Vol. 26(1995) *ACTA PHYSICA POLONICA B* No 12

WHAT IS BETWEEN FERMI–DIRAC AND BOSE–EINSTEIN STATISTICS?*

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(Received December 27, 1995)

We overview the properties of a quantum gas of particles with the intermediate statistics defined by Haldane. Although this statistics has no direct connection to the symmetry of the multiparticle wave function, the statistical distribution function interpolates continuously between the Fermi–Dirac and the Bose–Einstein limits. We present an explicit solution of the transcendental equation for the distribution function in a general case, as well as determine the thermodynamic properties in both low- and high-temperature limits.

PACS numbers: 05.30. -d, 71.10. +x

1. Introduction

Statistical properties of quantum many-body systems are determined by calculating thermal averages with a proper distribution function, which represents the mean number of particles occupying each of the single-particle states. For the three dimensional systems of noninteracting particles this distribution function is given by either the Fermi–Dirac (FD) or the Bose–Einstein (BE) function, depending on whether the spin of the particles is half-integer or integer, respectively [1].

In the systems with either strong interactions or of low space dimensionality ($d < 3$) the situation changes drastically. It turns out that the population of single particle quantum level can be given by a function which

* Presented at the XXXV Cracow School of Theoretical Physics, Zakopane, Poland, June 4–14, 1995.

is of neither FD, nor BE form. The reason for that may be twofold. First of all, in the space of two dimensions the symmetry of the many body wave function is not necessarily either even or odd [2]. When two identical particles are exchanged the wave function acquires an arbitrary phase factor $e^{i\alpha}$, where α is so called statistical angle. In other words, the proper symmetry group in this case is the braid group for which the irreducible representations are $\chi(\alpha) = e^{i\alpha}$. The particles obeying the fractional statistics are called *anyons*. On the other hand, one can imagine that the interactions between particles are sufficiently strong, *e.g.* singular, that number of available effective single-particle occupancies of each of the levels depends on the number of particles already present in that state. This property can also lead to the departure from FD and BE statistics in arbitrary spatial dimensions [3].

There is no universal theory of statistical properties of the systems described above. However, there are some exactly solvable models of interacting particles for which the momentum distribution functions are modified [4]. One of the best known cases is the one dimensional Calogero–Sutherland model for which the statistical properties of quasiparticles change from BE to FD limits as the interaction increases [5].

Having in mind those exactly solvable examples and the fact that, in general, the statistics of elementary excitations (quasiparticles) in correlated systems can be arbitrary, we believe it is valuable to study the thermal properties of particles with nonstandard distribution functions. In particular, in this paper we address the question how thermodynamic properties evolve when the statistics changes from BE to FD limits.

Our paper is organized as follows. In the first Section we introduce three interpolation schemes between BE and FD statistics. Then, in Section 2 and 3 we discuss the high and the low temperature properties of such systems. In Section 5 we present general phase diagrams for those three systems of particles.

2. Routes of interpolating between Fermi–Dirac and Bose–Einstein distribution functions

For noninteracting bosons the momentum distribution function is

$$\bar{n}(\varepsilon_i) = \frac{1}{e^{\beta(\varepsilon_i-\mu)} - 1}, \tag{2.1}$$

whereas for ideal fermions we have

$$\bar{n}(\varepsilon_i) = \frac{1}{e^{\beta(\varepsilon_i-\mu)} + 1}. \tag{2.2}$$

In these functions, and hereinafter, ε_i denotes the energy of a single particle state i, $\beta = 1/k_B T$, where T is the temperature, and μ is the chemical potential. Probably, the simplest modification of FD and BE functions can be carried out phenomenologically in the following manner (route 1)

$$\bar{n}_\gamma(\varepsilon_i) = \frac{1}{e^{\beta(\varepsilon_i - \mu)} + \gamma}, \tag{2.3}$$

where γ is an arbitrary real number in the range $(-1, 1)$. The limit -1 corresponds to BE statistics and $\gamma = 1$ to FD one. For other allowed values of γ we have an intermediate statistics. In particular, the limit $\gamma = 0$ corresponds to the classical, *i.e.* Boltzman, distribution function. Particles with arbitrary γ statistics are called γ-*ons*. We show later that this ad hoc proposal represents a high-temperature approximation to a general distribution function.

As far as we know, it is impossible to derive the distribution function (2.3) exactly [6, 7]. However, starting from the modified Hilbert–Fock space with creation and annihilation operators obeying so called q-mutator algebra, *i.e.* $a_i a_j^+ - e^{i\alpha} a_j^+ a_i = 0$ for $i \neq j$, there is a way to recover the function (2.3) approximately where $\gamma = \cos\alpha$ [7]. It is also interesting to observe that the distribution function (2.3) can be obtained if we suppose that the number of ways of putting N_i particles into G_i states is given by

$$w_i = \frac{1}{(N_i)!}\left(\frac{(G_i)!}{(G_i - \gamma N_i)!}\right)^{1/\gamma}. \tag{2.4}$$

Then, using the standard combinatoric approach to quantum statistics [1] we recover Eq. (2.3).

