensions Eq. (2.15): the eigenvectors will then satisfy (decoupled) equations of the form Eq. (2.7), with anomalous dimensions given by the corresponding eigenvalues. To order N^0 the eigenvalues are

$$\lambda_+ = 2C_A \frac{1}{N} - \left[\frac{11}{6}C_A + \frac{2}{3}T_R n_f - \frac{4C_F}{3C_A}T_R n_f \right] + O(N);$$

$$\lambda_- = -\frac{4C_F}{3C_A}T_R n_f + O(N), \tag{2.20}$$

corresponding to the eigenvectors $v_\pm = (Q_\pm, G_\pm)$ given by

$$Q_+ = \frac{2T_R}{3C_A}n_f N G_+ + O(N^2)$$

$$Q_- = -\frac{C_A}{C_F}G_- + O(N); \tag{2.21}$$

the singlet quark and gluon distributions are then given by $Q = Q_+ + Q_-$ and $G = G_+ + G_-$. Notice that, according to standard perturbation theory, when the eigenvalues are determined up to next-to-leading order (in N), only the leading nontrivial term ought to be kept in the expression for the eigenvector.

Transforming to x space with the help of Eq. (2.17) we thus get

$$\left[\frac{\partial^2}{\partial\xi\partial\zeta} + \delta_+ \frac{\partial}{\partial\xi} - \gamma^2 \right] G_+(\xi, \zeta) = 0 \,,$$

$$\left[\frac{\partial}{\partial\zeta} + \delta_- \right] G_-(\xi, \zeta) = 0 \,,$$

$$\left[\frac{\partial^2}{\partial\xi\partial\zeta} + \tilde\delta \frac{\partial}{\partial\xi} - \tilde\gamma^2 \right] q(\xi, \zeta) = 0 \,, \tag{2.22}$$

with

$$\gamma^2 = \frac{12}{\beta_0} \,, \quad \tilde\gamma^2 = \frac{8}{3\beta_0} \,,$$

$$\delta_+ = \frac{11 + \frac{2n_f}{27}}{\beta_0} \,, \quad \delta_- = \frac{16n_f}{27\beta_0} \,, \quad \tilde\delta = \frac{4}{3\beta_0} \,. \tag{2.23}$$

The eigenvector conditions Eq. (2.21) become

$$Q_+(\xi, \zeta) = \frac{n_f}{9} \frac{\partial G_+(\xi, \zeta)}{\partial \xi}; \qquad Q_-(\xi, \zeta) = -\frac{9}{4} G_-(\xi, \zeta). \qquad (2.24)$$

The eigenvector condition for v_+ can be conveniently rewritten by differentiating both sides with respect to ζ and using the expression Eq. (2.22) for $\partial^2 G_+/\partial \xi \partial \zeta$: the term proportional to $\partial G_+/\partial \xi$ can then be neglected since it only gives a subleading correction to the eigenvector, and we get

$$\frac{\partial Q_+(\xi, \zeta)}{\partial \zeta} = \frac{n_f}{9} \gamma^2 G_+(\xi, \zeta). \qquad (2.25)$$

Only the "large" eigenvalue λ_+ is singular at small N, hence, at small x and large Q^2, the "small" eigenvalue may be neglected. The singlet quark and gluon distributions are then no longer independent, but rather related by the large eigenvector condition Eq. (2.24). If we had kept only singular terms in the anomalous dimensions, the small eigenvalue would have vanished, the large eigenvector would have coincided with the gluon, and the singlet quark distribution would have vanished. This is also apparent by direct inspection of the small x evolution equation (2.22), which would have coincided with Eq. (2.18) with all terms proportional to $\partial/\partial \xi$ on the right hand side neglected. When the constant terms are retained, the large eigenvector contains a mixture of gluons and singlet quarks, obeying the evolution equation Eq. (2.22) for G_+ (and Q_+). This has the same form as the gluon equation in Eq. (2.18), but with the inhomogeneous term neglected, and a slightly different value of the coefficient of $\partial/\partial \xi$. The singlet quark contribution Q_+ is however subleading, being determined in terms of the gluon contribution G_+ according to Eq. (2.25), which coincides with the singlet quark equation in Eq. (2.22).

It thus appears that the behaviour of the singlet and nonsinglet components of F_2 at small x and large Q^2 are entirely determined by those of $G_+(\xi, \zeta)$ and $q(\xi, \zeta)$ respectively which, in turn, are determined by the evolution equations Eq. (2.22). These are recognized as a two-dimensional wave equations, $i.e.$ two-dimensional Klein–Gordon equations written in light-cone coordinates $(\xi, \zeta) = x \pm t$ with imaginary mass. This immediately implies several general properties of their solution:

(i) The equations are essentially symmetrical in ξ and ζ, so $G_+(\xi, \zeta)$ and $q(\xi, \zeta)$ evolve ('propagate') equally in both ξ and ζ ($i.e.$ in x and Q^2), up to the (small) asymmetry induced by the 'damping' term proportional to δ. Any further asymmetry in ξ and ζ must thus come from the boundary conditions.

(*ii*) The propagation is 'time-like', into the forward 'light-cone' at the origin $(\xi, \zeta) = (0, 0)$, along the 'characteristics' ξ = constant and ζ = constant (see Fig. 2).

(*iii*) At a given point (ξ, ζ), $G_+(\xi, \zeta)$ and $q(\xi, \zeta)$ depend only on their respective boundary conditions contained within the backward light cone formed by the two characteristics through (ξ, ζ).

(*iv*) Because the equations are linear, contributions to $G_+(\xi, \zeta)$ or $q(\xi, \zeta)$ from different parts of the boundaries are simply added together.

(*v*) Since the 'mass' terms are negative, the propagation is 'tachyonic'; this means that both $G_+(\xi, \zeta)$ and $q(\xi, \zeta)$ are unstable, growing exponentially rather than oscillating.

(*vi*) Since δ and $\bar{\delta}$ are both positive, the damping terms ensure that, at fixed ξ, $G_+(\xi, \zeta)$ and $q(\xi, \zeta)$ eventually fall with increasing ζ.

It is also straightforward to obtain the general solution to the evolution equations: they are simple examples of the characteristic Goursat problem, in which the solution is entirely determined by the knowledge of boundary conditions along two characteristics, and can be written explicitly in terms of a particular solution to the equation. The latter is easily found by observing that, setting $z = 2\gamma \sqrt{\xi \zeta}$, the first and third of Eqs (2.22) coincide with the Bessel equation, the appropriate solution of which is the Bessel function

$$I_0(z) \equiv \sum_0^\infty \frac{(\tfrac{1}{4}z^2)^n}{(n!)^2} \underset{z \to \infty}{\sim} \frac{1}{\sqrt{2\pi z}} e^z \left(1 + O\left(\frac{1}{z}\right)\right). \qquad (2.26)$$

The general solutions are thus found to be

$$G_+(\xi, \zeta) = I_0\left(2\gamma\sqrt{\xi\zeta}\right) e^{-\delta+\zeta} G_+(0,0)$$

$$+ \int_0^\xi d\xi' I_0\left(2\gamma\sqrt{(\xi-\xi')\zeta}\right) e^{-\delta+\zeta} \frac{\partial}{\partial\xi'} G_+(\xi',0)$$

$$+ \int_0^\zeta d\zeta' I_0\left(2\gamma\sqrt{\xi(\zeta-\zeta')}\right) e^{\delta+(\zeta'-\zeta)} \left(\frac{\partial}{\partial\zeta'} G_+(0,\zeta') + \delta_+ G_+(0,\zeta')\right),$$

$$G_-(\xi, \zeta) = e^{-\delta-\zeta} G_-(\xi,0),$$

$$q(\xi, \zeta) = I_0\left(2\bar{\gamma}\sqrt{\xi\zeta}\right) e^{-\bar{\delta}\zeta} q(0,0) + \int_0^\xi d\xi' I_0\left(2\bar{\gamma}\sqrt{(\xi-\xi')\zeta}\right) e^{-\bar{\delta}\zeta} \frac{\partial}{\partial\xi'} q(\xi',0)$$

$$+ \int_0^\zeta d\zeta' I_0\left(2\bar{\gamma}\sqrt{\xi(\zeta-\zeta')}\right) e^{\bar{\delta}(\zeta'-\zeta)} \left(\frac{\partial}{\partial\zeta'} q(0,\zeta') + \bar{\delta} q(0,\zeta')\right). \qquad (2.27)$$

These solutions display explicitly the symmetric nature of the evolution. It is interesting to compare them with the standard solution to the Altarelli–Parisi equations at larger x, where boundary conditions are imposed at a scale t_0 for x' such that $x \leq x' \leq 1$. Since the structure function vanishes kinematically at $x = 1$ the boundary condition on the lower boundary is trivial, and evolution takes place from the initial parton distributions assigned at a given t_0 forwards in t. In the present case, instead, the two boundaries are treated symmetrically, and evolution takes place as much with respect to x as it does with respect to t.

The asymptotic behaviour of $G_+(\xi, \zeta)$ or $q(\xi, \zeta)$ at small ξ and large ζ, $i.e.$ far away from the boundary, will in general depend on the form of the boundary conditions. Due to the linearity of the equation, we may consider each boundary separately. Contributions from each boundary are generated by fluctuations of the functions G_+ or q on that boundary. If these fluctuations are sufficiently well localized close to the origin, then far from the boundary we can use a multipole expansion, expanding the argument of the Bessel functions in powers of ξ'/ξ ($i.e.$, the distance from the boundary over the spread of the source) for the left boundary and ζ'/ζ for the lower boundary. All the contributions from higher moments of the boundary fluctuations are then seen to be suppressed by powers of the light-cone distance from the origin

$$\sigma \equiv \sqrt{\xi\zeta}\,, \tag{2.28}$$

while the leading contribution is simply given by the strength of the source at the origin, and its asymptotic behaviour is determined by that of the Bessel function (2.26):

$$G_+(\rho, \sigma) \underset{\sigma \to \infty}{\sim} \mathcal{N} \frac{1}{\sqrt{4\pi\gamma\sigma}} \exp\left\{2\gamma\sigma - \delta_+\left(\frac{\sigma}{\rho}\right)\right\}\left(1 + O\left(\frac{1}{\sigma}\right)\right),$$

$$q(\rho, \sigma) \underset{\sigma \to \infty}{\sim} \tilde{\mathcal{N}} \frac{1}{\sqrt{4\pi\tilde{\gamma}\sigma}} \exp\left\{2\tilde{\gamma}\sigma - \tilde{\delta}\left(\frac{\sigma}{\rho}\right)\right\}\left(1 + O\left(\frac{1}{\sigma}\right)\right), \tag{2.29}$$

where we have introduced the hyperbolic coordinate orthogonal to σ, namely

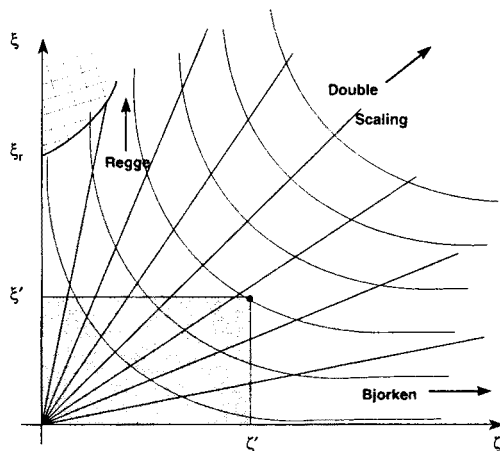$$\rho \equiv \sqrt{\frac{\xi}{\zeta}} \tag{2.30}$$

(see Fig. 2).

