HEAVY QUARKS, RENORMALONS AND LATTICES*

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I review the non-perturbative method, developed with Martinelli, for defining the higher dimensional operators which appear in the Heavy Quark Effective Theory (HQET), such that their matrix elements are free of renormalon singularities, and diverge at most logarithmically with the ultra-violet cut-off. Matrix elements of these operators can be computed numerically in lattice simulations of the HQET. The procedures are illustrated by presenting physical definitions of the binding energy ($\bar{\Lambda}$) and of the kinetic energy ($-\lambda_1/2m_Q$) of the heavy quark in a hadron. This allows for a definition of a "subtracted pole mass", whose inverse can be used as the expansion parameter in applications of the HQET. I also present some numerical results for the binding energy ($\bar{\Lambda} = 190 \pm 30$ MeV), and the kinetic energy ($|\lambda_1| = |\langle B|\bar{h}\vec{D}^2 h|B\rangle|/(2M_B) < 1.0$ GeV^2), obtained with Crisafulli, Gimenez and Martinelli.

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I would like to start by saying what a pleasure and an honour it is to be able to participate in this celebration of Kacper Zalewski's 60th birthday. The great respect and affection which his colleagues and students have for him is evident for all to see. I am very happy to be able to wish Kacper **STO LAT** and many further enjoyable years of scientific exploration.

1. Introduction

At this conference we have seen the importance of heavy quark physics in determining the parameters of the standard model of particle physics, and in providing tests of the theory and possible signatures of new physics.

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In order to interpret fully the experimental data on weak decays of heavy hadrons, it is necessary to understand and quantify the strong interaction effects in these decays. These effects are non-perturbative, and lattice QCD and numerical simulations provide the opportunity of evaluating the corresponding operator matrix elements from first principles. The status of lattice studies of weak decays of heavy mesons was reviewed at last year's Zakopane Summer School in Theoretical Physics by Lellouch [1]. In this talk I will review an important question in heavy quark physics which has received much attention during the last year, that of the definition of the higher dimensional operators which are present when one tries to evaluate strong interaction effects beyond the leading order in the Heavy Quark Effective Theory, and the evaluation of their matrix elements. Interest in this subject was stimulated by the recent observation that the matrix elements of these operators, and the Wilson coefficient functions which match the effective theory with QCD, contain renormalon singularities [2, 3]. This implies that the operators must be defined non-pertubatively. Together with Maiani and Martinelli we had arrived at the same conclusion when trying to compute the matrix elements of the higher dimensional operators in the effective theory using lattice simulations [4]. Our arguments were based on the need for the non-pertubative subtraction of the "power" divergences (*i.e.* ultraviolet singularities which diverge as powers of the cut-off), which we will show is analogous to the problem of the subtraction of renormalons. Martinelli and I have recently proposed a non-pertubative definition of higher dimensional operators, such that their matrix elements, and the corresponding coefficient functions, are free of renormalon singularities and power divergences [5]. Below I will review these ideas, illustrating the method by considering the binding energy ($\bar{\Lambda}$) and the kinetic energy ($-\lambda_1/2m_Q$) of a heavy quark in a heavy hadron. I will also present some numerical results for these quantities obtained together with Crisafulli, Gimenez and Martinelli [6, 7].

The Heavy Quark Effective Theory (HQET) has developed during the last few years into a very useful tool for the study of strong interaction effects in heavy quark physics [8–11] (for a comprehensive review see Ref. [12]). At this meeting the formalism of the HQET, and its applications to exclusive and inclusive decays of heavy hadrons, has been reviewed in the talks of Mannel [13] and Bigi [14]. In the HQET physical quantities are studied systematically as series in inverse powers of the mass of the heavy quark[1]. In particular, local composite operators of QCD, whose matrix elements contain the long-distance gluonic effects in physical processes, are expanded

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[1] In $B \to D$ decays the expansion may be performed simultaneously in the inverse powers of the masses of both heavy quarks.

in terms of the operators of the HQET

$$O^{\rm QCD} = \sum_{n,\alpha} \frac{C_{n,\alpha}(m_Q/\mu)}{m_Q^n} O^{\rm HQET}_{n,\alpha}(\mu)\,, \qquad (1)$$

where m_Q is the mass of the heavy quark and μ is the renormalisation scale used in defining the renormalised operators $O^{\rm HQET}_{n,\alpha}(\mu)$. For the moment we assume that the operators $O^{\rm HQET}_{n,\alpha}(\mu)$ are renormalised in some scheme based on the dimensional regularisation of ultra-violet divergences, such as the $\overline{MS}$ scheme. The QCD operators $O^{\rm QCD}$ generally also depend on some renormalisation scale, which we take to be different from μ, and we do not exhibit this dependence explicitly. As an important example of the operators $O^{\rm QCD}$ consider the vector and axial vector currents, $V^\mu = \bar{Q}\gamma^\mu q$ and $A^\mu = \bar{Q}\gamma^\mu\gamma^5 q$, where Q and q represent the fields of the heavy and light quark respectively. From the matrix elements of these currents we obtain the leptonic decay constants of heavy mesons and the form-factors of semi-leptonic $B \to \pi$ and ρ decays. Other examples have been presented in the talks by Mannel [13] and Bigi [14]. Below we will consider the expansion of the inverse heavy quark propagator in inverse powers of the mass of the heavy quark, which corresponds to taking the Dirac operator, $i\not{D} + m_Q$, for $O^{\rm QCD}$.

Renormalon singularities lead to ambiguities in the Wilson coefficient functions $C_{n,\alpha}$, and in the matrix elements of the operators $O^{\rm HQET}_{n,\alpha}$. Although these ambiguities cancel (see Ref. [15] and references therein, and Refs [2, 3, 16, 17]), the presence of renormalons requires an alternative definition of the renormalised operators $O^{\rm HQET}_{n,\alpha}$, if the HQET is to be applicable beyond leading order in the heavy quark mass, and in particular if the coefficient functions are to be calculable using perturbation theory. The renormalon singularities and the corresponding ambiguities which we are considering here are those induced by the introduction of the expansion in Eq. (1). Of course the matrix elements of the QCD operator $O^{\rm QCD}$ will themselves contain non-perturbative long-distance effects, but, as usual, these effects do not appear in the coefficient functions, but only in the matrix elements of the operators of the HQET.

The use of a hard (dimensionful) ultra-violet cut-off Λ in the effective theory (instead of dimensional regularisation) leads to matrix elements and coefficient functions which are free of renormalon ambiguities. This point has been stressed by Bigi and his collaborators [3]. However in this case the matrix elements in the effective theory diverge as powers of the cut-off Λ. The subtraction of these power divergences cannot be performed using perturbation theory; such a subtraction reintroduces renormalon ambiguities in the matrix elements and coefficient functions. This point will be discussed in detail in Section 4 below.