The thermodynamic potential $\Xi_\gamma(z, V, T)$ for γ-ons is given by

$$\Xi_\gamma(z, V, T) = \prod_i (1 + \gamma z e^{\beta\varepsilon_i})^{1/\gamma}, \tag{2.5}$$

where $z = e^{\beta\mu}$ is the fugacity. In Eqs (2.4), (2.5) we supposed that $\gamma \neq 0$. The case of $\gamma = 0$ must be treated separately. The equation of state for such gas of ideal γ-ons is

$$\frac{pV}{k_B T} = \ln \Xi_\gamma(z, V, T) = \frac{1}{\gamma}\sum_i \ln(1 + \gamma z e^{\beta\varepsilon_i}), \tag{2.6}$$

where p is the pressure and V is the volume of the system. In order to eliminate z from this formula we need another equation which determines the total number of particles N

$$N = z\frac{\partial}{\partial z}\ln \Xi_\gamma(z, V, T) = \sum_i \frac{1}{\frac{1}{z}e^{\beta\varepsilon_i} + \gamma} = \sum_i \bar{n}_\gamma(\varepsilon_i). \tag{2.7}$$

In deriving the last equation we could also convince ourselves that the thermodynamic potential (2.5) is really consistent with the distribution function (2.3). Eqs (2.6) and (2.7) describe the whole thermodynamics of γ-ons. We are going to explore them in the next sections.

On the other hand, one can interpolate between FD and BE limits using the state counting arguments. This provides us with the second possibility (route 2). Namely, let the number of single particle state D_i in the i-th quantum level depends on the number of particles in this group according to the formula [3, 8]

$$D_i(N_j) = G_i - g(N_i - 1)\,. \tag{2.8}$$

Also, suppose that the total number of states of N-body system for a given configuration $\{N_i\}$ is [3]

$$W(\{N_i\}) = \prod_i \frac{(G_i - g(N_i - 1) - 1 + N_i)!}{(N_i)!(G_i - g(N_i - 1) - 1)!}\,, \tag{2.9}$$

then using the Boltzman equation for the entropy $S = k_B \ln \Gamma(\varepsilon)$, where $\Gamma(\varepsilon) = \sum_{\{N_i\}} W(\{N_i\})$ is the total number of states, we can derive the momentum distribution function $\bar{n}_g(\varepsilon_i)$ [8]. In Eqs (2.8) and (2.9) G_i is the bare number of single particle states (when no particles are present). g is so called statistical interaction, or statistical parameter, which in fact parameterize the distribution function. We note that for $g = 0$, the number of states (2.9) is the same as for the ideal bosons. The case $g = 1$ corresponds to free fermions. However, for $0 \leq g \leq 1$, we can again obtain an arbitrary distribution function. Physical interpretation of g is follows. Ideal bosons do not obey exclusion principle, so the number of states in the i-th energy group would not change if we put another particle into this system and, therefore, $D_i(N_j) = G_i$. Fermions obey Pauli exclusion principle, which states that at most only one fermion can be present in a given quantum state. Then, if we put another fermion into the i-th group the number of accessible state must decrease by one. Hence, $D_i(N_j) = G_i - N_i + 1$ and $g = 1$. For arbitrary g between zero and one we have a partial exclusion principle and equation (2.9) directly leads to fractional exclusion statistics in arbitrary spatial dimensions. Explicitly, the distribution function for any g is given by the following transcendental equation [8]

$$\bar{n}_g(\varepsilon_i)e^{\beta(\varepsilon_i - \mu)} = (1 + (1 - g)\bar{n}_g(\varepsilon_i))^{1-g}(1 - g\bar{n}_g(\varepsilon_i))^g\,. \tag{2.10}$$

Of course, taking $g = 0$ or $g = 1$ in this equation we recover distribution functions (2.1) and (2.2). Particles obeying statistics with arbitrary g are called *g-ons*. From the last equation we find that the distribution function can be written in the following form

$$\bar{n}_g(\varepsilon_i) = \frac{1}{w_g(\varepsilon_i) + g}, \tag{2.11}$$

where $w_g(\varepsilon_i)$ must satisfy the equation

$$e^{\beta(\varepsilon_i-\mu)} = (w_g(\varepsilon_i) + 1)^{1-g}(w_g(\varepsilon_i))^g. \tag{2.12}$$

As in the standard statistical mechanics [1] we can find the thermodynamic potential for g-ons $\Xi_g(z, V, T)$ and, hence, the equation of state is

$$\frac{pV}{k_B T} = -\sum_i \ln\left(\frac{1 + (1-g)\bar{n}_g(\varepsilon_i)}{1 - g\bar{n}_g(\varepsilon_i)}\right). \tag{2.13}$$

The equation for the total number of g-ons is

$$N = \sum_i \frac{1}{w_g(\varepsilon_i) + g}. \tag{2.14}$$

The function $\bar{n}_g(\varepsilon_i)$ and $w_g(\varepsilon_i)$ are given by Eqs (2.10) and (2.12), respectively. Again, these functions (2.13) and (2.14) with (2.10) and (2.12) determine the whole thermodynamics of g-ons.