Fig. 2

The (ξ,ζ) plane, showing the backward light cone at the point (ξ',ζ'), curves of constant σ (the hyperbolae) and lines of constant ρ. The origin is chosen in such a way that the small x approximation of the evolution equations is valid for positive ξ, and perturbation theory breaks down for negative ζ where α_s grows too large. The hatched area $x \lesssim x_\tau \exp(-\alpha_s(t_0)^2/\alpha_s(t)^2)$ indicates the region where parton recombination effects are expected [11] to lead to breakdown of simple perturbative evolution due to higher twist corrections.

The origin of double scaling is now clear: because of the isotropy of the evolution equation, its solution asymptotically only depends on the scaling variable σ, *i.e.*, its level curves are hyperbolae in the (ξ, ζ) plane (see Fig. 2), and do not depend on the direction in which the propagation occurs (ρ scaling). Furthermore, because the solutions (2.29) are asymptotically independent of the boundary conditions, the dependence on σ is given by a universal rise, stronger than any power of ξ but weaker than any inverse power of x. The universal form of this rise reflects the underlying dynamical mechanism which generates it, namely, in the singlet case the collinear singularity in the triple gluon vertex which is responsible for the singularity in the anomalous dimension γ_{gg} (Eq. (2.15)) whose strength determines the coefficient γ^2 (Eq. (2.23)), and in the nonsinglet the corresponding singularity in $\gamma_{qq}^{\mathrm{NS}}$ due to collinear gluon bremsstrahlung, whose strength determines the coefficient $\tilde\gamma^2$. In an Abelian theory the former singularity clearly vanishes, so the singlet distributions would no longer grow so strongly, behaving instead more like the nonsinglet.

Using Eq. (2.25) to determine Q, and the expression Eq. (2.5) of F_2, and neglecting the nonsinglet contribution, since this is suppressed by a factor of approximately $e^{-\xi}$, it is easy to show that the asymptotic behaviour of F_2^p (or F_2^n) will also have a double scaling form, namely:

$$F_2^p \underset{\sigma \to \infty}{\sim} \frac{5n_f}{162} \frac{\gamma}{\rho} G_+(\sigma, \rho) \left(1 + O\left(\frac{1}{\sigma}\right) + O\left(\frac{1}{\rho}\right)\right), \qquad (2.31)$$

with G_+ given by Eq. (2.29). This behaviour holds up to corrections of order $1/\sigma$, from the subasymptotic form of the Bessel function and the boundary corrections, up to terms of relative order σ/ρ^2 in the exponent, from higher order contributions to the small N expansion Eq. (2.15) of the anomalous dimensions, and up to corrections of order $1/\rho$, from higher order contributions to the eigenvector equation. It thus holds in the limit $\sigma \to \infty$ along any curve such that also $\rho \to \infty$, such as for example the curve $\xi \propto \zeta^{1+\epsilon}$ with $\epsilon > 0$: that is, far from the boundaries, and provided the increase of $\ln(1/x)$ is more rapid than that of $\ln t$.

All of this, however, hinges on the assumption that the solution Eq. (2.27) to the wave equation may be treated in the multipole expansion, *i.e.*, that the fluctuations on the boundaries fall off away from the origin. If this does not happen, then the multipole expansion is not valid, the asymptotic behaviour Eq. (2.29) does not hold, and the chain of arguments leading to the scaling form of F_2 Eq. (2.31) breaks down. Specifically, assume for example that the boundary conditions at $\zeta = 0$ are exponentially rising functions of ξ: $G_+(\xi, 0) \sim \exp \lambda\xi = x^{-\lambda}$, $q(\xi, 0) \sim \exp \tilde{\lambda}\xi = x^{-\tilde{\lambda}}$. The boundary integral may then be evaluated by the saddle point method, is dominated by a nontrivial saddle-point, and gives the asymptotic behaviour

$$G_+(\sigma, \rho) \underset{\sigma \to \infty}{\sim} \mathcal{N}' \exp \left\{ \lambda\sigma\rho + \left(\frac{\gamma^2}{\lambda} - \delta_+\right)\left(\frac{\sigma}{\rho}\right)\right\} \left(1 + O\left(\frac{1}{\sigma}\right) + O\left(\frac{1}{\rho}\right)\right),$$

$$q(\sigma, \rho) \underset{\sigma \to \infty}{\sim} \tilde{\mathcal{N}}' \exp \left\{ \tilde{\lambda}\sigma\rho + \left(\frac{\tilde{\gamma}^2}{\tilde{\lambda}} - \tilde{\delta}\right)\left(\frac{\sigma}{\rho}\right)\right\} \left(1 + O\left(\frac{1}{\sigma}\right) + O\left(\frac{1}{\rho}\right)\right).$$
$$(2.32)$$

The strong growth on the boundary is thus preserved by the evolution. Since the evolution equations are linear, this behaviour should be added to the dynamically generated contributions (2.29). Since it is power-like, the rise of the boundary condition will eventually dominate the dynamically generated rise Eq. (2.29) when ξ is large enough. However the universal behaviour Eq. (2.29) is still dominant when ζ is large enough that the nontrivial saddle point leading to Eq. (2.32) is no longer dominant, *i.e.* whenever $\rho \lesssim (\gamma/\lambda)$, $\rho \lesssim (\tilde{\gamma}/\tilde{\lambda})$ respectively. Similar contributions to (2.32) would arise from exponentially rising functions of ζ on the lower boundaries.

In practice, we may choose x_0 close to the turning point of the evolution, so that the lower boundary condition is reasonably flat. Furthermore Regge theory suggests that the left hand boundary condition is given by $F_2 \sim x^{1-\alpha(0)}$, where $\alpha(0)$ is the intercept of the appropriate Regge trajectory. In the singlet channel, this is the pomeron trajectory, with [2] $\alpha_P(0) \simeq 1.08$, so the left hand boundary condition for G_+ should also be soft, and the asymptotic behaviour of F_2^p and F_2^n given by (2.31), at least for $\rho \lesssim \gamma/(\alpha_P(0)-1) \simeq 15$. However in the nonsinglet channel, the appropriate trajectory is that of the ρ, and $\alpha_\rho(0) \simeq (1/2)$. So the left hand boundary condition for the nonsinglet, $q \sim x^{-\alpha_\rho}$, is hard, and asymptotically (2.29) will be dominated by (2.32), whence, for $\rho \gtrsim, \tilde{\gamma}/\tilde{\lambda} \simeq 1$

$$F_2^p - F_2^n \underset{\sigma \to \infty}{\sim} \tilde{\mathcal{N}}'' \exp\left\{(\tilde{\lambda} - 1)\sigma\rho + \left(\frac{\tilde{\gamma}^2}{\tilde{\lambda}} - \tilde{\delta}\right)\left(\frac{\sigma}{\rho}\right)\right\}\left(1 + O\left(\frac{1}{\sigma}\right) + O\left(\frac{1}{\rho}\right)\right),$$

$$(2.33)$$

with $\tilde{\lambda} \simeq (1/2)$. Recent data from NMC [12] seem to be in good qualitative agreement with this prediction, but deuteron data in the kinematic region explored at HERA would be necessary to confirm it more precisely.

It was suggested some time ago [13] that the singlet boundary condition at Q_0^2 might not be given by the intercept of the pomeron trajectory, but could rather rise very steeply as $x^{-\lambda_L}$, with $\lambda_L = 4\ln 2(C_A/\pi)\alpha_s \simeq (1/2)$. This steep initial rise (sometimes called the 'hard pomeron') was supposed to incorporate, in an admittedly rather heuristic manner, the higher order perturbative effects at small x described by the BFKL equation [14]. With such a boundary condition, the double scaling rise (2.31) in F_2^p would be masked by the stronger rise of G_+ as given by (2.32), so

$$F_2^p \underset{\sigma \to \infty}{\sim} \mathcal{N}'' \exp\left\{\lambda_L\sigma\rho + \left(\frac{\gamma^2}{\lambda_L} - \delta\right)\left(\frac{\sigma}{\rho}\right)\right\}\left(1 + O\left(\frac{1}{\sigma}\right) + O\left(\frac{1}{\rho}\right)\right), \quad (2.34)$$

whenever $\rho \gtrsim \gamma/\lambda_L \simeq 2$. This prediction is nonuniversal, in the sense that the precise slope of the rise cannot be predicted since λ_L depends on α_s, and it is not known at which scale α_s should be determined. Furthermore it is qualitatively different in form from the double scaling rise (2.31): in particular the rise at large ξ, fixed ζ is now no longer accompanied by a corresponding rise at large ζ and fixed (though large) ξ. We will come back on the issue of applicability of leading order computations and the relative importance of higher order corrections in the next section. First however we will see whether the HERA data support the universal double scaling prediction (2.31), or prefer the more phenomenological suggestion (2.34).

The scaling plots in Fig. 1 display the measured values of F_2, with the subasymptotic corrections in Eq. (2.31) rescaled out, *i.e.* $R'_F F_2$ with

$$R'_F(\sigma, \rho) = \exp\left(\delta\left(\frac{\sigma}{\rho}\right) + \tfrac{1}{2}\ln\gamma\sigma + \ln\left(\frac{\rho}{\gamma}\right)\right). \quad (2.35)$$

The scaling variables are computed with $x_0 = 0.1$, $Q_0 = 1$ GeV, $\Lambda = 263$ MeV, and $\delta = \delta_+$ Eq. (2.23) with $n_f = 4$. The data cover a wide span in ρ: for instance the data in Fig. 1c have $1 \lesssim \rho \lesssim 5$. Nevertheless they all fall on the same line, and display a slope which agrees very well with the predicted asymptotic value $2\gamma = 2.4$. If the boundary condition were hard, Eq. (2.32) shows that the leading behaviour would also be a linear rise of $\ln F_2$ in σ, but now with a slope which is not universal (as it depends on λ), and strongly ρ dependent. Hence, the data should not fall on a single line, and the agreement of the observed slope with the calculated value of 2γ could only be a coincidence. The fact that the data display double scaling thus allows us to exclude the possibility of power-like boundary conditions to leading-order (or, as we will se in a moment, next-to-leading order) evolution at a very high confidence level [8].