In this talk I will review the approach recently proposed by Martinelli and myself [5]. In contrast to previous approaches, we propose a non-perturbative definition of the renormalised operators $O^{\rm HQET}_{n,\alpha}$, such that they are free of both renormalon ambiguities and power divergences. This allows us to present "physical" definitions of parameters such as the binding energy ($\bar{\Lambda}$) or the kinetic energy ($-\lambda_1/2m_Q$) of the heavy quark in a heavy hadron. By "physical" we mean that they do not depend on any regularisation or renormalisation scale, nor on the method used to regulate the ultra-violet divergences (although they do depend on the renormalisation prescription used to define them).

An immediate, and related, question concerns the choice of the mass parameter m_Q used in the expansion. The pole mass is ambiguous due to the presence of infrared renormalon singularities [2, 3], and so a different definition of m_Q is required. We introduce the definition of a "subtracted pole mass", m_Q^S, from which the renormalon ambiguities have been subtracted non-perturbatively. m_Q^S has the attractive property that it can be determined from the mass of a hadron H containing the heavy quark Q, by calculations performed entirely within the HQET. For each such hadron H we define a parameter $\bar{\Lambda} \equiv m_H - m_Q^S$, and $\bar{\Lambda} \to$ constant as $m_Q \to \infty$ (for simplicity of notation we suppress the label H on $\bar{\Lambda}$). $\bar{\Lambda}$ remains finite when the ultraviolet cut-off is taken to infinity, and can be computed numerically in lattice simulations of the HQET. We discuss the question of the definition and evaluation of $\bar{\Lambda}$ to leading order in Section 2, and to $O(1/m_Q)$ in Section 3.

As a further illustration of our general procedure we study in Section 3 the matrix elements which determine the kinetic energy of the heavy quark in a hadron[2]

$$\lambda_1 = \frac{\langle H|\bar{h}\vec{D}^2 h|H\rangle}{2m_H}, \qquad (2)$$

where h represents the heavy quark field in the HQET, and H is a hadron (of mass m_H) containing a heavy quark (we suppress the label H on λ_1). The λ_1's are important ingredients in the study of the spectroscopy and inclusive decays of heavy hadrons. The chromomagnetic operator $\bar{h}\sigma_{ij}G^{ij}h$, where G_{ij} are the spatial components of the gluon field strength tensor, appears at the same order of the heavy quark expansion as the kinetic energy operator. However the spin structure of the chromomagnetic operator ensures that its matrix elements are free of renormalon singularities.

[2] Here we consider the effective theory for a heavy quark at rest. The generalisation to an arbitrary four-velocity is straightforward, although there are considerable subtleties in formulating the effective theory at nonzero velocity in Euclidean space [18, 19].

In Ref. [4], together with Maiani and Martinelli, we pointed out that in the lattice formulation of the HQET the evaluation of higher order terms in $1/m_Q$ involves the appearance of ultra-violet singularities which diverge as inverse powers of the lattice spacing a. This is due to the fact that the higher order terms generally involve matrix elements of higher dimensional operators which can mix with lower dimensional ones (*e.g.* $\bar{h}\vec{D}^2 h$ can mix with $1/a\,\bar{h}D_4 h$ and $1/a^2\,\bar{h}h$). In Ref. [4] it was further stressed that these divergences must be subtracted non-perturbatively, since factors such as

$$\frac{1}{a}\exp\left(-\int^{g_0(a)}\frac{dg'}{\beta(g')}\right) = \Lambda_{\rm QCD}\,, \tag{3}$$

which do not appear in perturbation theory, give non-vanishing contributions as $a \to 0$. Renormalons represent an explicit example of non-perturbative effects of this kind. Below we explain how the subtraction of power divergences and of renormalon singularities, can be performed non-perturbatively, illustrating our ideas with the evaluation of $\bar{\Lambda}$ and λ_1.

Much of the discussion in this talk is presented within the framework of lattice field theory, however all the theoretical questions addressed below have to be faced with any ultra-violet regularisation scheme [20]. Moreover, our proposed definitions of subtracted higher dimensional operators in general, and of $\bar{\Lambda}$ and λ_1 in particular, are in fact independent of the regularisation. Our main aim in this talk is to provide an understanding of these theoretical issues. In addition however, lattice simulations provide the opportunity for the evaluation of the parameters of the HQET, such as $\bar{\Lambda}$ and λ_1, from first principles. In the following sections we also explain how this can be done in principle, and also present some results from an exploratory numerical study Ref. [6] (some preliminary numerical results have been presented in Ref. [7]). However, instead of using lattice simulations, it is also possible to use other non-perturbative methods, such as QCD sum rules (reviewed in this meeting by Narison [21]), to compute the matrix elements of the operators defined by our prescription. These matrix elements are free of renormalons and power divergences.

We use the following notation for ultra-violet cutoffs and renormalisation scales. We denote by M the scale used to define the renormalised operators in QCD ($O^{\rm QCD}$), and by μ that to define the operators in the HQET (the $O^{\rm HQET}_{n,\alpha}$). Thus $\mu \ll m_Q \ll M$. It will also be convenient to consider bare operators in the HQET, and we denote by Λ the corresponding ultra-violet cut-off. Where the discussion is particular to the lattice formulation of the HQET we replace Λ by a^{-1}.

The plan of the remainder of this talk is as follows. In the next section we discuss the definition of $\bar{\Lambda}$, and explain how it can be computed

in lattice simulations. We explain our general procedure for the definition of subtracted higher dimensional operators in Section 3, using the matrix elements λ_1 as examples. In both these sections we present some results from an exploratory lattice simulation. Section 4 contains a discussion of "matching", *i.e.* the determination of the coefficient functions $C_{n,\alpha}$ corresponding to our definition of the operators. Finally in Section 5 we present our conclusions.

2. The binding energy —$\bar{\Lambda}$

In this section we discuss the definition of the "subtracted pole mass", m_Q^S, from which the ambiguities from the leading renormalon singularities have been subtracted non-perturbatively (see Subsection 2.2 below). m_Q^S can thus be used as the expansion parameter in the HQET. We also explain how m_Q^S, and the corresponding $\bar{\Lambda}$ parameter, can be computed in lattice simulations, and in Subsection 2.3 we present some numerical results. We start however by reviewing very briefly some of the relevant points concerning the renormalon singularities in the propagator of the heavy quark.