Finally, there is another way of changing FD and BE statistics (route 3). Namely, suppose that the number of particles in each single particle quantum state can be equal to p, where p is a natural number, *i.e.*

$$N_i = 0, 1, 2, \ldots p. \tag{2.15}$$

The case $p = 1$ corresponds to free fermions and $p = \infty$ gives BE statistics. The thermodynamic potential for a given p is [9]

$$\Xi_p(z, V, T) = \prod_i \sum_{n_i=0}^{p} e^{-\beta(\varepsilon_i-\mu)n_i} = \prod_i \frac{1 - e^{(p+1)\beta(\varepsilon_i-\mu)}}{1 - e^{-\beta(\varepsilon_i-\mu)}}. \tag{2.16}$$

To derive this we used the method of grand canonical ensemble. With the aid of the property $\bar{n}_p(\varepsilon_i) = z\partial/\partial z \ln \Xi_p$ we can obtain the distribution function

$$\bar{n}_p(\varepsilon_i) = \frac{1}{e^{\beta(\varepsilon_i-\mu)} - 1} - \frac{p+1}{e^{(p+1)\beta(\varepsilon_i-\mu)} - 1}. \tag{2.17}$$

The equation of state for a gas of so called *p-ons* is

$$\frac{pV}{k_B T} = \sum_i \ln \frac{1 - z^{p+1}e^{-\beta\varepsilon_i(p+1)}}{1 - ze^{-\beta\ \varepsilon_i}}, \tag{2.18}$$

and equation for their number is

$$N=\sum_i\left(\frac{1}{e^{\beta(\varepsilon_i-\mu)}-1}-\frac{p+1}{e^{(p+1)\beta(\varepsilon_i-\mu)}-1}\right). \tag{2.19}$$

Let us summarize this Section. We introduced three schemes of modified quantum statistics and, as a result, we obtained three families of new nonstandard statistical distribution functions which are parameterized by some numbers (γ, g, p). In these cases the statistics interpolate between FD and BE limits as the statistical parameters are changed. In the next sections we are going to discuss thermal properties of such systems as a function of these parameters.

3. High temperature properties

Let us consider first the ideal γ-ons with parabolic dispersion relation $\varepsilon_k = \frac{\hbar^2k^2}{2m}$ moving in the d-dimensional space. Then, we can expand the functions (2.6) and (2.7) in powers of γz, and performing the simple Gaussian integrals we find that

$$\frac{p}{k_BT}=\frac{1}{\lambda^d}\frac{1}{\gamma}f_{\frac{d}{2}+1}(\gamma,z), \tag{3.1}$$

$$n=\frac{N}{V}=\frac{1}{\lambda^d}\frac{1}{\gamma}f_{\frac{d}{2}}(\gamma,z), \tag{3.2}$$

where the function $f_k(\gamma,z)$ is defined as

$$f_k(\gamma,z)=\sum_{i=1}^{\infty}(-1)^{i+1}\frac{(\gamma z)^i}{i^k}. \tag{3.3}$$

$\lambda=\sqrt{2\pi\hbar^2/mk_BT}$ is the thermal wavelength, and n is the particle density. Of course, one can simply check that these equations reduce to those for ideal fermions or bosons as $\gamma=1$ or $\gamma=-1$, respectively.

Now, considering the high temperature limit, *i.e.* a limit for which $n\lambda^d\gamma \ll 1$, we can solve Eq. (2.3) finding the fugacity

$$z\simeq n\lambda^d+\frac{\gamma(n\lambda^d)^2}{2^{d/2}}. \tag{3.4}$$

Then, substituting this into Eq. (3.1) we obtain the high-temperature (low-density) limit of the equation of state

$$\frac{p}{k_BT}=n(1+A_2^{\gamma}n+\cdots), \tag{3.5}$$

where A_2^γ is a second virial coefficient

$$A_2^\gamma \equiv \gamma \frac{\lambda^d}{2^{d/2+1}}\,. \tag{3.6}$$

The character of the first quantum correction to the classical gas depends on the sign of γ. For $\gamma > 0$, A_2^γ is positive and, effectively, particles repel each other. In this case the pressure is higher than for the classical gas. For $\gamma < 0$ the second virial coefficient is negative which means that the pressure is lowered. In this range of statistical parameter γ particles tend to collapse because they effectively attract each other. Roughly speaking, for $\gamma > 0$ the particles resemble fermions and for $\gamma < 0$ they look like bosons. At the boundary ($\gamma = 0$) between these two limits there is no quantum correction to the classical behavior.

We can also calculate the internal energy $U = (d/2)pV$ and, hence, the specific heat of γ-ons at high temperatures. The result is

$$C_V = \frac{k_B d}{2} n(1 + A_2^\gamma n(1 - d/2) + \cdots)\,. \tag{3.7}$$

In the one dimensional case the specific heat is decreasing function of temperature for positive γ. For $d = 3$ the behavior is opposite whereas, for $d = 2$ the specific heat does not depend on temperature. Also, we see that as $T \to \infty$ the specific heat goes to its classical value $(d/2)R$, where R is a gas constant ($R = nk_B$), and this result is independent of γ.