The scaling plots in Fig. 1 also show that the slope of the rise of $\ln F_2$ is significantly smaller than the asymptotic one when σ is not too large. This suggests that scaling violations may already be important here. Even more dramatic scaling violations are seen if one considers data at low Q^2. Both effects are illustrated in Fig. 3 (i), which displays F_2 after complete rescaling of the leading asymptotic behaviour, *i.e.* by a factor

$$R_F(\sigma, \rho) = \exp\left(-2\gamma\sigma + \delta\left(\frac{\sigma}{\rho}\right) + \tfrac{1}{2}\ln\gamma\sigma + \ln\left(\frac{\rho}{\gamma}\right)\right). \tag{2.36}$$

It is apparent that the data display a systematic drop in σ (the asymptotic double scaling line is approached from above), and the recent (albeit preliminary) data with low Q^2 ($2 \lesssim \rho \lesssim 5$ and $1.3 \lesssim \sigma \lesssim 1.8$) do not seem to scale at all.

This leads us to consider scaling violations, the simplest of which appear when two-loop corrections are included in the Altarelli–Parisi equations. The leading singularities of the two loop anomalous dimensions [16] in $\overline{\text{MS}}$ scheme are

$$\gamma_S^{(2)}(N) = \frac{1}{N}\begin{pmatrix} (\tfrac{4}{3}C_F - \tfrac{46}{9}C_A)T_Rn_f & C_FC_A - \tfrac{40}{9}C_FT_Rn_f \\ \tfrac{40}{9}C_AT_Rn_f & \tfrac{40}{9}C_FT_Rn_f \end{pmatrix} + O(1),$$

$$\gamma_{\text{NS}}^{(2)}(N) = \frac{C_F^2}{(N+1)^3} + O((N+1)^{-2}). \tag{2.37}$$

Note that all the entries in the two loop singlet anomalous dimension are singular, and the nonsinglet is more singular than at one loop. Taking the inverse Mellin transform of (2.37), using for the nonsinglet the result $\mathcal{M}\left[1/2\,(1/x)\ln^2(1/x)\right] = 1/N^3$, and including them in the Altarelli–Parisi equations (2.3) gives wave equations similar to (2.18), but with additional
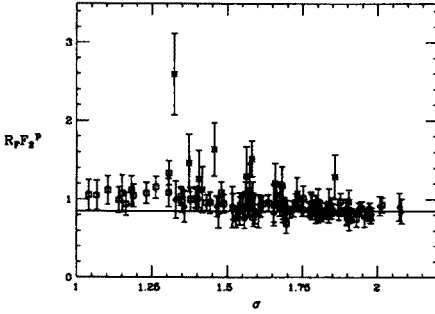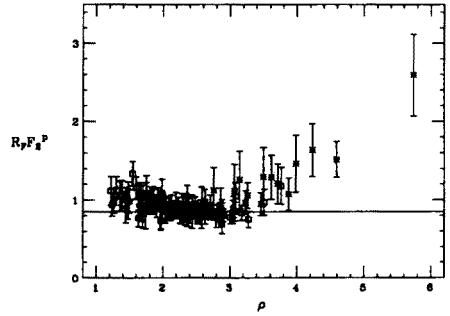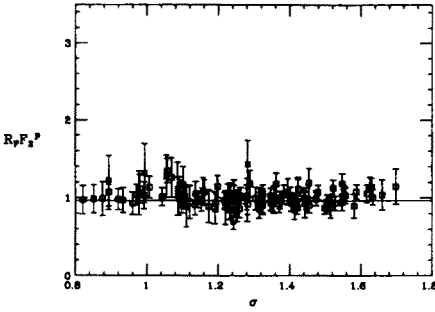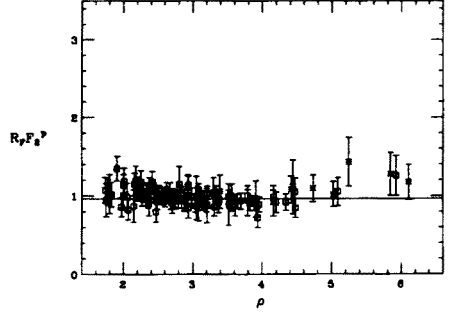
Fig. 3(i)a   Fig. 3 (i)b

Fig. 3(ii)a   Fig. 3(ii)b

Double scaling plots of $R_F F_2^p$ against (a) σ and (b) ρ, (i) with rescaling of the LO asymptotic behaviour Eq. (2.36); (ii) with NLO rescaling Eq. (2.42). The stars are preliminary low Q^2 ($1.5\text{GeV}^2 \leq Q^2 \leq 15\text{GeV}^2$, $3.5 \ 10^{-5} < x < 4.0 \ 10^{-3}$) points from ZEUS [15].

terms of $O(\alpha_s)$ on the right hand side. Linearizing these two loop corrections, the wave equations (2.22) become (suppressing subasymptotic contributions)

$$\left[\frac{\partial^2}{\partial\xi\partial\zeta} + \delta_+ \frac{\partial}{\partial\xi} - \gamma^2\right]G_+(\xi,\zeta) = \epsilon_+ \alpha_s(t_0)e^{-\zeta}G_+(\xi,\zeta),$$

$$\left[\frac{\partial}{\partial\zeta} + \delta_-\right]G_-(\xi,\zeta) = \epsilon_- \alpha_s(t_0)e^{-\zeta}G_+(\xi,\zeta),$$

$$\left[\frac{\partial^2}{\partial\xi\partial\zeta} + \tilde{\delta}\frac{\partial}{\partial\xi} - \tilde{\gamma}^2\right]q(\xi,\zeta) = \tilde{\epsilon}\alpha_s(t_0)e^{-\zeta}\int_0^\xi d\xi' (\xi - \xi')q(\xi',\zeta),$$

$$(2.38)$$

with now, in place of (2.11),

$$\zeta \equiv \ln\left(\frac{\alpha_s(t_0)}{\alpha_s(t)}\right), \tag{2.39}$$

with α_s evaluated to two loops, and

$$\epsilon_+ = \left(\frac{103}{27}n_f + 3\frac{\beta_1}{\beta_0}\right)(\pi\beta_0)^{-1}, \quad \epsilon_- = \frac{26n_f}{3\pi\beta_0}, \quad \tilde{\epsilon} = \frac{16}{3\pi\beta_0}, \tag{2.40}$$

$\beta_1 = 102 - (38/3)n_f$ being the two loop coefficient of the β-function. It is not difficult to show that the three new parameters (2.40) are in fact all independent of the choice of factorization scheme. The eigenvector conditions (2.21), (2.24) and (2.25) are unchanged.

The general solutions of the three equations (2.38) are the same as those of the leading order equations, (2.27), but each with an additional contribution on the right hand side:

$$\epsilon_+ \frac{4\pi}{\beta_0} \int_0^\xi \int_0^\zeta d\xi' d\zeta' I_0\left(2\gamma\sqrt{(\xi-\xi')(\zeta-\zeta')}\right) e^{\delta_+(\zeta'-\zeta)-\zeta'} G_+(\xi',\zeta'),$$

$$\epsilon_- \frac{4\pi}{\beta_0} \int_0^\zeta d\zeta' e^{\delta_-(\zeta'-\zeta)-\zeta'} G_+(\xi',\zeta'),$$

$$\tilde{\epsilon}\frac{4\pi}{\beta_0} \int_0^\xi \int_0^{\xi'} \int_0^\zeta d\xi' d\xi'' d\zeta' I_0\left(2\gamma\sqrt{(\xi-\xi')(\zeta-\zeta')}\right) e^{\delta_+(\zeta'-\zeta)-\zeta'}(\xi'-\xi'')q(\xi'',\zeta'),$$

$$\tag{2.41}$$

respectively. Two loop corrections to the leading asymptotic behaviour are then found by substituting the leading behaviour into (2.41), and evaluating the asymptotic form of the integrals. Clearly the form of the correction will then depend on the form of the leading behaviour. For soft boundary conditions the double scaling behaviours (2.29) are corrected by extra factors

$$G_+(\rho,\sigma) \underset{\sigma\to\infty}{\sim} \mathcal{N}\frac{1}{\sqrt{4\pi\gamma\sigma}} \exp\left\{2\gamma\sigma - \delta_+\left(\frac{\sigma}{\rho}\right)\right\}$$
$$\times \left[1 - \epsilon_+\left(\alpha_s(t_0) - \alpha_s(t)\right)\frac{\rho}{\gamma}\right]\left(1 + O\left(\frac{1}{\sigma}\right)\right),$$

$$q(\rho,\sigma) \underset{\sigma\to\infty}{\sim} \tilde{\mathcal{N}}\frac{1}{\sqrt{4\pi\tilde{\gamma}\sigma}} \exp\left\{2\tilde{\gamma}\sigma - \tilde{\delta}\left(\frac{\sigma}{\rho}\right)\right\}$$
$$\times \left[1 - \tilde{\epsilon}\left(\alpha_s(t_0) - \alpha_s(t)\right)\frac{\rho^3}{\tilde{\gamma}^3}\right]\left(1 + O\left(\frac{1}{\sigma}\right)\right), \tag{2.42}$$

while the double scaling behaviour of F_2, (2.31), is corrected by a similar factor:

$$
F_2^p \underset{\sigma \to \infty}{\sim} \frac{5n_f}{162} \frac{\gamma}{\rho} \mathcal{N} \frac{1}{\sqrt{4\pi\gamma\sigma}} \exp\left\{2\gamma\sigma - \delta_+\left(\frac{\sigma}{\rho}\right)\right\}
$$
$$
\left[1 - \left(\epsilon_+\left(\alpha_s(t_0) - \alpha_s(t)\right) - \frac{9\epsilon_-}{n_f\gamma^2}\alpha_s(t)\right)\frac{\rho}{\gamma}\right]\left(1 + O\left(\frac{1}{\sigma}\right) + O\left(\frac{1}{\rho}\right)\right),
$$

$$
(2.43)
$$

the extra term in the square brackets coming from the subleading contribution to G_-. For hard boundary conditions, the asymptotic behaviours (2.32), (2.33), are corrected by the same factors (2.42), (2.43) respectively, but with ρ/γ, $\rho/\bar\gamma$ replaced by λ, λ respectively: the corrections are thus then x independent.