2.1. Renormalons in the heavy quark propagator

Our discussion follows closely the presentation by Beneke and Braun [2], in which these authors study the heavy quark propagator in the large N_f limit, where N_f is the number of light quark flavours. We refer the reader to Ref. [2] for more details, and also to Ref. [20] in which the discussion is extended to include the terms of $O(1/m_Q)$ in the inverse propagator. To leading nontrivial order in $1/N_f$, the renormalon singularities are obtained by summing over an arbitrary number of light quark loops in the gluon propagator. The Borel transformed gluon propagator (in the Landau gauge) is written as

$$D_{\mu\nu}^{AB}(k,u) = \sum_{n=0}^{\infty} \frac{1}{n!} D_{\mu\nu,n}^{AB}(k) \left(\frac{-4\pi\, u}{\beta_0\, \alpha_s}\right)^n$$
$$= i\delta^{AB} \left(\frac{e^C}{\mu^2}\right)^{-u} \frac{k_\mu k_\nu - k^2 g_{\mu\nu}}{(-k^2)^{2+u}}, \qquad (4)$$

where β_0 is the coefficient of $\alpha_s/4\pi$ in the first term of the β-function ($\beta_0 = -(11 - {}^2/_3 N_f)$), and A and B are colour labels. μ is the renormalisation scale and C is a scheme dependent constant. $D_{\mu\nu,n}^{AB}(k)$ is the contribution to the gluon propagator from the diagram with n quark loops. Of course in leading order in N_f it is only the term ${}^2/_3 N_f$ which appears, however

it is assumed that the replacement of $^2/_3 N_f$ by β_0 is a consistent one for identifying the singularities. This assumption is based on the intuition, that it is the infrared behaviour of the running coupling constant which is (at least partially) responsible for the singularities.

Feynman diagrams evaluated with the Borel transformed propagator (4) may have poles at positive values of u, rendering the inverse Borel transform (which requires an integral over positive values of u) ambiguous[3]. This is particularly true in Operator Product Expansions, where such singularities in the Wilson coefficient functions are cancelled by those in the matrix elements of higher dimensional, or higher twist, operators [15, 22]. The expansion in inverse powers of the mass of the heavy quark is an interesting example of this phenomenon. We start by considering the quark propagator.

The inverse quark propagator in QCD can be written in the form

$$S^{-1}(p,m) = \not{p} - m - \Sigma(p,m)\,, \tag{5}$$

where

$$\Sigma(p,m) = m\Sigma_1(p^2,m) + (\not{p} - m)\Sigma_2(p^2,m) \tag{6}$$

and m is the bare mass. We now write

$$p = m_Q v + k\,, \tag{7}$$

where m_Q is some well defined choice of the heavy quark mass, to be specified later. In all the explicit examples given in Sections 2.2 and 3 we will take $v = (1,\vec{0})$. It is also convenient to define the quark propagator sandwiched between projection operators

$$\frac{1+\not{v}}{2} S_P(k,m_Q) \equiv \frac{1+\not{v}}{2} S(p,m) \frac{1+\not{v}}{2}\,, \tag{8}$$

Then, [2],

$$S_P^{-1}(k,m_Q) = m_Q - m_{\rm pole}(m_Q,\mu) - C\left(\frac{m_Q}{\mu}\right) S_{\rm eff}^{-1}(v\cdot k,\mu) + O\left(\frac{(v\cdot k)^2}{m_Q}, \frac{k_\perp^2}{m_Q}, \frac{1}{N_f^2}\right), \tag{9}$$

where $S_{\rm eff}$ is the quark propagator in the HQET (whose action is given by $\bar{h} iv\cdot D h$; h represents the field of the heavy quark), μ is the renormalisation

[3] When the calculations are extended beyond the leading order in $1/N_f$, the poles become replaced by branch points of cut singularities.

scale, and $k_\perp^2 = k^2 - (v\cdot k)^2$. $S_{\rm eff}^{-1}$ is linearly divergent in perturbation theory. In dimensional regularisation the Borel transforms of both $m_{\rm pole}$ and $S_{\rm eff}^{-1}$ have renormalon singularities at $u = 1/2$, the infrared renormalon in $m_{\rm pole}$ cancels the ultra-violet renormalon in $S_{\rm eff}^{-1}$ on the r.h.s. of Eq. (9) [2]. The connection between power divergences and renormalons can be seen by noting that if $S_{\rm eff}^{-1}$ is linearly divergent in one-loop perturbation theory (which corresponds to using (4) with $u = 0$ as the gluon propagator), then its Borel transform is logarithmically divergent at $u = 1/2$, corresponding to a pole singularity at this point. In particular $S_{\rm eff}^{-1}(0,\mu)$ is not equal to zero, which is a signal of the presence of ultra-violet renormalon singularities at $u = 1/2$. A more detailed account of the correspondence between power divergences and renormalons is given in Ref. [20].

For the discussion below, it is convenient to re-express Eq. (9) in terms of the bare propagator in the effective theory with a hard cut-off Λ,

$$S_P^{-1}(k,m_Q) = m_Q - m_{\rm pole}(m_Q,\Lambda) - C\left(\frac{m_Q}{\Lambda}\right) S_{\rm eff}^{-1}(v\cdot k,\Lambda) + O\left(\frac{(v\cdot k)^2}{m_Q},\frac{k_\perp^2}{m_Q},\frac{1}{N_f^2}\right). \quad (10)$$

Now $m_{\rm pole}(m_Q,\Lambda)$ and $S_{\rm eff}^{-1}(v\cdot k,\Lambda)$ both diverge linearly with the cut-off, but their Borel transforms have no poles at $u = 1/2$. Thus we have replaced the problem of the ambiguities associated with renormalon singularities with that of determining the exponentially small terms when power divergences are present (see Eq. (3) and the corresponding discussion). By adding a residual mass counterterm $(\delta m\,\bar{h}h)$ to the action of the HQET, it is possible to impose the non-perturbative condition $S_{\rm eff}^{-1}(0,\Lambda) = 0$, thus removing the power divergences. In the following Subsection, we impose instead an equivalent condition on the static (*i.e.* $\vec{v} = 0$) heavy quark propagator in Euclidean configuration space ($S_{\rm eff}^{ij}(x;y)$, where i and j are colour labels) in the Landau gauge:

$$\lim_{t\to\infty}\frac{d}{dt}\ln\left(|S_{\rm eff}^{ii}(x;0)|\right) = 0\,, \quad (11)$$

where $t = x^0$ and $S_{\rm eff}^{ii}$ is the trace over the colour components of the propagator. In other words δm is chosen in such a way that the propagator tends to a constant at large times, and does not fall (or rise) exponentially with time. Since the operator we are subtracting $(\bar{h}h)$ is conserved, either of these conditions is sufficient to determine its coefficient δm fully. Matching the effective theory onto full QCD now implies that $m_{\rm pole}$ on the r.h.s. of Eq. (10) is replaced by a "subtracted pole mass", m_Q^S, from which the power