Next, we examine the high temperature properties of g-ons. The momentum distribution function for them is not given explicitly but via the solution of Eq. (2.10). However, we can find exactly this solution in term of Taylor power expansion [10]. To do this we solve Eq. (2.10) by utilizing the Legendre inversion theorem. The distribution function is in the form of the expansion

$$\bar{n}_g(\varepsilon_i) = \sum_{m=1}^{\infty} \frac{(mg + g - m)_m}{m!} \frac{(-1)^m}{\xi_i^{m+1}}\,, \tag{3.8}$$

where $\xi_i = e^{\beta(\varepsilon_i - \mu)}$ and $(a)_m = a(a+1)\cdots(a+m-1) = \Gamma(a+m)/\Gamma(a)$. For details see Appendix A. Then it is straightforward to find the equation of state for ideal g-ons

$$\frac{p}{k_B T} = \frac{1}{\lambda^d} f_{d/2+1}(g, z)\,, \tag{3.9}$$

and the equation for the total number of particles

$$n = \frac{N}{V} = \frac{1}{\lambda^d} f_{d/2}(g, z)\,. \tag{3.10}$$

Again, λ is the thermal wavelength and function $f_k(g,z)$ is defined as follows

$$f_k(g,z) = \sum_{m=0}^{\infty} (-1)^m \frac{(gm+g-m)_m}{m!} \frac{z^{m+1}}{(m+1)^k} . \tag{3.11}$$

Eliminating z from Eq. (3.9) with the aid of Eq. (3.10) we find the equation of state in the high temperature limit in the form

$$\frac{p}{k_B T} = n(1 + A_2^g n + \cdots) , \tag{3.12}$$

where the second virial coefficient for g-ons is

$$A_2^g = (2g-1) \frac{\lambda^d}{2^{d/2+1}} . \tag{3.13}$$

For $g > 1/2$ this coefficient is positive, so g-ons are like fermions. However, for $g < 1/2$ they resemble bosons in that sense that they tend to condense. We also see that identifying statistical parameters γ and g, *i.e.*

$$\gamma = 2g - 1 , \tag{3.14}$$

these two sorts of particles obeying fractional statistics are identical at high temperatures. Also, the whole results for the internal energy and the specific heat for γ-ons could be repeated for g-ons if we use the relation (3.14).

Finally, we examine the high temperature properties of p-ons. Again, the distribution function (2.17) can be expanded in power series and Gaussian integrals can be performed easily for the ideal-gas case. We find the equation of state and equation for particle number in the forms

$$\frac{p}{k_B T} = \frac{1}{\lambda^d} g_{d/2+1} g(p,z) , \tag{3.14}$$

$$n = \frac{N}{V} = \frac{1}{\lambda^d} g_{d/2}(p,z) . \tag{3.15}$$

In the present case the function $g_k(p,z)$ is following

$$g_k(p,z) = \sum_{m=1}^{\infty} (-1)^{mp} \frac{z^{m(p+1)}}{m^k} + \sum_{m=1}^{\infty} \frac{z^m}{m^k} . \tag{3.17}$$

If we take $p > 1$ then the fugacity z is

$$z \simeq n\lambda^d - \frac{(n\lambda^d)^2}{2^{d/2}} , \tag{3.18}$$

where we dropped all higher order terms. For $p = 1$ the sign in the second term on the right hand side of Eq. (3.18) would be positive. Substituting (3.18) into (3.15) and taking only terms of the order n^2 we have

$$\frac{p}{k_B T} = n(1 + A_2^p n + \cdots), \tag{3.19}$$

where

$$A_2^p = -\frac{\lambda^d}{2^{d/2+1}}. \tag{3.20}$$

We see that the second virial coefficient for p-ons does not depend on p and is always negative for $p > 1$. Therefore, we conclude that at high temperatures these particles resemble ordinary bosons.

To summarize this Section, we notice that for the first two statistical distribution functions we obtained the same behavior in the high temperature limit, whereas in the third case the result was different. In other words, g-ons and γ-ons behave similarly whereas p-ons always have the boson type corrections to classical limit. In the next section we check the low temperatures properties of those systems.

4. Low temperature properties

Low temperature properties of γ-ons must be discussed separately for positive and negative γ. Let us consider first the $\gamma > 0$ case.

At $T = 0$ the momentum distribution function reduces to the step function

$$\bar{n}_g(\varepsilon_i) = \frac{1}{\gamma}\theta(\mu - \varepsilon_i). \tag{4.1}$$

All states below the Fermi level are occupied with the mean population $1/\gamma$. As $\gamma \to 0$ this number diverges. For finite γ we see that the averaged number of particles at each quantum level could be fractional and greater than one. The states above the Fermi level are empty.