Two loop corrections will be most important for soft boundary conditions, and thus in the singlet channel, as the leading correction to the double scaling seen at HERA. To see their effect, the data are replotted [in Fig. 3 (ii)] with a new rescaling function,

$$
R_F^{(2)}(\sigma, \rho) = R_F(\sigma, \rho)\left[1 - \alpha_s(t_0)\left(\epsilon_+ - (\epsilon_+ + \frac{9\epsilon_-}{n_f\gamma^2})e^{-\sigma/\rho}\right)\frac{\rho}{\gamma}\right]^{-1}, \quad (2.44)
$$

where R_F is the leading order rescaling (2.36). It can be seen from the plots that the effect of the two loop correction is moderate (except at very low Q^2) but significant in the range of the present data: it increases the starting scale from $Q_0 \simeq 1$ GeV at leading order to around $Q_0 \simeq 1.5$ GeV, reduces the slope of the σ-plot by about 10%, and decreases the rise in the ρ-plot at large ρ and low Q^2. Thus most of the as yet observed scaling violations can be accounted for by the two loop correction, and conversely the effect of this correction can be clearly seen in the data. These results have been confirmed by numerical calculation using the full one and two loop anomalous dimensions [17–19].

The residual rise at large ρ in the low Q^2 data [15], if it turns out to be statistically significant, could be due to many different (and possibly competing) nonperturbative effects: a small rise in the (nonperturbative) boundary condition (even $x^{-0.08}$ has observable effects at $Q_0 = 1.5$GeV), nonperturbative effects due to the opening of the charm threshold, conventional higher twist effects, or even more novel higher twist effects such as parton recombination. However it could also be due to higher loop singularities. It is particularly important to consider these, since it is necessary, in the light of the above discussion about the 'hard pomeron' boundary condition above, to understand why they do not in fact spoil double scaling. Such an undertaking is possible since the precise form of the leading (and

some of the sub-leading) singularities are known: we will now explain in some detail how their effects may be properly included.

3. Perturbation theory at small x

In the previous section we have seen how double scaling appears as a generic feature of the solution to the LO or NLO Altarelli–Parisi equations, in the limit as ρ and σ grow large, i.e. as both $\ln(1/x)$ and $\ln t$ grow, provided the former grows faster than the latter. This prediction thus defines a "double scaling limit" of QCD, intermediate between the Regge limit (small x at fixed t), and the Bjorken limit (large t at fixed x). In the Regge limit perturbation theory fails and we are unable to calculate either the x or the t dependence of parton distributions. In the Bjorken limit perturbation theory holds and predicts the scale dependence of parton distributions, viewed as functions of x, but the x dependence itself depends on an incalculable initial condition. In the double scaling limit, the x and t dependence is universal and only depends on a single overall normalization.

We can thus divide the (ξ, ζ) plane in various regions (see Fig. 2): for low ζ (say, $\zeta < 0$ with a suitable choice the origin of coordinates) perturbation theory breaks down, whereas the Altarelli–Parisi equations hold for positive values of ζ, and become more and more accurate as ζ increases. On the other hand if ξ is also sufficiently large, the anomalous dimensions may be expanded around their leading singularities Eq. (2.15) so the Altarelli–Parisi equations take the small x form Eq. (2.18). However if ξ keeps increasing, perturbation theory eventually breaks down because higher twist corrections, necessary to ensure unitarity, must become get more important. Physically, one can understand this by noting that x can be interpreted as the momentum fraction carried by individual partons. When ξ is very large momentum is shared between an increasingly large number of partons; this corresponds to an increase of the parton density which cannot continue indefinitely and should eventually stop when partons start to recombine with each other [11]. This is expected to happen in the region $x \lesssim x_r \exp(-\alpha_s(t_0)^2/\alpha_s(t)^2)$, where x_r cannot be reliably computed but could be of order 10^{-5} at scales of a few GeV^2 [11]. If instead ζ keeps increasing we eventually get to a region where effects from the lower boundary of perturbative small x evolution are important: these reflect the shape of parton distributions propagated down from large values of x, and thus we get back to Bjorken scaling, with incalculable dependence on x.

Even in the remaining double scaling region, however, double scaling actually holds only with sufficiently soft boundary conditions; besides, we have derived it only in a LO or NLO calculation. The two issues are actually closely related, because of general arguments suggesting that when higher

order corrections are included a hard power-like rise of the singlet parton distributions may result. The underlying logic is the following: the leading logarithmic gluon-induced contributions to the deep-inelastic scattering cross section to all orders in $\alpha_s \log(1/x)$ may be summed up by the solution of an equation satisfied by the gluon distribution (BFKL equation) [14]. This equation is not consistent with the renormalization group, in that it does not sum logs of Q^2 and thus it does not include evolution in Q^2: in fact, it is derived at fixed coupling α_s. It therefore also does not separate leading twist from higher twist contributions. However, it is possible [20] to extract the leading twist contribution to the solution of this equation in Mellin space: as $Q^2 \to \infty$ the solution has the form

$$G(N, Q^2) = \left(\frac{Q^2}{Q_0^2}\right)^{\gamma(N, \alpha_s)} G(N, Q_0^2) \left[1 + O\left(\frac{Q_0^2}{Q^2}\right)\right], \qquad (3.1)$$

the extra term in the square brackets being higher twist. The leading twist piece is now the same as a solution to a (fixed coupling) renormalization group equation Eq. (2.7), with $\gamma(N, \alpha_s)$ identified as the anomalous dimension. This anomalous dimension is determined as the inverse of the function $\chi(x) = 2\psi(1) - \psi(x) - \psi(1 - x)$ (where $\psi(x)$ is the Euler function):

$$\chi\left[\gamma\left(N, \alpha_s\right)\right] = \frac{N}{\bar{\alpha}_s}, \qquad (3.2)$$

where $\bar{\alpha}_s \equiv C_A \alpha_s / \pi$.

The solution to (3.2), the Lipatov anomalous dimension $\gamma_{\rm L}$, thus turns out to be a function of $\bar{\alpha}/N$:

$$\gamma_{\rm L}\left(\frac{\bar{\alpha}_s}{N}\right) = \sum_{k=1}^{N} \gamma_{\rm L}^{(k)} \left(\frac{\bar{\alpha}}{N}\right)^k, \qquad (3.3)$$

with coefficients $\gamma_{\rm L}^{(k)}$ determined uniquely by Eq. (3.2). All the coefficients turn out to be positive, save $\gamma_{\rm L}^{(2)}$, $\gamma_{\rm L}^{(3)}$ and $\gamma_{\rm L}^{(5)}$ which vanish. Now, inverse powers of N are the Mellin transforms of logs of $1/x$:

$$\mathcal{M}\left[\frac{1}{x}\frac{1}{(k-1)!}\ln^{k-1}\frac{1}{x}\right] = \frac{1}{N^k}. \qquad (3.4)$$

One may then argue [20] that, since the leading logs of $1/x$ are summed by the BFKL equation which leads to Eq. (3.1), the coefficients $\gamma_{\rm L}^{(k)}$ in Eq. (3.3) must give the coefficient of the most singular term in expansion in powers of N of the k-th order contribution to the ordinary anomalous dimension

(leading singularity): these are by definition the leading logs in $1/x$ since in x space they correspond to the contributions with the largest number of logs at each perturbative order. This result may actually be proven rigorously by means of suitable factorization theorems [21]: the expansion of the Lipatov anomalous dimension Eq. (3.3) gives the coefficients of the leading singularities to all orders in α_s in the gluon anomalous dimension γ_{gg} (due to the fact that the BFKL equation describes gluon propagation and emission). Notice that these coefficients are factorization scheme independent. This result has several important consequences for our discussion. First, it shows explicitly that the double logarithms, of the form $(1/x)\alpha_s(\alpha_s \ln^2(1/x))^{n-1}$, which one might expect naively to arise in a perturbative expansion of splitting functions [22], are reduced in the gluon sector to single logarithms of the form $(1/x)\alpha_s(\alpha_s \ln(1/x))^{n-1}$, because many of the singularities cancel systematically. This cancellation actually occurs in the whole singlet sector [23] (though not in the nonsinglet [24]). This is in accordance with the calculation of the two loop singularities (2.37) (which in fact exhibit a further accidental cancellation in the singlet sector, which also occurs at three and five loops).

Despite this remarkable cancellation, at higher orders in α_s the singularity in the anomalous dimension is still growing strongly, albeit not quite so fast as one might have naively expected. Thus at small enough x the enhancement due to the extra powers of $\ln(1/x)$ in the corresponding splitting function may offset the suppression due to the extra powers of α_s, so that the inclusion of higher order corrections may be required in order to obtain accurate results. Furthermore, the function $\chi(x)$ has a symmetric minimum at $x = 1/2$, implying that the anomalous dimension $\gamma_L(\bar{\alpha}_s/N)$ has a square-root branch point there (there are also other branch points, and in fact the structure of $\gamma_L(\bar{\alpha}_s/N)$ in the complex plane is quite complicated [25]). The value of its argument such that $\gamma_L(\bar{\alpha}_s/N) = 1/2$ is $4\ln 2$, so the branch point is at

$$\lambda_L(\alpha) \equiv 4\ln 2\left(\frac{C_A \alpha}{\pi}\right). \tag{3.5}$$

This implies that Mellin-space parton distributions also have a singularity at $N = \lambda_L$, and correspondingly, in x space they should increase as $x \to 0$ as x^{λ_L}. This corresponds to a rather strong rise for realistic values of α_s, and would certainly spoil the observed double scaling behaviour if it occurred already at presently attainable values of x. However, it is not clear that this simple argument is consistent with the renormalization group: for example, the result of using a power like behaviour of this form as a boundary condition for LO or NLO perturbative evolution depends on the scale Q_0 at which the boundary is set. Moreover, it is not clear how

this behaviour should be matched to conventional perturbative evolution at larger x, nor indeed in which region of the x-t plane it should become important.