divergences (and indeed all the dependence on Λ) have been subtracted. m_Q^S is therefore a natural choice for the expansion parameter m_Q of the HQET. The implementation of the above subtraction requires the non-perturbative evaluation of the quark propagator in the HQET because of the arguments given in the discussion of Eq. (3). Lattice simulations provide the possibility for such an evaluation, and in the following Subsection we discuss an explicit definition of m_Q^S based on Eq. (11), and of the corresponding $\bar{\Lambda}$ parameter, $\bar{\Lambda} = m_H - m_Q^S$, where H is a hadron containing one heavy quark Q. In spite of the fact that the entire discussion of the following Subsection is in the framework of lattice field theory, the value of $\bar{\Lambda}$ is determined by its definition based on the condition (11), and is independent of the method of regularisation.

2.2. Definition of $\bar{\Lambda}$

In this Subsection we present our definition of $\bar{\Lambda}$ explicitly. This definition is based on the imposition of the condition in Eq. (11). We also explain how $\bar{\Lambda}$ might be evaluated using the lattice formulation of the HQET. The discussion is presented in Euclidean space, and we take as the action of the HQET in the static (*i.e.* $\vec{v} = 0$) case

$$\mathcal{L}_{\text{eff}} = \bar{h} D_4 h \,, \tag{12}$$

where h represents the field of the heavy quark. Consider the correlation function

$$C(t) = \sum_{\vec{x}} \langle 0 | \, J_\Gamma(\vec{x}, t) \; \bar{J}_\Gamma^\dagger(\vec{0}, 0) \, | 0 \rangle \,, \tag{13}$$

where $J_\Gamma^\dagger$ and J_Γ are interpolating operators which can create or annihilate a meson state in the HQET. For example, we may take $J = \bar{h} \Gamma q$ where q represents the field of the light quark and Γ is one of the Dirac matrices[4]. For sufficiently large times, so that only the ground state contributes significantly to the correlation function,

$$C(t) \to Z^2 \exp(-\mathcal{E} t) \,, \tag{15}$$

where the constant Z is independent of the time. The exponent $\mathcal{E}$ is equivalent to the definition of $\bar{\Lambda}$ proposed in Ref. [23]

$$\bar{\Lambda} = \frac{-\partial_4 \langle 0 | \, \bar{h} \Gamma q \, | M \rangle}{\langle 0 | \, \bar{h} \Gamma q \, | M \rangle} \,. \tag{15}$$

[4] The generalisation to heavy baryons is straightforward.

However $\mathcal{E}$, and hence $\bar{\Lambda}$ defined through Eq. (15), is not a physical quantity, since it diverges linearly as $a \to 0$ (as can be demonstrated in one-loop perturbation theory). If the matrix elements in Eq. (15) are defined in the $\overline{MS}$ scheme, then the corresponding definition of $\bar{\Lambda}$ contains an ambiguity of $O(\Lambda_{\rm QCD})$. The linear divergence in $\mathcal{E}$ appears due to the mixing of the operator $\bar{h}D_4h$ in the effective action, with the lower dimensional operator $\bar{h}h$. It is possible to subtract this divergence non-perturbatively by adding a residual mass term $\delta m\,\bar{h}h$ to the action, where δm is determined by imposing a suitable renormalisation condition, such as that in Eq. (11). We now state explicitly our proposal for a physical definition of $\bar{\Lambda}$, and for its non-perturbative evaluation:

(i) Evaluate the heavy quark propagator in the theory defined by the action (12), in some fixed gauge (the Landau gauge say). In practice this will be done using lattice simulations. The heavy quark propagator in a smooth gauge, such as the Landau gauge, is of the form (for $t > 0$)

$$S_h^{ij}(\vec{x},t;\vec{0},0) = \langle\, 0 \,|\, h^i(x)\bar{h}^j(0) \,|\, 0 \,\rangle = \delta^3(\vec{x})\,\delta^{ij}A(t)\exp(-\lambda t)\,, \qquad (17)$$

where the ultra-violet divergences associated with the quark mass are contained in the exponent λ. i and j are colour labels, and unless specifically required they will be suppressed below. $A(t)$ satisfies the condition

$$\lim_{t\to\infty}\frac{d}{dt}\ln\{|A(t)|\} = 0\,. \qquad (17)$$

In the explicit examples below we will use the following lattice covariant derivative

$$D_4 f(\vec{x},t) = \frac{1}{a}\Big(f(\vec{x},t) - U_4^\dagger(\vec{x},t-a)f(\vec{x},t-a)\Big)\,, \qquad (18)$$

where $\{U_\mu(\vec{x},t)\}$ are the link variables. Other lattice definitions of D_4 are also acceptable.

(ii) Add a residual mass term to the effective action[5]

$$\mathcal{L}'_{\rm eff} = \frac{1}{1+\delta m\,a}\left(\bar{h}D_4h + \delta m\,\bar{h}h\right)\,, \qquad (19)$$

where δm will be specified in *(iii)* below. With the new action, the heavy quark propagator (now denoted by S'_h) is given by

$$S'_h(\vec{x},t;\vec{0},0) = S_h(\vec{x},t;\vec{0},0)\,\exp\left(-\ln(1+\delta m\,a)\frac{t}{a}\right)\,. \qquad (20)$$

[5] The normalization factor $1/(1+\delta m\,a)$ is introduced for convenience, as will become apparent below.

This result demonstrates that the divergences associated with mass renormalisation do indeed exponentiate.

(iii) We fix the residual mass counterterm δm by the condition

$$-\nu = \frac{\ln(1+\delta m\, a)}{a} \equiv \lim_{t\to\infty} \frac{1}{a} \ln\left(\frac{|S_h^{ii}(\vec{x}, t+a; \vec{0}, 0)|}{|S_h^{ii}(\vec{x}, t; \vec{0}, 0)|}\right) \tag{21}$$

ν can be computed in numerical simulations by studying the logarithm on the right hand side of Eq. (21) as a function of t.

(iv) The "physical" definition of $\bar{\Lambda}$ is

$$\bar{\Lambda} \equiv \mathcal{E} - \nu = \mathcal{E} + \frac{1}{a} \ln(1 + \delta m\, a) \tag{22}$$

$\bar{\Lambda}$ is finite in the limit $a \to 0$ and is free of renormalon ambiguities. The corresponding "subtracted" pole mass m_Q^S is defined by

$$m_Q^S \equiv m_H - \bar{\Lambda}. \tag{23}$$

m_Q^S is independent of a, and contains no renormalon ambiguities. Thus from the computed value of m_Q^S one can determine $m_{\overline{MS}}(\mu)$, or any other short-distance definition of the quark mass (up to uncertainties which are now of $O(1/m_Q)$), using perturbation theory. This will be discussed in Section 4.