For free particles we can find the Fermi energy $\varepsilon_F^\gamma = \mu(T = 0)$. Namely, using Eq. (4.1) we calculate that for a given density of particles

$$\varepsilon_F^\gamma = \left(\frac{nd|\gamma|}{2\Omega}\right)^{\frac{2}{d}}, \tag{4.2}$$

where $\Omega = \frac{1}{(2\pi)^{d/2}\Gamma(d/2)}(\frac{m}{\hbar^2})^{d/2}$. The length of the Fermi vector $|\boldsymbol{k}_F| = \sqrt{\frac{2m\varepsilon_F^\gamma}{\hbar^2}}$ depends on the value of γ and so does the volume enclosed by the Fermi surface. In particular we see that as $\gamma \to 0$ the Fermi vector disappears and all particles are on the lowest quantum level.

The existence of the Fermi surface implies that the low temperature properties of a system are affected only by the low energy particle-hole excitations close to it. In other words, only the particles from the vicinity of the Fermi surface give contributions to the thermodynamics in this limit. Therefore, we perform the low temperature (Sommerfeld type) expansion for any thermal quantity. However, in order to do this we have to make some extra modification in the distribution function (2.3). Let us rewrite it as follows

$$\frac{1}{e^{\beta(\varepsilon-\mu)}+\gamma} = \frac{1}{\gamma}\frac{1}{e^{\beta(\varepsilon-\mu^*)}+1}, \tag{4.3}$$

where $\mu^* \equiv \mu + k_B T \ln\gamma$. Now, we can expand the thermodynamic functions around the point μ^*. The general scheme of calculation is presented in Appendix B. Here we quote only the final results.

The lowest corrections to the chemical potential are

$$\mu = \varepsilon_F^\gamma - k_B T \ln\gamma - \frac{\pi^2}{6}\frac{d-2}{2\varepsilon_F^\gamma}\frac{1}{|\gamma|}(k_B T)^2 . \tag{4.4}$$

For ordinary fermions ($\gamma = 1$) we had the first correction to the chemical potential proportional to T^2. Now, the linear in temperature correction has appeared. This is a direct consequence of the modified form of the distribution function (2.3). However, this term disappears in the formula for the internal energy. Namely, as was shown in Appendix B the internal energy is

$$\frac{U}{V} = \bar{\varepsilon}_\gamma + \frac{\Omega}{|\gamma|}\left(\frac{nd|\gamma|}{2\Omega}\right)^{1-\frac{2}{d}}(k_B T)^2 + \dots . \tag{4.5}$$

The general form is similar to that for fermions. The only difference is the factor $1/\gamma$ which enhances the density of states at the Fermi level. Also, the specific heat

$$C_V = K_\gamma T \tag{4.6}$$

is linear function of T. The coefficient $K_\gamma \sim (1/\gamma)^{2/d}$ diverges as $\gamma \to 0$.

So, we proved that the ground state and the low temperatures properties of γ-ons for $\gamma > 0$ resemble properties of ordinary fermions. However, it is not the case for $\gamma < 0$.

For negative statistical parameter γ the distribution function (2.3) is singular when $\varepsilon_i + k_B T \ln\gamma - \mu = 0$. This may be a signal of the Bose–Einstien condensation (BEC) at least in dimensions greater than two. Let us consider this case.

The equation for total number of particles is

$$n = n_0 + \frac{1}{\lambda^d|\gamma|} g_{1/d+1}(|\gamma|, z), \tag{4.7}$$

where n_0 is the number of particles at the first quantum level (with $\boldsymbol{k} = 0$) divided by V. The function g_k is given by

$$g_k(|\gamma|, z) = \sum_{m=1}^{\infty} \frac{(|\gamma|z)^m}{m^k}. \tag{4.8}$$

BEC occurs when $|\gamma|z = 1$ and then the state $\boldsymbol{k} = 0$ starts to be microscopically occupied. This gives that in three dimensional space we have the following condition [1]

$$\lambda^3 n|\gamma| = 2.612. \tag{4.9}$$

Hence, we find the critical temperature

$$k_B T_c^\gamma = \frac{2\pi\hbar^2}{m}\left(\frac{n|\gamma|}{2.612}\right)^{2/3}. \tag{4.10}$$

This temperature is smaller for particles with fractional bosonic statistics than for ordinary bosons. Below T_c the contribution to the internal energy comes from particles which are outside the condensate and we can easily find that

$$\frac{U}{V} = \frac{\Gamma(d/2+1)\xi(d/2+1)}{\Gamma(d/2)\xi(d/2)} n(k_B T)\left(\frac{T}{T_c^\gamma}\right)^{d/2}. \tag{4.11}$$

Hence, the specific heat is

$$C_V = \frac{\Gamma(d/2+1)\xi(d/2+1)}{\Gamma(d/2)\xi(d/2)}\left(\frac{d+2}{2}\right) k_B \left(\frac{T}{T_c^\gamma}\right)^{d/2}, \tag{4.12}$$

and the equation of state is

$$p = \frac{d}{2}\frac{\Gamma(d/2+1)\xi(d/2+1)}{\Gamma(d/2)\xi(d/2)} n(k_B T)\left(\frac{T}{T_c^\gamma}\right)^{d/2}. \tag{4.13}$$

The dependence on γ is hidden in T_c^γ only. On Eq. (4.10) we see that as $|\gamma| \to 0$ then $T_c^\gamma \to 0$ too, and hence, the thermal functions (4.11)–(4.13) diverges. This expresses the fact that the limit $\gamma = 0$ is a nonanalytic boundary between particles with $\gamma > 0$ and with $\gamma < 0$. In other words, this particular value of γ establishes a true phase boundary between two phases of noninteracting γ-ons.