We need thus to keep into account higher order singularities in N in a way consistent with the renormalization group. This can be done [26] by reorganizing the perturbative expansion of the anomalous dimensions, or, equivalently, the splitting functions used in the Altarelli–Parisi equations. To understand how this works, it is convenient to classify the contributions to anomalous dimensions by expanding the anomalous dimension used in renormalization group equations Eq. (2.7) in powers of α_s, and then each order in powers of N (see Fig. 4):

$$\frac{\alpha_s}{2\pi}\gamma(N,\alpha) = \sum_{m=1}^{\infty} \alpha^m \sum_{n=-\infty}^{m} A_n^m N^{-n} = \sum_{m=1}^{\infty} \alpha^m \left(\sum_{n=1}^{m} A_n^m N^{-n} + \bar{\gamma}_N^{(m)} \right),$$

(3.6)

where the numerical coefficients A_n^m are given by Eq. (3.3), and in the last step we have separated out the regular part of the anomalous dimension $\bar{\gamma}_N^{(m)}$. Using Eqs (2.17), (3.4) this is seen to corresponds to expanding the associated splitting functions as

$$\frac{\alpha_s}{2\pi}P(x,t) = \sum_{m=1}^{\infty} \left(\alpha_s(t) \right)^m \left(\frac{1}{x}\sum_{n=1}^{m} A_n^m \frac{\ln^{n-1}\frac{1}{x}}{(n-1)!} + \bar{P}^{(m)}(x) \right),$$

(3.7)

where $\bar{P}^{(m)}(x)$ are regular as $x \to 0$.

Solving renormalization group equations sums all leading logs of the scale which appears in the equation: for instance, upon solving Eq. (2.7), the anomalous dimension gets exponentiated according to Eq. (2.10). At LO only the term with $m = 1$ is included in the anomalous dimension Eq. (3.6), which is thus linear in α_s, so this amounts to summing up all contributions to the deep-inelastic cross sections where each extra power of α_s is accompanied by a power of $\ln Q^2$. In fact, because the LO anomalous dimension has a $1/x$ singularity (which leads to a factor of $\ln(1/x)$ in the cross section upon integration) some logs of $1/x$ are also summed, but $\ln(1/x)$ is not considered leading, in that factors of α_s may or may not be accompanied by $\ln(1/x)$. Thus, if the LO in the expansion Eq. (3.6) of the anomalous dimension is used, all logs of the form

$$\alpha_s^p(\ln Q^2)^q \left(\ln\frac{1}{x} \right)^r$$

(3.8)

with $q = p$, and $0 \le r \le p$ are summed up. At NLO the anomalous dimension includes both linear and quadratic terms in α_s, and thus all logs

Eq. (3.8) with $q \leq p \leq 2q$ are summed (there are at most twice as many powers of α_s as powers of $\ln Q^2$). In standard perturbative computations the NLO solution is however then linearized, by expanding the exponential of the NLO term of the anomalous dimensions in powers of α_s and retaining only the leading nontrivial term (for instance, the NLO asymptotic correction Eq. (2.43) was derived in this way). This means that only terms with $q \leq p \leq q + 1$, i.e. $p = q + 1$ are included at NLO. Furthermore, there is an extra power of $\ln(1/x)$ in the NLO anomalous dimension, so $0 \leq r \leq p$ as at LO. At NNLO yet an extra power of α_s per power of $\ln Q^2$ is allowed, and so forth. We will refer to this as the "large x" expansion.

This is not the only way to organize the perturbative expansion, however. We might instead want for instance to consider $\ln(1/x)$ as leading: this would be appropriate at very small x. Then, all terms where each extra power of α_s is accompanied by a power of $\ln(1/x)$ should be included at LO. These are the leading singularities to all orders in α_s (see Fig. 4b): it is thus convenient to reorganize the expansion of the anomalous dimension as

$$\frac{\alpha_s}{2\pi}\gamma(N,\alpha) = \sum_{m=1}^{\infty} \alpha^{m-1}\left(\sum_{n=2-m}^{\infty} A_n^{n+m-1}\left(\frac{\alpha}{N}\right)^n \right), \qquad (3.9)$$

which corresponds to the splitting function

$$\frac{\alpha_s}{2\pi}P(x,t) = \sum_{m=1}^{\infty} (\alpha_s(t))^{m-1}\left(\frac{1}{x}\sum_{n=1}^{\infty} A_n^{n+m-1}\frac{(\alpha_s(t))^n \ln^{n-1}\frac{1}{x}}{(n-1)!} \right.$$
$$\left. + \sum_{q=0}^{m-2} A_{-q}^{m-q-1}(\alpha_s(t))^{-q}\frac{d^q}{d\xi^q}\delta(1-x) \right). \qquad (3.10)$$

Subsequent orders are still labelled by the index m of the outer sum. The LO is the sum of leading singularities, and sums all logs Eq. (3.8) with $r = p$ and $1 \leq q \leq p$. (terms with $q = 0$ are not included because at least one power of $\ln Q^2$ is produced by integration of the renormalization group equation). In NLO terms with an extra overall factor of α_s are included in the anomalous dimension, so the solution contains terms with $r < p \leq r + q$. Upon linearization, only terms with $p = r + 1$ are kept, while still $1 \leq q \leq p$. There is then complete symmetry between this "small x" expansion and the large x expansion, with the roles of the two logs interchanged.

The small x approximation to LO evolution discussed in the previous section corresponds to only retaining the most singular terms in the LO anomalous dimension: it thus corresponds to taking the intersection of the small x and large x leading order terms, i.e. the pivotal term with $m = n = 1$ in Eq. (3.7) or (3.10), which sums all logs with $p = q = r$. The

symmetry of double scaling reflects this double logarithmic approximation. The terms contributing to the coefficient δ Eq. (2.23) are large x corrections (*i.e.* corresponding to $r < p = q$) and so forth. The fact that double scaling is observed indicates that the HERA data are taken in a region where the two logs start being equally important.

This suggests that in this region the most convenient way of organizing the perturbative expansion is one where the two logs are treated symmetrically at each order ("double leading" expansion). To do this, the anomalous dimensions are expanded as (see Fig. 4c)

$$\frac{\alpha_s}{2\pi}\gamma(N,\alpha) = \sum_{m=1}^{\infty} \alpha^{m-1}\left(\sum_{n=1}^{\infty} A_n^{n+m-1}\left(\frac{\alpha}{N}\right)^n + \alpha\bar{\gamma}_N^{(m)}\right). \qquad (3.11)$$

In LO (*i.e.* when $m = 1$) all terms with $1 \le q \le p$, $0 \le r \le p$, and $1 \le p \le q + r$ are summed. If all cross terms (*i.e.* those containing a product of a contribution to $\bar{\gamma}_N^{(m)}$ times a singular contribution) are linearized then the solution includes all contributions where each extra power of α_s is accompanied by a log of either $1/x$ or Q^2 or both.

In NLO an overall extra power of α_s will be allowed, and so forth.

Of course, a variety of other expansions which interpolate between these could be constructed. The crucial point here however is that all these expansions are consistent with the renormalization group: in each case we may define

$$\frac{\alpha_s}{2\pi}\gamma(N,\alpha) = \sum_{m=1}^{\infty} \alpha^m \gamma_m(N,\alpha); \qquad (3.12)$$

each term in this expansion is then of order α_s compared to the previous one and, in particular, a change of scale at k-th order may be compensated by adjusting the $k + 1$-th order terms. The expansion Eq. (3.12) may thus be treated using the standard machinery used to perform NLO or higher order QCD computations. However, if we choose an expansion which is appropriate at small x (say, the extreme small x one (3.9)), then $\gamma_{m+1}(N,\alpha)$ is of the same order as $\gamma_m(N,\alpha)$ since they both sum up the relevant logarithms, hence the $m + 1$-th order contribution to γ is genuinely of order α_s as compared to the m-th order one. This is not achieved by an *ad hoc* "resummation" of a particular class of contributions, but simply by organizing the perturbative renormalization group in a different, equally consistent way.

There is a very important issue which must still be addressed however: namely, in the large x expansion Eq. (3.6) the series in n which defines the anomalous dimension at each perturbative order (in m) of course converges — in fact, we defined this series by expanding out an expression
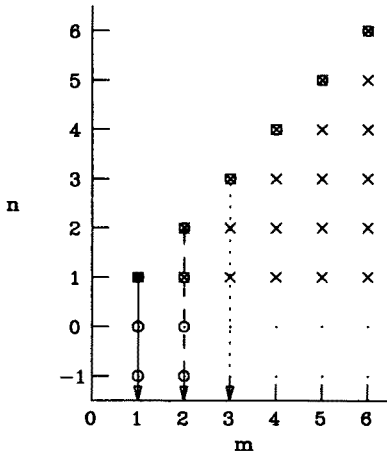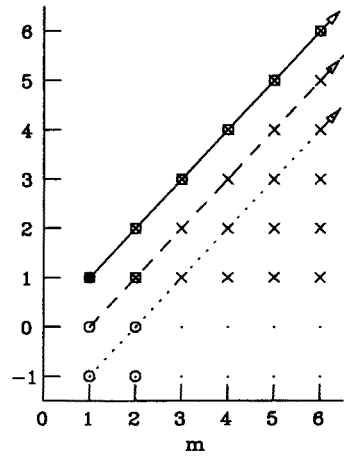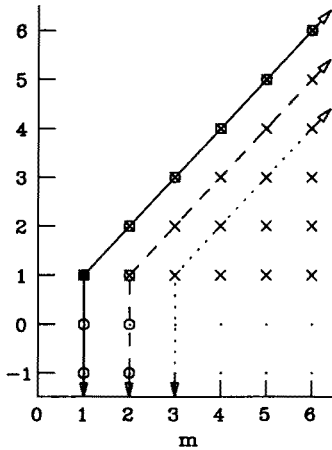
Fig. 4a

Fig. 4b

Fig. 4c

The terms summed in the various expansions of the anomalous dimensions and associated splitting functions: a) the standard (large-x) expansion (3.6), b) the small-x expansion (3.9), and c) the 'double-leading' expansion (3.11). Leading, sub-leading and sub-subleading terms are indicated by the solid, dashed and dotted lines respectively; m denotes the order in α, while n denotes the order in $1/N$. Singular terms are marked as crosses, while terms whose coefficients known at present (for γ_N^{gg}) are marked by circles: the term which leads to double scaling is marked with a filled square.

given as a function of N. The summation of leading singularities Eq. (3.3), however, does not converge for all N: as mentioned earlier, $\gamma_{\mathrm{L}}(N, \alpha_s)$ has a branch-point, implying that the series has a finite radius of convergence, and diverges when $N < 4 \ln \bar{\alpha}$ (the other two branch points on the first sheet [25] being also inside this circle). In fact it is not even (Borel) resummable for $\mathrm{Re}\, N < 4 \ln \bar{\alpha}$, since the integral over the Borel transformed series (which has infinite radius of convergence) diverges at the upper limit. This seems to pose an insurmountable problem for the perturbative approach to small x evolution: the series which defines the leading coefficient in expansion (3.9) of the anomalous dimension, which was supposed to be useful for small N, is actually only well defined when N is sufficiently large. This apparent inconsistency seems to have led many to the conclusion that conventional perturbative evolution breaks down at small x.