The counterterm ν defined above is gauge-invariant, in spite of the fact that it is calculated from the heavy quark propagator in a fixed gauge. The argument goes as follows. The linear divergence is eliminated from any correlation function, *i.e.* for any external state, by subtracting from the action (12) a term proportional to the gauge-invariant operator $\bar{h}h$. Since in this way one eliminates all divergences both for quark and hadron external states, the coefficient of the mixing has to be gauge-invariant. This must be true also for the finite non-perturbative term which accompanies the linear divergence. The gauge-invariance of the linearly divergent term has been checked explicitly in one-loop perturbation theory. The same argument can be applied to the evaluation of the subtraction constants which appear in the definition of a finite kinetic energy operator, to be discussed in the following section.

It may appear more natural to define $\bar{\Lambda}$ using Eqs (21) and (22), but with ν determined at a small value of t, (t^* say, with $1/t^* \gg \Lambda_{\rm QCD}$). In particular it may seem that the value of $\bar{\Lambda}$ obtained from a measurement at "short distances" can be used more reliably to determine some standard

short distance mass in QCD ($m_{\overline{MS}}(\mu)$ say, with $\mu \gg \Lambda_{\rm QCD}$) using perturbation theory. This is not the case however, since in addition to the non-perturbative contribution of $O(\Lambda_{\rm QCD})$ to $\bar{\Lambda}$, there is a perturbative contribution which is proportional to $1/t^*$,

$$-\nu_{\rm pert}(t^*) \equiv \frac{1}{a} \ln \left(\frac{|S^{ii}_{\rm pert}(\vec{x}, t^* + a; \vec{0}, 0)|}{|S^{ii}_{\rm pert}(\vec{x}, t^*; \vec{0}, 0)|} \right)$$

$$= -\frac{1}{a} \frac{\alpha_s C_F}{4\pi} \gamma_\psi \ln\left(1 + \frac{a}{t^*}\right) + O(\alpha_s^2), \qquad (24)$$

$$\to -\frac{\alpha_s C_F}{4\pi} \frac{\gamma_\psi}{t^*} + O(\alpha_s^2) \qquad (\text{as } a \to 0), \qquad (25)$$

where $S_{\rm pert}$ is the heavy quark propagator in perturbation theory, which for $t > 0$ takes the form

$$S^{ij}_{\rm pert}(\vec{x}, t; \vec{0}, 0) = \delta^{(3)}(\vec{x})\, \delta^{ij} \left[1 - \frac{\alpha_s C_F}{4\pi} \left(\gamma_\psi \ln(t/a) + c_t \right) + \ldots \right] \qquad (26)$$

and the anomalous dimension of the heavy quark field (γ_ψ) and c_t are constants (in the Landau gauge $\gamma_\psi = -6$). C_F is the eigenvalue of the quadratic Casimir operator in the fundamental representation ($C_F = {}^4/_3$). In order to determine $\bar{\Lambda}$, from the propagator computed at a finite value of t^*, the perturbative contribution must be subtracted. The evaluation of the term proportional to $1/t^*$ in perturbation theory (which in practice can only be performed up to some low order), becomes less accurate as t^* is decreased. The reason is that, although the calculation of the coefficient of the term proportional to $1/t^*$ becomes more accurate as t^* decreases, the presence of the factor $1/t^*$ implies that the subtraction becomes larger numerically and that the error due to (unknown) higher order pertubative corrections also increases, reducing the accuracy of the result for $\bar{\Lambda}$. For this reason we propose to define δm from Eq. (21), *i.e.* from measurements of the propagator at large values of t. In some simulations it may not be possible to compute the propagators at sufficiently large values of t for a plateau to be reached (*i.e.* for the ratio of the propagators on the right hand side of Eq. (21) to be independent of t). In those cases it may be necessary to determine m_Q^S from measurements taken at intermediate values of t, and to perform the subtraction of the terms proportional to $1/t$, either by using perturbation theory or by fitting $-\nu$ to a function of t and extracting the asymptotic value (*i.e.* the value as $t \to \infty$). In the following Subsection we show that the latter method can be used to give a precise determination of $\bar{\Lambda}$, and hence of m_Q^S.

2.3. Numerical results for $\bar{\Lambda}$

In this Subsection we present our results for $\bar{\Lambda}$ [6]. We have determined ν from simulations on a $16^3 \times 32$ lattice at $\beta = 6.0$ (300 gluon configurations) and on a $20^3 \times 32$ lattice at $\beta = 6.2$ (50 gluon configurations). In order to obtain $\bar{\Lambda}$ using Eq. (22), we have combined these results for ν, with those for $\mathcal{E}$ obtained by the APE collaboration at the same values of the lattice spacing [24, 25]. All these simulations are performed using the $O(a)$ improved SW-Clover action [26], for which the errors due to the lattice spacing are reduced from $O(a)$ to $O(\alpha_s a)$ [27].

In Figs 1 and 2 we plot the results obtained for $a\nu(t)$, defined by

$$a\,\nu(t) = -\ln\left(\frac{|S_h^{ii}(\vec{x}, t+a; \vec{0}, 0)|}{|S_h^{ii}(\vec{x}, t; \vec{0}, 0)|}\right), \tag{27}$$

as a function of the time. ν is the asymptotic value of $\nu(t)$ at large times. We average the propagators in Eq. (27) over all values of $\vec{x}$, which improves the statistical accuracy. In order to determine the asymptotic value a fit of the results for $\nu(t)$ has to be made. We have tried a number of theoretically motivated forms including $\nu(t) = \nu + \gamma/t$ (as suggested by perturbation theory), and $\nu(t) = \nu + \gamma' \ln((t+a)/t)/a$ [6]. The asymptotic value is stable against the different fitting functions, and we include the spread of results in our quoted error. Our results for the mass counterterm are

$$a\nu = 0.520 \pm 0.006 \pm 0.010 \quad \text{at} \quad \beta = 6.0\,, \tag{28}$$

$$a\nu = 0.445 \pm 0.008 \pm 0.010 \quad \text{at} \quad \beta = 6.2\,, \tag{29}$$

where in each case the first error is statistical, and the second is systematic, being estimated by the spread of results obtained with different fitting functions and performing the fits using different ranges in t.