In the next part we are going to discuss the low temperature properties of g-ons. At $T = 0$ the momentum distribution function reduces to

$$\bar{n}_g(\varepsilon_i) = \frac{1}{g}\Theta(\mu - \varepsilon_i), \tag{4.14}$$

for all $0 < g \leq 1$. This means that for all positive g we have a true Fermi surface in the reciprocal space which separates occupied and unoccupied states. Note also, that the case $g = 0$ must be treated separately because then $\bar{n}_g(\varepsilon_i)$ is infinite. Again, as in the case of γ-ons with $\gamma > 0$, the main contribution to the low energy properties is given by the particle-hole excitations around the Fermi surface. Performing the Sommerfeld-type expansion we find that (see Appendix B and also [11])

$$\mu = \varepsilon_F^g - \frac{d-2}{2\varepsilon_F^g} g C_1(g)(k_B T)^2 . \tag{4.15}$$

In the present case there is no linear temperature corrections to the chemical potential. We also calculate the internal energy

$$\frac{U}{V} = \bar{\varepsilon}_g + \Omega(\varepsilon_F^g)^{1-2/d} C_1(g)(k_B T)^2 , \tag{4.16}$$

where now the Fermi energy is

$$\varepsilon_F^g = \left(\frac{ndg}{2\Omega}\right)^{\frac{2}{d}} . \tag{4.17}$$

Hence, we obtain the specific heat

$$C_V = K_g T , \tag{4.18}$$

where $K_g \sim (1/g)^{2/d-1} C_1(g)$. The low temperature properties of g-ons resemble those of ideal fermions with modified density of states by the factor $1/g$. We also see that the case $g = 0$ can not be reached analytically from the case $g > 0$. This is due to the possibility of BEC of ideal bosons ($g = 0$).

As an additional problem it was interesting to consider BEC for ideal g-ons with $g < 0$. As we proved rigorously in the Appendix C, there is no BEC in this case. So, only true bosons can develop a macroscopic occupation of the lowest ($\boldsymbol{k} = 0$) quantum level at finite temperatures.

Finally, we consider p-ons at low temperatures. At $T = 0$ the distribution function (2.17) is

$$\bar{n}_p(\varepsilon_i) = p\Theta(\mu - \varepsilon_i) . \tag{4.19}$$

We see that, although both functions in Eq. (2.17) diverge at $\varepsilon_i = \mu$ the final result is well behaved; those two singularities cancel each other. The last result means that p-ons possess a sharp Fermi surface with a wavevector

$$|k_F^p| = \sqrt{\frac{2m}{\hbar^2}} \left(\frac{nd}{2\Omega p}\right)^{1/d} . \tag{4.20}$$

This property allows us to perform again the Sommerfeld (low temperature) expansion for thermodynamic quantities. This was done in Appendix B. We note that for any $1 \leq p < \infty$ the specific heat is linear with temperature, *i.e.*

$$C_V = K_p T , \tag{4.21}$$

where now $K_p \sim \frac{p}{p+1} p^{2/d-1}$. So, we conclude that at very low temperatures p-ons resemble free fermions with modified density of states.

5. Conclusions

In this paper we studied the thermodynamic properties of particles obeying three different statistics. Each of them interpolates the distribution function between the BE and the FD limits. We compared both the high and the low temperature properties of γ-ons, g-ons and p-ons. We showed that γ-ons behave like fermions when $\gamma > 0$ and like bosons when $\gamma < 0$ at all temperatures. However, their properties are not the same because their thermal functions are parameterized by γ which changes the number of states. We also showed that in between of the boselike and the fermilike regimes, there is a true boundary condition ($\gamma = 0$), where the statistics is classical. However, when we approached this limit from the left or the right sides the thermal functions diverge as $1/\gamma$. In Fig. 1 we draw the schematic phase diagram for γ-ons for different values of γ.

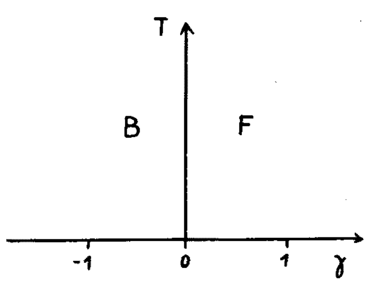

Fig. 1. Schematic phase diagram for γ-ons. The letters F and B characterize the fermilike and boselike type of behavior, respectively.

The thermodynamic properties of g-ons turned out to be quite different. At high temperatures we noted two classes of behavior: fermilike for $g > 1/2$ and boselike for $g < 1/2$. For $g = 1/2$ the statistics was classical. However, at low temperatures g-ons resemble ordinary fermions with modified density of states for all $0 < g \leq 1$. This means that there must be a crossover temperature when the typical behavior changes its character. The schematic phase diagram is provided in Fig. 2.