The dilemma is resolved by the observation that the physically relevant quantity, namely the splitting function $P_{gg}(x)$, is instead given at leading order in the expansion (3.10) by the series

$$\frac{\alpha_s}{2\pi} P_{gg}(x,t) = \frac{\alpha_s(t)}{x} \sum_{n=1}^{\infty} (A_n^n)^{gg} \frac{\left(\alpha_s(t) \ln \frac{1}{x}\right)^{n-1}}{(n-1)!} \equiv \frac{\lambda_s(t)}{x} B\left(\lambda_s(t) \ln \frac{1}{x}\right),$$
(3.13)

where $\lambda(\alpha_s)$ is given by Eq. (3.5)

$$B(u) = \sum_{n=1}^{\infty} \frac{a_n}{(n-1)!} u^{n-1}$$
(3.14)

$$a_n = (A_n^n)^{gg} \left(4 \ln 2 \, \frac{C_A}{\pi}\right)^{-n}.$$
(3.15)

Because the series $\sum_{n=1}^{\infty} a_n v^n$ has radius of convergence one, the radius of convergence of $B(u)$ is infinite, and thus the series which defines the splitting function Eq. (3.13) converges uniformly on any finite intervals of x and t which exclude $x = 0$: the only reason for the bad behaviour of the series (3.3) is that when transforming to Mellin space one attempts to integrate all the way down to $x = 0$, and this is not possible for singlet distributions because the total number of partons diverges there. If instead the parton distributions are evolved using the Altarelli–Parisi equations (2.3), the splitting functions are only required over the physically accessible region $x > x_{\mathrm{min}}$, and no convergence problems arise. Indeed, because the series for the splitting function is convergent it is only necessary, when working in a physically accessible region to a certain level of accuracy, to retain a finite number of terms. Since then the only singularities in the Lipatov anomalous dimension are poles at the origin, it follows that the cuts which arise to all orders are strictly unphysical.

Similar considerations will apply to the subsequent orders in the expansion of the anomalous dimension Eq. (3.9): if the coefficients in the expansion of the anomalous dimension have a nonvanishing radius of convergence, then the corresponding splitting functions will converge for all $x > 0$. This will be true provided there is no singularity (or accumulation of singularities) in the anomalous dimensions at $N = \infty$ or, equivalently, at $\alpha_s = 0$. This is a standard assumption in perturbative QCD (at least order by order in perturbation theory), so it is natural to conjecture that this will be true for all values of m in Eq. (3.9).

So far we considered the behaviour of the γ_{gg} anomalous dimension. We already know that at LO only γ_{gg} and γ_{gq} display a singularity at $N = 0$. In fact, it may be proven [21] that the leading singularities in γ_{gg} and γ_{gq} are related to all orders, according to the so-called color-charge relation

$$\gamma_{gq} = \frac{C_F}{C_A}\gamma_{gg} + O\left[\alpha_s\left(\frac{\alpha_s}{N}\right)^n\right], \qquad (3.16)$$

while the coefficients of the leading singularities in the quark anomalous dimension γ_{qq}^S and γ_{qg} vanish to all orders. The coefficients of the NL singularities (*i.e.* the NLO in the small x expansion Eq. (3.9)) have been calculated recently [27]: beyond the (nonsingular) lowest order in α_s [given in Eq. (2.15)] they also satisfy a color charge relation, namely

$$\gamma_{qq}^S = \frac{C_F}{C_A}\left(\gamma_{qg} - \frac{4}{3}Tn_f\right) + O\left[\alpha_s\left(\frac{\alpha_s}{N}\right)^n\right]. \qquad (3.17)$$

The series expansions for these NLO anomalous dimensions have the same radius of convergence as that of the Lipatov anomalous dimension, though the form of the branch point singularity is now scheme dependent. The NLO coefficients in the expansion of the gluon anomalous dimensions are still unknown.

All the NLO coefficients, being subleading, are scheme dependent: given their expression in the parton scheme, where coefficient functions satisfy Eq. (2.2), their expressions in any other factorization scheme such as the $\overline{\text{MS}}$ scheme may be determined, along with the corresponding coefficient functions. In fact, the freedom of choosing a factorization scheme turns out [28] to be wider in all expansions other than the standard large x expansion Eq. (3.6): this is due to the fact that the normalization of the gluon beyond order α_s may be modified by change of scheme. It follows that, besides the usual freedom to perform a scheme change which modifies the F_2 coefficient functions, there is now also the possibility of performing a scheme change which does not affect the F_2 coefficient functions, but changes the definition of the gluon distribution (while leaving all LO anomalous dimensions

unaffected, as a scheme change ought to). Such a redefinition, in particular, affects the quark anomalous dimensions, which only start at NLO in the small x expansion: for instance, it can be used [29] to remove a singularity which this anomalous dimension has at NLO in the parton scheme [27] at the Lipatov point $N = \lambda_{\mathrm{L}}(\alpha_s)$. The nonsingular anomalous dimensions thus obtained has been argued [29] to have a more direct physical interpretation in that they correspond to specifying the initial parton distributions in accordance with Regge theory. It is also possible to find a scheme in which the quark anomalous dimensions vanish [30].

This freedom, however, could in principle be completely pinned down by requiring that the evolution equations conserve momentum, which implies the constraints

$$\gamma_{qg}(1, \alpha_s) + \gamma_{gg}(1, \alpha_s) = 0, \qquad \gamma_{qq}^{\mathrm{S}}(1, \alpha_s) + \gamma_{gq}(1, \alpha_s) = 0, \qquad (3.18)$$

in analogy to what happens at large x, where the gluon normalization (which in that case is just a constant) is fixed by the same requirement. These constraints, which are necessary consistency conditions that must be satisfied order by order in the expansion of anomalous dimensions, cannot be satisfied at LO in the small x expansion, where however they do not apply because the gluon decouples from F_2 (which therefore does not evolve). At NLO they fix uniquely the NLO coefficients in the small x expansion Eq. (3.9) of γ_{gg} in terms of the LO coefficients of γ_{gg} and the NLO coefficients of γ_{qg} [19]: substituting Eq. (3.9) in Eq. (3.18) implies immediately

$$(A_n^{n+1})_{qg} + (A_n^{n+1})_{gg} + (A_{n+1}^{n+1})_{gg} = 0. \qquad (3.19)$$

It can be proven [28] that the freedom of choosing the gluon normalization by a change of scheme is in one-to-one correspondence with the values of the NLO coefficients in γ_{gg}: of course, whatever the choice of gluon normalization there exists a set of values of $(A_n^{n+1})_{gg}$ which satisfies Eq. (3.18); more interestingly, whatever the coefficients $(A_n^{n+1})_{gg}$ turn out to be when they will be calculated, there exist a choice of normalization that will enforce Eq. (3.18).[1] At present we have to content ourselves with parametrizing our ignorance of these coefficients by the choice of gluon normalization, and estimating the corresponding uncertainty by varying this normalization.

In the nonsinglet channel the inclusion of the leading singularities is more problematic. The singularity at $N = -1$ in the anomalous dimension

---

[1] Strictly speaking this is only possible when $n \geq 2$ in Eq. (3.18): the scheme-independent $O(\alpha_s)$ terms violate momentum conservation in the small x expansion. This violation is of course asymptotically subleading as $x \to 0$, when this expansion is supposed to hold. Exact momentum conservation can be obtained order by order in the double leading expansion Eq. (3.11).

$\gamma_{qq}^{\rm NS}$ is known [24] to be stronger than that at $N = 0$ of the singlet anomalous dimensions, *i.e.* to be double-logarithmic:

$$\frac{\alpha_s}{2\pi}\gamma_{qq}^{\rm NS}(N,\alpha) = \sum_{m=1}^{\infty} \alpha^m \sum_{n=-\infty}^{2m-1} \tilde{A}_n^m (N+1)^{-n}. \tag{3.20}$$

This is in agreement with the two loop result (2.37), and is indeed the generic expected behaviour of anomalous dimensions in the small N limit. The behaviour of the nonsinglet distributions which is obtained by including these singularities to all orders in the coupling has been computed [24], and leads to a power-like growth: $q_{\rm NS} \sim x^{-\lambda_{\rm KL}}$ with $\lambda_{\rm KL} = \sqrt{8\alpha/3\pi}$, with an almost identical result for charge-conjugation odd distributions. This is in fact very similar to the growth predicted by Regge theory and discussed in the previous section. However, it is obtained with fixed coupling and is not yet consistent with the renormalization group. Since the corresponding factorization theorems are as yet unproven, it is still unclear whether it will be possible to treat this case in the same way as the singlet one, making it consistent with the renormalization group by using the results of Ref. [24] to derive the leading term (namely the coefficients $\tilde{A}_{2m-1}^m$ [31]) in the small x expansion of the nonsinglet splitting function which can then be used in the Altarelli–Parisi equation in the same way as in the singlet sector. What is clear is that the effect of such a procedure would be relatively undramatic, given that the boundary condition is already hard, with $\tilde{\lambda} \sim \lambda_{\rm KL}$. It follows that the singlet contribution to F_2^p still dominates the nonsinglet at small x, and thus for the rest of the discussion we ignore it and consider only the singlet.

We can now finally study the small x behaviour of singlet parton distributions: using evolution equations which include logarithmic effects in $\ln(1/x)$ to all orders we will be able to assess in which region the simple LO double scaling predictions are reproduced. Even though in the HERA region the double leading expansion is probably more appropriate, let us consider the extreme small x expansion Eq. (3.9), both because it is somewhat simpler, and because it allows us to test the stability of double scaling in an extreme case: if the LO correction in this expansion does not spoil scaling, no other corrections will since all the most singular corrections are included, and they all have the same sign.