We now combine these results for ν with those for the bare binding energy $\mathcal{E}$ from the APE collaboration [24, 25], to determine $\bar{\Lambda}$ using Eq. (22). The APE collaboration find $a\mathcal{E} = 0.61(1)$ at $\beta = 6.0$ and $0.52(1)$ at $\beta = 6.2$. In order to convert the values of $\bar{\Lambda}$ obtained in this way in lattice units to those in physical units we must determine the lattice spacing a. In quenched simulations there is typically an $O(10\%)$ uncertainty in this, corresponding to the spread of values one obtains from using different physical quantities to set the scale. For this talk we will take $a^{-1} = 2.0(2)$ GeV at $\beta = 6.0$ and $a^{-1} = 2.9(3)$ GeV at $\beta = 6.2$. Thus we obtain

$$\bar{\Lambda} = 180 \pm 35 \text{ MeV} \quad \text{at} \quad \beta = 6.0\,, \tag{30}$$

$$\bar{\Lambda} = 220 \pm 55 \text{ MeV} \quad \text{at} \quad \beta = 6.2\,. \tag{31}$$

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[6] The fits obtained with this latter form are shown in Figs 1 and 2.

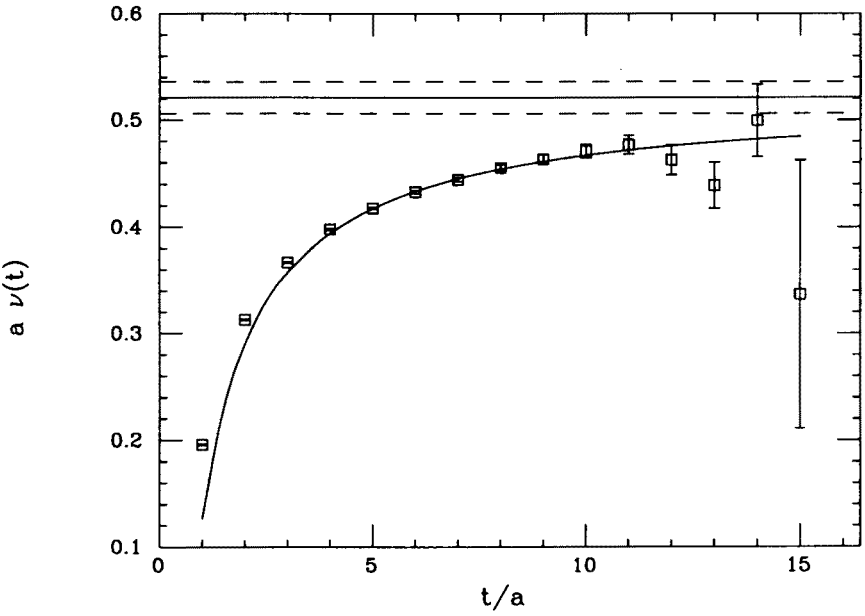

Fig. 1. $\nu(t)$ as a function of t from the simulation at $\beta = 6.0$. The fit shown in this figure is to the function $\nu(t) = \nu + \gamma' \ln((t+a)/t)/a$, where γ' is a parameter.

Combining these two results, we take as our best estimate of $\bar{\Lambda}$

$$\bar{\Lambda} = 190 \pm 30 \text{ MeV}. \tag{32}$$

This value is lower than many used in phenomenological applications of the HQET, however it is the value corresponding to the definition of $\bar{\Lambda}$ we use.

The results obtained above are fully consistent with the preliminary ones from 36 configurations on a $16^3 \times 32$ lattice at $\beta = 6.0$ which have been presented in Ref. [7], ($a\nu = 0.50 \pm 0.01 \pm 0.02$ to be compared with Eq. (28) above).

Finally in this section we make some remarks about the numerical results, stressing in particular the non-perturbative nature of the effects.

(i) The contribution from one-loop perturbation theory to $a\nu$ is about $2.12\alpha_s$, which is significantly lower than the values determined non-perturbatively (Eqs (28) and (29)). For example, we estimate this contribution using the boosted coupling $\alpha_s = 6/4\pi\beta u_0^4$, where u_0 is a measure of the average link variable, for which we take $u_0 = 1/8\kappa_{\rm crit}$, where $\kappa_{\rm crit}$ is the value of the hopping parameter at which the light quarks are massless [28]. With this value of the coupling constant, the one loop contribution to $a\nu$ is about 0.31 (0.28) at $\beta = 6.0$ (6.2), to be

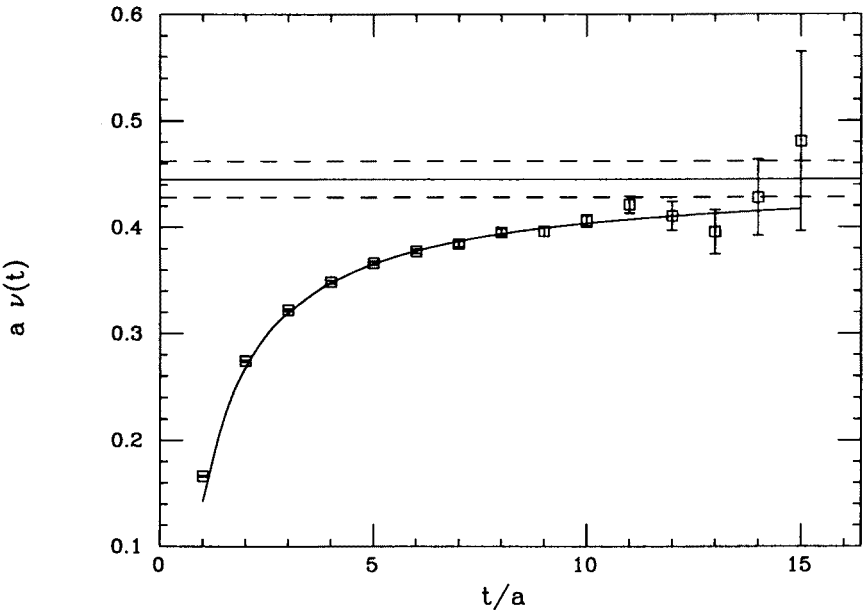

Fig. 2. $\nu(t)$ as a function of t from the simulation at $\beta = 6.2$. The fit shown in this figure is to the function $\nu(t) = \nu + \gamma' \ln((t+a)/t)/a$, where γ' is a parameter.

compared to the values in Eqs (28) and (29). Other standard choices for the coupling constant give similar values. We also note that the dependence of the measured values of $a\nu$ on the lattice spacing in Eqs (28) and (29) is stronger than would have been expected from perturbation theory.