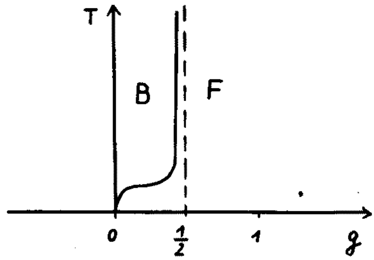

Fig. 2. Schematic phase diagram for p-ons.

Finally, we considered p-ons and we showed that for $p > 1$ they look like bosons at very high temperature. On the other hand, they resemble fermions at low temperatures if $p < \infty$. So, again there must exist a crossover temperature from the bosonic to the fermionic behavior. The phase diagram for p-ons is shown in Fig. 3.

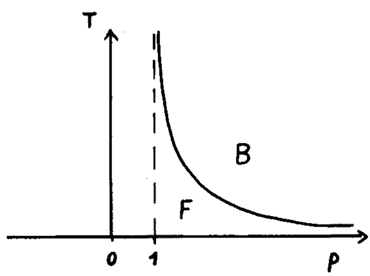

Fig. 3. Schematic phase diagram for p-ons.

In conclusion, we showed that different routes of interpolating between FD and BE limits correspond to different ways of approaching the statistical mechanics of particles with a nonstandard statistics. Our methods 2 and 3 are based on the combinatorial approach, which is not directly related to the symmetry properties of the many-body wave function with respect to the particle transposition. In particular, the knowledge of the second virial coefficient is not sufficient to determine the full thermodynamics. For example, although we know the virial coefficient for anyons, it is impossible distinguish which statistics describes them at low temperatures, γ-ons or g-ons. We think that none of them is the proper one. However, anyons in the strong magnetic field seem to be described by g-on statistical mechanics [12]. Hence, the Haldane (g-on) statistics described here as route 2 seems to be applicable in some situations.

Appendix A

In this Appendix we show how to solve Eq. (2.10) using the Lagrange inversion theorem. Suppose, that we have a transcendental equation

$$y = a + x\phi(y)\,, \tag{A.1}$$

and we want to find y as an unknown variable. The variables x and a are supposed to be given. The solution can be represented by infinite power series

$$y = a + \sum_{n=1}^{\infty} \frac{x^n}{n!} \frac{d^{n-1}}{dy^{n-1}} [\phi^n(a)]\,. \tag{A.2}$$

Noting, that Eq. (2.10) has the same formal structure as (A.1) with $a = 0$ we use Eq. (A.2) to obtain Eq. (3.8).

Appendix B

In this Appendix we present the low temperature expansion for any thermodynamic function calculated with arbitrary statistics. The only limitation is that this thermal function must be well behaved in the vicinity of chemical potential. Let us consider the following integral

$$I = \int_0^{\infty} H(\varepsilon) n(\varepsilon - \mu) d\varepsilon\,. \tag{B.1}$$

Introducing the new variable $y = \beta\varepsilon - \beta\mu$ and manipulate with the integrals we obtain

$$\begin{aligned} I =& \frac{1}{\beta} \int_0^{\infty} dy H(\frac{y}{\beta} + \mu) n(y) \\ &+ \frac{1}{\beta} \int_{-\beta\mu}^{0} dy H(\frac{y}{\beta} + \mu)(n(y) - a) + \frac{1}{\beta} \int_{-\beta\mu}^{0} dy H(\frac{y}{\beta} + \mu) a\,, \end{aligned} \tag{B.2}$$

where the constant a must be chosen properly to the statistics. For example, for γ-ons we should take $a = 1$, g-on statistics needs $a = 1/g$ whereas for p-ons $a = p$. At low temperatures we can approximate the second integral extending the lower limit of integration to $-\infty$. Then, changing the variable in this integral $y \to -y$ and expanding both functions $H(\mu \pm k_B T y)$ around the point μ we obtain

$$I = a \int_0^{\mu} H(\varepsilon) d\varepsilon + \sum_{i=0}^{\infty} C_i(a) \frac{1}{i!} \frac{d^i H(\mu)}{d\varepsilon^i} (k_B T)^{i+1}, \tag{B.3}$$

where the coefficients $C_i(a)$ are defined as follows

$$C_i(a) = \int_0^{\infty} dy y^i (n(y) + (-1)^i (n(-y) - a)). \tag{B.4}$$

If $a = 1$ and $n(y)$ is γ-ons momentum distribution function we obtain that the even coefficients are zero $C_{2n}(1) = 0$ and the odd coefficients are given by

$$C_{2n-1}(1) = (-1)^{n-1} (2\pi)^{2n} \frac{B_{2n}}{2n} (1 - 2^{1-2n}), \tag{B.5}$$

where B_{2n} are the Brilouin numbers. So, γ-ons have almost the same expansion as ideal fermions.