In the small x expansion at LO the quark anomalous dimensions vanish, hence the quark sector may be neglected, and the gluon evolution equation is found using the splitting function P_{gg} Eq. (3.10) in the Altarelli–Parisi

equation Eq. (2.3):

$$\frac{\partial}{\partial \zeta} G(\xi, \zeta) = \frac{4C_A}{\beta_0} \sum_{n=1}^{\infty} a_n \frac{\lambda_s(\zeta)^{n-1}}{(n-1)!} \int_{-\xi_0}^{\xi} d\xi' (\xi - \xi')^{n-1} G(\xi', \zeta)$$

$$= \gamma^2 \sum_{n=0}^{\infty} a_{n+1} \lambda_s(\zeta)^n \int_{-\xi_0}^{\xi} d\xi_1 \int_{-\xi_0}^{\xi_1} d\xi_2 \dots \int_{-\xi_0}^{\xi_n} d\xi' \, G(\xi', \zeta), \qquad (3.21)$$

where we have used the one loop form of $\alpha_s(t)$ as appropriate for a LO calculation, the coefficients a_n are given by Eq. (3.15), and γ is as in Eq. (2.23). Since we retain only singular contributions to the splitting functions the lower limit $\xi_0 = \ln(1/x_0)$ in the integrations on the right hand side can be consistently set to zero. Differentiating both sides w.r. to ξ this can again be cast in the form of a wave equation

$$\frac{\partial^2}{\partial \xi \partial \zeta} G(\xi, \zeta) = \gamma^2 G(\xi, \zeta)$$

$$+ \gamma^2 \sum_{n=1}^{\infty} a_{n+1} \lambda_s(\zeta)^n \int_0^{\xi} d\xi_1 \int_0^{\xi_1} d\xi_2 \dots \int_0^{\xi_{n-1}} d\xi' \, G(\xi', \zeta)$$

$$= \gamma^2 G(\xi, \zeta) + \gamma^2 \sum_{n=1}^{\infty} a_{n+1} \frac{\lambda_s(\zeta)^n}{(n-1)!} \int_0^{\xi} d\xi' (\xi - \xi')^{n-1} G(\xi', \zeta),$$

$$(3.22)$$

which, when only the first singularity is retained, reduces to Eq. (2.22) with $\delta = 0$ (the term contributing to δ is NLO in the small x expansion).

The quark does not evolve at LO, hence, in order to determine the evolution of F_2, we must go to NLO. This can be done in complete analogy to the derivation of the double scaling evolution equations Eq. (2.22) in the previous section: in fact the derivation there can be viewed as a simplified version of a NLO calculation in the small x scheme, where only the (order α_s) contributions with $n = 2 - m$ to the anomalous dimensions Eq. (3.9) are retained. Thus, we first diagonalize the anomalous dimension matrix, which up to NLO has eigenvalues

$$\lambda_+(\alpha) = \gamma_0^{gg} + \alpha \big(\gamma_{gg}^1 + \frac{\gamma_{gq}^0}{\gamma_{gg}^0} \gamma_{qg}^1 \big) + O(\alpha^2),$$

$$\lambda_-(\alpha) = \alpha \big(\gamma_{qq}^1 - \frac{\gamma_{gq}^0}{\gamma_{gg}^0} \gamma_{qg}^1 \big) + O(\alpha^2). \qquad (3.23)$$

The corresponding eigenvectors are given by

$$Q_+ = \alpha \frac{\gamma_{qg}^1}{\gamma_{gg}^0} G_+ + O(\alpha^2), \qquad Q_- = -\frac{\gamma_{gg}^0}{\gamma_{gq}^0} G_- + O(\alpha). \qquad (3.24)$$

where the anomalous dimension is expanded as in Eq. (3.12). Notice that the large eigenvector condition does not depend on the unknown NLO gluon anomalous dimensions, and the small eigenvector condition has still the simple form of Eq. (2.24), thanks to the color-charge relation Eq. (3.16).

Transforming to x space, the large eigenvector component G_+ is seen to satisfy Eq. (3.22), plus $O(\alpha_s)$ corrections, which lead to the δ term of Eq. (2.22) and to $O(\alpha_s)$ corrections to the coefficients a_n in Eq. (3.22). The quark equation is then found differentiating the large eigenvector condition with respect to ζ and using the equation for G_+, with the result

$$\frac{\partial}{\partial \zeta} Q_+(\xi, \zeta) = \frac{n_f}{9} \gamma^2 \Big[G_+(\xi, \zeta) + \sum_{n=1}^{\infty} \tilde{a}_n \lambda_s(\zeta)^n \int_0^\xi d\xi' \frac{(\xi - \xi')^{n-1}}{(n-1)!} G_+(\xi', \zeta) \Big],$$

$$(3.25)$$

where the coefficients $\tilde{a}_n$ re given by

$$\tilde{a}_n = (A_n^n)^{qg} (4 \ln 2 \, C_A/\pi)^{-n}. \qquad (3.26)$$

Although these evolution equations cannot be solved in closed form, a solution can be developed perturbatively in the usual way by noting, as we did at two loops (2.41), that the solution (2.27) for G_+ now acquires an extra set of terms

$$\gamma^2 \sum_{n=1}^{\infty} \frac{a_{n+1}}{(n-1)!} \int_0^\xi \int_0^{\xi'} \int_0^\zeta d\xi' d\xi'' d\zeta'$$

$$\times I_0 \big(2\gamma \sqrt{(\xi - \xi')(\zeta - \zeta')} \big) \lambda_s(\zeta')^n (\xi' - \xi'')^{n-1} G_+(\xi'', \zeta') \quad (3.27)$$

on the right hand side. The full solution may then be developed iteratively by substitution of lower order solutions, leading to a power series expansion in terms of integrals over Bessel functions, which can then be evaluated numerically. A similar expansion may be obtained in Mellin space by integrating explicitly the singlet renormalization group equation (2.7), and then performing the inverse transform (2.12) by choosing a contour which encircles the singularities in the region $|N| < \lambda_L$ [32].

For present purposes, however, we derive some simple analytic estimates which demonstrate analytically the nature of the solution [26]. The

key observation which dictates the structure of the solution is the simple property of Bessel function $z^n I_{n-1}(z) = d\big(z^n I_n(z)\big)/dz$. This implies that if we determine iteratively the solution to the full gluon equation Eq. (3.22) by then Eq. (3.22) with the perturbation Eq. (3.27) takes at lowest order of the iteration the approximate form

$$\frac{\partial^2}{\partial\xi\partial\zeta}G(\xi,\zeta) \simeq \gamma^2 G(\xi,\zeta) + \frac{2}{\beta_0}\sum_{n=1}^{\infty} a_{n+1}\left(\frac{\rho\lambda_s(\zeta)}{\gamma}\right)^n I_n(2\gamma\sigma). \qquad (3.28)$$

But all Bessel functions in this expansion depend only on the scaling variable σ and have the same asymptotic behaviour Eq. (2.26) (which only sets in more slowly for higher values of n). Since $\rho^n I_n(2\gamma\sigma)$ is bounded above by $\xi^n I_0(2\gamma\sigma)$ it follows that double scaling will always set in asymptotically provided the series $\sum_{n=1}^{\infty} a_{n+1}\big(\xi\lambda_s(\zeta)/\gamma\big)^n$ converges uniformly, that is provided $\xi < (\gamma/\lambda_s(\zeta))$. This is much wider region than that in which double scaling would hold if we were to impose a "hard" boundary condition of the form $x^{-\lambda}$ at $\zeta = 0$, namely $\xi < (\gamma/\lambda_s(0))$, because $\lambda(\zeta)$ rapidly falls as ζ increases. Such a boundary condition is precisely that which the simple argument based on the location of the singularity in γ_L Eq. (3.3) would suggest.

To see more clearly how such a power-like behaviour could arise, consider the evolution equation Eq. (2.22) in the region of extremely small x, where the higher order terms give the dominant contribution to the series on the right hand side. Because the series (3.14) has unit radius of convergence it follows that $\lim_{n\to\infty} (a_{n+1}/a_n) \to 1$. But then, setting $a_{n+1} \approx a_n$ in the sum in Eq. (3.22), shifting the summation index by one unit, and using Eq. (3.21), we have

$$\frac{2}{\beta_0}\sum_{n=1}^{\infty} a_{n+1}\lambda_s(\zeta)^n \int_0^\xi d\xi_1 \int_0^{\xi_1} d\xi_2 \cdots \int_0^{\xi_{n-1}} d\xi' \, G(\xi',\zeta) = \lambda_s(\zeta)\frac{\partial G}{\partial\zeta}. \qquad (3.29)$$

Hence, asymptotically as $\xi \to \infty$ the evolution equation (3.22) becomes simply

$$\frac{\partial^2}{\partial\xi\partial\zeta}G(\xi,\zeta) - \lambda_s(\zeta)\frac{\partial G}{\partial\zeta} = \gamma^2 G(\xi,\zeta), \qquad (3.30)$$

and the summation of all leading singularities leads to a damping term in the wave equation.

We can now see how the $x^{-\lambda}$ behaviour obtains in the fixed-coupling limit: if the coupling is frozen, then λ is just a constant, and the solution

to Eq. (3.30) is given in terms of the (double scaling) solution $G_0(\xi, \zeta)$ to the original wave equation

$$G(\xi, \zeta) = e^{\lambda \xi} G_0(\xi, \zeta) = x^{-\lambda} G_0(\xi, \zeta). \tag{3.31}$$

However there is no reason to fix the coupling. A solution with running coupling can be derived by the saddle point method [26], and turns out to give the double scaling behaviour (up to a small correction)

$$G(\xi, \zeta) \sim \frac{1}{\sqrt{\sigma}} \exp\left(2\gamma\sigma + (\lambda(0) - \lambda(\zeta)\rho^2\right), \tag{3.32}$$

throughout the region $\xi \ll \frac{\gamma^2}{(\lambda(0) - \lambda)^2} \zeta^3$. When ξ is extremely large the power-like behaviour

$$G(\xi, \zeta) \sim \frac{1}{\xi}\left(\frac{\xi}{\lambda(0) - \lambda(\zeta)}\right)^{\gamma^2/\lambda_0} \exp\left(\xi\lambda(0) + \frac{\gamma^2 \zeta}{\lambda(0)}\right) \tag{3.33}$$

is found instead. This has essentially the "hard" form of Eq. (3.31), with the large value of λ evaluated at the initial scale, but it is confined to the extremely small region $\xi \gg (\gamma^2/\lambda(0)^2)e^\zeta$. In fact this is still an overestimate of the region where the power-like behaviour should arise: approximating the evolution equation with Eq. (3.30) ignores the fact that the asymptotic behaviour of the coefficients a_n only sets in rather slowly, and, by the time it does, $a_n \ll a_1$. If this effect is taken into account the power-like region is further reduced.

Even though of course all these estimates are only based on the asymptotic form of the coefficients a_n, and thus they will not accurately reproduce subasymptotic and specifically subexponential corrections (such as the factor of $1/\sqrt{\sigma}$ in Eq. (3.32)) and normalizations, they do correctly estimate the leading behaviours and their region of validity, as confirmed by a full numerical analysis. This allows us to conclude that, even though a power like behaviour of the form Eq. (3.31) is generated very close to the $\zeta = 0$ boundary, it very rapidly dies off due to the running of the coupling, rather than spreading in the whole $\xi > (\gamma/\lambda(0))$ region as it would do if it were input to LO evolution. Furthermore, any hard power-like boundary condition will be hidden by this rise (unless it is even stronger than it): since $\lambda(0)$ is very large for reasonable choices of the starting scale (for instance $\lambda(4 \text{ GeV}^2) \approx 0.8$) this means than scaling will appear very rapidly for all boundary conditions, except unreasonably hard ones such as $x^{-\lambda}$ with $\lambda \lesssim -0.8$ at $Q^2 = 4 \text{ GeV}^2$.