(ii) The FNAL group of Duncan et al. has studied the bare binding energy $\mathcal{E}$ as a function of the lattice spacing, and their results are consistent with a linear dependence of the form [29]:

$$a\mathcal{E}(a) = X + a\bar{\Lambda}_{\rm FNAL}\,. \tag{33}$$

The value these authors obtain for $\bar{\Lambda}_{\rm FNAL}$ is 0.481(25) GeV, which is almost 300 MeV larger than that for $\bar{\Lambda}$ in Eq. (32). We interpret the difference as being due to different definitions of $\bar{\Lambda}$, which differ by finite terms of $O(\Lambda_{\rm QCD})$. Our definition was designed to facilitate the matching of the effective theory and full QCD (see Section 4 below). This also underlines the obvious point that before using a value of $\bar{\Lambda}$ from some determination, the matching procedure must be consistent with the definition of $\bar{\Lambda}$ which is being used.

3. The Kinetic energy — λ_1

In this section we present our general proposal for the elimination of power divergences and renormalons from the matrix elements of higher dimensional operators. We illustrate our method by considering explicitly the kinetic energy operator $\bar{h}\vec{D}^2 h$.

3.1. Definition of the subtracted operator $\bar{h}\vec{D}^2_S h$

The matrix elements of the operator $\bar{h}\vec{D}^2 h$ contain power divergences because it can mix with the lower dimensional operators $\bar{h}D_4 h$ and $\bar{h}h$ [7]. A subtracted kinetic energy operator, one which is free of power divergences, is of the form

$$\bar{h}\vec{D}^2_S h = \bar{h}\vec{D}^2 h - \frac{c_1}{1+\delta m\, a}(\bar{h}D_4 h + \delta m \bar{h}h) - c_2 \bar{h}h\,, \tag{34}$$

where the constants c_1 and c_2 are fixed by imposing appropriate renormalisation conditions. We propose to define c_1 and c_2 by imposing that the matrix element of $\bar{h}\vec{D}^2_S h$ between quark states with $\vec{k} = 0$ (where k is the momentum of the quark), and in the Landau gauge, vanishes

$$\langle h(\vec{k}=0)|\bar{h}\vec{D}^2_S h|h(\vec{k}=0)\rangle = 0\,. \tag{35}$$

Although not unique, this is perhaps the most intuitive definition of the kinetic energy of the heavy quark in a hadron. Specifically we determine the constants c_1 and c_2 by using[8]

$$c_1 + c_2 t_x = \frac{\sum_{\vec{x},\vec{y}} \sum_{t_y=0,t_x} \langle\, 0 \,|\, h(\vec{x},t_x)\, \bar{h}(\vec{y},t_y) \vec{D}^2_y h(\vec{y},t_y)\, \bar{h}(\vec{0},0) \,|\, 0\,\rangle}{\sum_{\vec{x}} \langle\, 0 \,|\, h(\vec{x},t_x) \bar{h}(\vec{0},0) \,|\, 0\,\rangle}\,. \tag{36}$$

In deriving Eq. (36) it is implied that we are using the action $\mathcal{L}'$ given in Eq. (19) which contains the residual mass counterterm. However, from the discussion in the previous section we can readily see that this is equivalent to using the action $\mathcal{L}$ given in Eq. (12) which has no residual mass term. The only difference in using the actions $\mathcal{L}'$ and $\mathcal{L}$ is a factor $\exp(\nu t_x)$ in both the numerator and denominator of the right hand side of Eq. (36).

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[7] The relation between power divergences in matrix elements of the particular operator $\bar{h}\vec{D}^2 h$ and renormalon ambiguities in dimensional regularisation is subtle and not fully understood. A detailed discussion of this subject can be found in Ref. [20].

[8] From now on we will work in lattice units, setting $a = 1$.

For some important applications it is only the constant c_2 which is required. This is because

$$\frac{1}{1+\delta m}(\bar{h}D_4 h + \delta m \bar{h}h)S'_h(x;y) = \delta^4(x-y)\,, \tag{37}$$

so that if the operators in a correlation function are separated (*i.e.* up to contact terms), then the term proportional to c_1 vanishes. c_2 can also be determined directly by eliminating the sum over t_y in Eq. (36):

$$c_2 = \frac{\sum_{\vec{x},\vec{y}} \langle\, 0 \,|\, h(\vec{x},t_x)\, \bar{h}(\vec{y},t_y)\vec{D}_y^2 h(\vec{y},t_y)\, \bar{h}(\vec{0},0) \,|\, 0 \,\rangle}{\sum_{\vec{x}} \langle\, 0 \,|\, h(\vec{x},t_x)\bar{h}(\vec{0},0) \,|\, 0 \,\rangle}\,, \tag{38}$$

for $t_y \neq 0, t_x$.

Having defined the subtracted operator $\bar{h}\vec{D}_s^2 h$, λ_1 can be determined from a computation of two- and three-point correlation functions in the standard way. Consider the meson three-point correlation function (the extension of this discussion to baryons is entirely straightforward)

$$C_{\vec{D}_S^2}(t_x,t_y) = \sum_{\vec{x}} \langle 0|J_\Gamma(\vec{x},t_x)\; \bar{h}(\vec{y},t_y)\vec{D}_S^2 h(\vec{y},t_y) J_\Gamma^\dagger(\vec{0},0)|0\rangle\,. \tag{39}$$

For sufficiently large values of t_y and $t_x - t_y$

$$C_{\vec{D}_S^2}(t_x,t_y) \;\to Z^2\lambda_1 \exp\left(-(\mathcal{E}-\nu)t_x\right)\,, \tag{40}$$

where λ_1 is defined by Eq. (2), using the subtracted kinetic energy operator,

$$\lambda_1 = \frac{\langle H|\bar{h}\vec{D}_S^2 h|H\rangle}{2m_H}\,, \tag{41}$$

and H is the lightest meson state which can be created by the operator $J_\Gamma^\dagger$. A convenient way to extract λ_1 is to consider the ratio

$$R(t_x,t_y) = \frac{C_{\vec{D}_S^2}(t_x,t_y)}{C(t_x)} \to \lambda_1\,. \tag{42}$$

As usual λ_1 must be evaluated in an interval in which $R(t_x,t_y)$ is independent of the times t_y and t_x, so that the contribution from the excited states can be neglected. By the same argument as was given after Eq. (36), the ratio in Eq. (42) can be evaluated using the action $\mathcal{L}$ of Eq. (12) with no

residual mass term. In the present case the difference between the matrix elements of the subtracted and unsubtracted operators is given by

$$\lambda_1 \equiv \frac{\langle H|\bar{h}\vec{D}^2_S h|H\rangle}{2m_H} = \frac{\langle H|\bar{h}\vec{D}^2 h|H\rangle}{2m_H} - c_2\,. \tag{43}$$