Similar results we obtain for p-ons. Using the distribution function (2.17) we find that

$$C_{2n-1}(p) = (-1)^{n-1} (2\pi)^{2n} \frac{B_{2n}}{2n} (1 - (p+1)^{1-2n}) \tag{B.6}$$

and $C_{2n} = 0$. Note that for $p = \infty$ we obtain the coefficients for the Sommerfeld expansion of ideal bosons provided that they do not condense.

For g-ons it is impossible to find the analytic formula for $C_k(g)$. We only know that $C_0(g) = 0$ [11] and the rest ones are finite.

The formalism developed above is useful in calculating the chemical potential and the internal energy. Namely, the equation for μ is

$$n = \frac{N}{V} = \Omega \int_0^{\infty} d\varepsilon \varepsilon^{d/2-1} n(\varepsilon), \tag{B.7}$$

and for the energy

$$\frac{E}{V} = \Omega \int_0^{\infty} d\varepsilon \varepsilon^{d/2} n(\varepsilon), \tag{B.8}$$

where

$$\Omega = \frac{1}{(2\pi)^{d/2}\Gamma(d/2)}\left(\frac{m}{\hbar^2}\right)^{d/2} . \tag{B.9}$$

Expanding Eq. (B.7) to first nontrivial order as in Eq. (B.3) we have

$$n \simeq \Omega(a\int_0^\mu d\varepsilon\varepsilon^{d/2-1} + C_1(a)(d/2-1)(\mu)^{d/2-2}(k_BT)^2) , \tag{B.10}$$

and then supposing that the correction to the chemical potential is small $\mu = \varepsilon_F^a + \delta\mu$ we can find that

$$\mu = \varepsilon_F^a\left(1-(d/2-1)\frac{C_1(a)}{a}\left(\frac{k_BT}{\varepsilon_F^a}\right)^2\right) . \tag{B.11}$$

Similarly, we find the low temperature internal energy

$$\frac{E}{V} = \bar{\varepsilon} + \Omega(\varepsilon_F^a)^{d/2-1}C_1(a)(k_BT)^2 , \tag{B.12}$$

where $\bar{\varepsilon}_a = \frac{2\Omega a}{d+2}(\varepsilon_F^a)^{d/2+1}$. The last equation allows us to calculate the specific heat which is

$$C_V = K_aT . \tag{B.13}$$

where $K_a = 2\Omega(\frac{nd}{2a\Omega})^{1-2/d}C_1(a)k_B^2$.

Appendix C

In this Appendix we extend our discussion of g-ons to the case with negative g (*i.e.* we suppose that $g \le 0$). In principle, the value of g can be arbitrary and it seems very interesting to investigate how the particles look like with "*attractive*" statistical interaction. The most exciting question deals with the BEC in such system. However, as we show below, the condensation does not occur in such gas.

The equation for the momentum distribution function is following (*cf.* (2.10))

$$\bar{n}_g(\varepsilon_i)\xi = (1+(1+\eta)\bar{n}_g(\varepsilon_i))^{1+\eta}(1+\eta\bar{n}_g(\varepsilon_i))^{-\eta} , \tag{C.1}$$

where we define $g = -\eta$, with $\eta > 0$, and $\xi = e^{\beta(\varepsilon_i-\mu)}$. Using the method developed in Ref. [10 (see also Appendix A) we find the solution of Eq. (C.1) in the form of the infinite power series, *i.e.*

$$\bar{n}_g(\xi) = \sum_{m=0}^{\infty}\frac{(m\eta+\eta+m)_m}{m!}\frac{1}{\xi^{m+1}} . \tag{C.2}$$

This series is absolutely converged for $\xi > \frac{(\eta+2)^{\eta+2}}{(\eta+1)^{\eta+1}}$. Next, we find the equation for the total particle number

$$\frac{N}{V} = \frac{1}{V}\bar{n}_0 + \frac{1}{\lambda^d} g_{d/2}(\eta, z), \tag{C.3}$$

where the function $g_k(\eta, z)$ is given by the following series

$$g_k(\eta, z) = \sum_{m=0}^{\infty} \frac{(\eta m + \eta + m)_m}{m!} \frac{z^{m+1}}{(m+1)^k}. \tag{C.4}$$

$\bar{n}_0$ is the population of the lowest energy level and we treat it separately in case of BEC. Eq. (C.3) is absolutely converged for all z in the range

$$0 \le z \le z_R \equiv \frac{(\eta+1)^{\eta+1}}{(\eta+2)^{\eta+2}}. \tag{C.5}$$

On the other hand, momentum distribution function can be written in the form (2.11) where $w(\xi)$ obeys the equation

$$w^{-\eta}(1+w)^{1+\eta} = \frac{1}{z}e^{\beta\varepsilon}. \tag{C.6}$$

Condition for BEC is $w = \eta$. Substituting this into Eq. (C.1) and taking the maximal value of possible $z = z_R$ we obtain the allowed ε

$$\beta\varepsilon = 2(1+\eta)\ln(1+\eta) - (2+\eta)\ln(2+\eta) - \eta\ln\eta. \tag{C.7}$$

We see that in order to have BEC ε must be negative but we have supposed that $\varepsilon \ge 0$, so we obtain contradiction and this proves the absence of BEC for g-ons.

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