Turning finally to the quark equation, we may take advantage of the fact that, throughout most of the (ξ, ζ) plane, $\bar{G}(\xi, \zeta) \simeq \mathcal{N} I_0(2\gamma\sigma)$: substituting this in the r.h.s. of Eq. (3.25), and neglecting the small eigenvector we get

$$\frac{\partial}{\partial \zeta} Q(\xi, \zeta) \simeq \frac{n_f}{9} \gamma^2 \mathcal{N} \sum_{n=0}^{\infty} \tilde{a}_n \left(\frac{\rho \lambda_s(\zeta)}{\gamma} \right)^n I_n(2\gamma\sigma). \qquad (3.34)$$

It follows that when $\xi \ll (\gamma^2/\lambda_0^2)\zeta e^{2\zeta}$, $Q(\xi, \zeta)$ still scales; for $\xi \gg (\gamma^2/\lambda_0^2)\zeta e^{2\zeta}$, if we set $\tilde{a}_n = 1$ for all n, (3.34) may be simplified yet further by using the relation

$$\sum_{-\infty}^{\infty} t^n I_n(z) = \exp\left(\tfrac{1}{2}z(t + t^{-1})\right),$$

to give

$$Q(\xi, \zeta) \sim \mathcal{N}\zeta \exp\left(\xi\lambda_0 + \frac{\gamma^2\zeta}{\lambda_0} \right). \qquad (3.35)$$

The quark anomalous dimensions may thus still produce a growth of F_2 close to the boundary, which however again does not spread in the (ξ, ζ) plane due to the running of the coupling. The actual size of the region where this growth will appear depends on the size of the coefficients $\tilde{a}_n$. This, in turn, strongly depend on the choice of gluon normalization which, as discussed above, can be modified by changing factorization scheme. The latter could be fixed using momentum conservation [28] if we knew the NLO small x gluon anomalous dimensions. Since we do not, the effect of the quark anomalous dimensions on the relative normalization of quark and gluon distributions (and thus on the relative size of F_2 and F_L) turns out to be strongly dependent on the choice of factorization scheme: while the $\tilde{a}_n$ corresponding to the $\overline{\text{MS}}$ or DIS schemes [27] give large (but very different) effects, the Q_0-schemes [29], being less singular, lead to substantially smaller effects, while in some schemes [30] there is almost no effect at all [33] except at very small x very close to the boundary.

These analytic results are all closely supported by detailed numerical investigations, either retaining only the singular terms in the anomalous dimensions [26, 32], or including also the full one and two loop contributions in the double leading scheme [19, 33] or using some other procedure [25]. When care is taken to consistently factorize residual nonperturbative effects into the boundary conditions, all these analyses reach essentially the same conclusions. The leading (Lipatov) singularities have only a negligible effect throughout the measured region, so small that there is still no empirical evidence for them, not only because they don't affect the shape of F_2 in most

of the (ξ, ζ) plane, but also because the coefficients a_n are so small. The subleading quark anomalous dimensions have little effect of the shape of F_2, explaining the success of conventional perturbation theory and double scaling [26, 19]. However they can have a substantial effect on the scale Q_0 at which soft initial distributions are input, raising it to around 2GeV in the $\overline{\text{MS}}$ scheme [26, 25], and on the relative normalization of quark and gluon, and thus on the size of F_L as deduced from the measured F_2 [26, 32]. However these latter effects are strongly factorization scheme dependent, both in sign and magnitude [33], and this scheme dependence could only be resolved theoretically by a complete calculation of the subleading singularities of the gluon anomalous dimension, or phenomenologically by a direct measurement of F_L.

4. Outlook

The observation of double asymptotic scaling at HERA is a striking success of a perturbative QCD prediction made now more than 20 years ago [6]. The explanation of the effect turns out to be somewhat subtler than it might seem at first: even though it is a direct consequence of the singularity structure of the leading-order perturbative evolution equations, its stability in a region where higher order effects might naively be expected to be important is due to the all order cancellation of double logarithmic singularities, the unexpected smallness of the coefficients of the remaining single logarithmic singularities, and then finally the reduction in their impact when the effect of the running coupling is included in a way consistent with the renormalization group.

This result has interesting ramifications from both the phenomenological and the theoretical point of view, which have just started to be explored. The success of NLO perturbation theory in the HERA region strongly suggests that this may be an ideal place to perform precision tests of perturbative QCD, since most of the poorly known low-energy effects (such as higher twists) are either absent or negligible here. For example, α_s can be measured from existing HERA data [19] with a precision comparable to that of all other existing deep-inelastic experiments combined. As more data in the very small x, not-so-small Q^2 region become available, they may shed light on subtle issues of scheme dependence, and provide information on the behaviour of parton distribution which are input to perturbative evolution. On the more theoretical side, it would be desirable to derive perturbation theory at small x in a way which treats the two large scales symmetrically from the outset, rather than solving renormalization group equations with respect to one scale while including the summation of the other scale in the anomalous dimensions, as we did here. This, besides being interesting for

its own sake, could shed light on the dynamics of perturbative QCD in the high energy regime.

S.F. thanks Maciej Nowak for organizing an interesting interdisciplinary school, and K. Golec-Biernat and L. Lipatov for stimulating discussions during the school.

REFERENCES

[1] See G. Altarelli, *Phys. Rep.* **81**, 1 (1981) for a general introduction and notation.

[2] See P. Landshoff, hep-ph/9410250 and references therein.

[3] ZEUS Collab., *Phys. Lett.* **B316**, 412 (1993); H1 Collab., *Nucl. Phys.* **B407**, 515 (1993).

[4] ZEUS Collaboration, *Zeit. Phys.* **C65**, 379 (1995); H1 Collaboration, *Nucl. Phys.* **B439**, 471 (1995).

[5] F. Eisele, Summary talk at the Europhysics Conference, Brussels, August 1995.

[6] A. De Rújula *et al.*, *Phys. Rev.* **D10**, 1649 (1974).

[7] R.D. Ball, S. Forte, *Phys. Lett.* **B335**, 77 (1994).

[8] R.D. Ball, S. Forte, *Phys. Lett.* **B336**, 77 (1994).

[9] G. Wolf, for the ZEUS Collab., talk at the International Workshop on Deep Inelastic Scattering. Eilat, Israel, February 1994; K. Müller, for the H1 Collab., talk at the 29th Rencontre de Moriond, March 1994.

[10] G. Altarelli, G. Parisi, *Nucl. Phys.* **B126**, 298 (1977).

[11] L.V. Gribov, E.M. Levin, M.G. Ryskin, *Phys. Rep.* **100**, 1 (1983).

[12] NMC Collaboration, M. Arneodo *et al.*, *Phys. Rev.* **D50**, 1 (1994).

[13] J. Kwieciński *et al.*, *Phys. Rev.* **D42**, 3645 (1990).

[14] L.N. Lipatov, *Sov. Jour. Nucl. Phys.* **23**, 338 (1976); V.S. Fadin, E.A. Kuraev, L.N. Lipatov, *Phys. Lett.* **60B**, 50 (1975); *Zh.E.T.F.* **71**, 840 (1976); **72**, 377 (1977); *Sov. Phys. JETP* **44**, 443 (1976); **45**, 199 (1977); Y.Y. Balitski, L.N. Lipatov, *Sov. Jour. Nucl. Phys.* **28**, 822 (1978). For a pedagogical introduction see V. Del Duca, preprint DESY 95-023, hep-ph/9503226.

[15] ZEUS Collaboration, DESY 95-193, hep-ex/9510009.

[16] E.G. Floratos *et al.*, *Nucl. Phys.* **B192**, 417 (1981).

[17] R.D. Ball, S. Forte, CERN-TH.7421/94, hep-ph/9409373, to be published in *The Heart of the Matter*, VIth Rencontre de Blois, June 1994, Editions Frontières.

[18] R.D. Ball, S. Forte, in the Proceedings of *QCD94*, Montpellier, July 1994 (*Nucl. Phys. B (Proc. Suppl.)* **39B,C**, 25 (1995)).

[19] R.D. Ball, S. Forte, *Phys. Lett.* **B358**, 365 (1995).

[20] T. Jaroszewicz, *Phys. Lett.* **116B**, 291 (1982).

[21] S. Catani, F. Fiorani, G. Marchesini, *Phys. Lett.* **B234**, 339 (1990); *Nucl. Phys.* **B336**, 18 (1990); S. Catani, F. Fiorani, G. Marchesini,

G. Oriani, *Nucl. Phys.* **B361**, 645 (1991); S. Catani, M. Ciafaloni, F. Haut-mann, *Phys. Lett.* **B242**, 97 (1990); *Nucl. Phys.* **B366**, 135 (1991).

[22] D. Gross in the proceedings of the XVII Intern. Conf. on High Energy Physics, London, 1974 (published by SRC, Rutherford Lab.) and lectures given at Les Houches, Session XXVIII, 1975, published in *Methods in Field Theory*, eds R. Balian and J. Zinn-Justin, North-Holland, 1976.

[23] S. Catani, M. Ciafaloni, F. Hautmann, *Phys. Lett.* **B307**, 147 (1993).

[24] R. Kirschner, L. Lipatov, *Nucl. Phys.* **B213**, 122 (1983).

[25] R.K. Ellis, F. Hautmann, B.R. Webber, *Phys. Lett.* **B348**, 582 (1995).

[26] R.D. Ball, S. Forte, *Phys. Lett.* **B351**, 313 (1995).

[27] S. Catani, F. Hautmann, *Phys. Lett.* **B315**, 157 (1993); *Nucl. Phys.* **B427**, 475 (1994).

[28] R.D. Ball, S. Forte, *Phys. Lett.* **B359**, 362 (1995).

[29] M. Ciafaloni, *Phys. Lett.* **B356**, 74 (1995).

[30] S. Catani, DFF 226/5/95, hep-ph/9506357.

[31] B.I. Ermolaev, S.I. Manayenkov, M.G. Ryskin, DESY 95-017, hep-ph/9502262.

[32] J.R. Forshaw, R.G. Roberts, R.S. Thorne, *Phys. Lett.* **B356**, 79 (1995).

[33] S. Forte, R.D. Ball, CERN-TH/95-184, hep-ph/9507211, to be published in the proceedings of the Workshop on Deep Inelastic Scattering and QCD (DIS 95), Paris, April 1995.