We conclude this section by presenting the definition of the subtracted quark mass up to, and including the terms of $O(1/m_Q)$,

$$\tilde{M}_B = m^S_Q + \mathcal{E} - \nu - \frac{\lambda_1}{2m^S_Q}\,, \tag{44}$$

where $\tilde{M}_B$ is the spin-averaged mass, $\tilde{M}_B = (M_B + 3M_{B^*})/4$, which has no contribution from the chromomagnetic operator. Eq. (44) must be modified to include the effects of perturbative corrections. We denote the renormalised kinetic energy operator in some continuum renormalisation scheme by $\bar{h}D^2_{\rm cont}h$. In one loop perturbation theory we have

$$\bar{h}D^2_{\rm cont}h = \left(1 + \frac{\alpha_s}{4\pi}X_{\vec{D}^2_S}\right)\bar{h}D^2_S h\,, \tag{45}$$

from which we derive

$$\tilde{M}_B = m^S_Q + \mathcal{E} - \nu - \left(1 + \frac{\alpha_s}{4\pi}X_{\vec{D}^2_S}\right)\frac{\lambda_1}{2m^S_Q}\,. \tag{46}$$

The term proportional to $X_{\vec{D}^2_S}$ in Eq. (46) is absent in continuum formulations of the HQET, and is a manifestation of the lack of reparametrisation invariance in the lattice version. It has been calculated in Ref. [4].

Although we have restricted our explicit discussion to the kinetic energy operator, clearly the same techniques can be applied to a wide class of operators. This includes, for example, the operators whose matrix elements determine the $1/m_Q$ corrections to exclusive leptonic and semileptonic decays of B-mesons [12]. In each case one can construct linear combinations of higher and lower dimensional operators which are free of power divergences and renormalon ambiguities, by imposing appropriate normalisation conditions for matrix elements between quark states in a fixed gauge, and at given momenta. This approach is a particular application of the general method for the non-perturbative normalisation of lattice operators proposed in Ref. [30].

3.2. Preliminary numerical results for λ_1

Our numerical results for λ_1 are not as precise as for the binding energy $\bar{\Lambda}$. We have performed an exploratory computation using 36 gauge field configurations on a $16^3 \times 32$ lattice at $\beta = 6.0$, and the preliminary results were presented in Ref. [7]. For the determination of λ_1 it is only the constant c_2 which is required (see Eq. (43) above) for which we find

$$a^2 c_2 = -0.82 \pm 0.12\,, \tag{47}$$

where we have exhibited explicitly the powers of the lattice spacing. For the bare kinetic energy of a pseudoscalar meson we find

$$a^2 \frac{\langle H|\bar{h}\vec{D}^2 h|H\rangle}{2m_H} = -0.75 \pm 0.15\,, \tag{48}$$

from which $a^2 c_2$ (see Eq. (47)) has to be subtracted in order to obtain λ_1. Thus the subtraction proves to be a very large one, and at present we are only able to determine the loose upper bound

$$|\lambda_1| < 1.0\,\mathrm{GeV}^2\,. \tag{49}$$

This calculation will be repeated with several hundred configurations in order to decrease the bound significantly, or to obtain a signal for λ_1.

4. Matching

In the preceding sections we have proposed a method for defining higher-dimensional operators $O^{\mathrm{HQET}}_{n,\alpha}$, whose matrix elements are free of (ultraviolet) renormalon ambiguities and power divergences, and which can be computed in lattice simulations. In order to derive physical predictions from these matrix elements it is necessary to calculate the corresponding coefficient functions, *i.e.* the $C_{n,\alpha}$ of Eq. (1). The subtraction of the ultraviolet renormalons from the matrix elements, implies the elimination of the corresponding infrared renormalons from the coefficient functions, which can therefore be computed in perturbation theory [22, 31, 32]. To illustrate the "matching" procedure consider a simple situation for which we can write Eq. (1) as:

$$\begin{aligned} O^{\mathrm{QCD}}(M) = C_1\left(\frac{m_Q}{M}, \frac{\Lambda}{M}\right) O_1^{\mathrm{HQET}}(\Lambda) \\ + \frac{1}{m_Q} C_2\left(\frac{m_Q}{M}, \frac{\Lambda}{M}\right) O_2^{\mathrm{HQET}}(\Lambda) + O\left(\frac{1}{m_Q^2}\right)\,, \end{aligned} \tag{50}$$

i.e., where there is a single operator in each of the first two terms of the heavy quark expansion (the discussion can readily be extended to the general case). In Eq. (50) we have exhibited explicitly the dependence on the scale M used to normalize the QCD operators $O^{\rm QCD}(M)$ (*i.e.* the operators in the full theory). Λ is the (hard) ultra-violet cut-off in the effective theory (for example in the lattice formulation of the HQET $\Lambda = a^{-1}$), and the operators on the right-hand side of Eq. (50) are bare operators. m_Q can be considered as the subtracted pole mass m_Q^S defined in Section 2.

Note that although we have chosen to write Eq. (50) in terms of the bare operators in the effective theory, this equation is equivalent to Eq. (1), which is written using renormalised operators. From the bare operators (computed in lattice simulations for example), one can determine the corresponding ones renormalised in a given (continuum) scheme by using perturbation theory, or by some non-perturbative method (such as that given in Ref. [30]). The determination of the matrix elements of renormalised operators of the effective theory may be a convenient intermediate step, but it is not necessary. We choose instead to compute the matrix elements of the QCD operators $O^{\rm QCD}$ directly from the bare operators of the effective theory.

With a hard cut-off the Borel transforms of the coefficient function C_1 and the matrix elements of $O_2^{\rm HQET}$ do not have renormalon singularities at $u = {}^1/_2$. For example, in the large N_f limit, the Borel transform of C_1 in the vicinity of $u = 1/2$ will have the structure

$$\tilde{C}_1(\frac{m_Q}{M}, \frac{\Lambda}{M}, u) \propto \frac{1}{(1-2u)} \left[\left(\frac{m_Q}{M}\right)^{-2u} - \frac{\Lambda}{m_Q} \left(\frac{\Lambda}{M}\right)^{-2u} \right], \qquad (51)$$

where the tilde denotes the Borel transform. Although the residue of the pole vanishes at $u = {}^1/_2$, both C_1 and the matrix elements of $O_2^{\rm HQET}$ contain terms which, in perturbation theory, diverge linearly with the ultra-violet cut-off Λ. The use of a hard factorisation scale to organize the heavy quark expansion has been suggested by Bigi *et al.* [3] (see also Ref. [33] for a very recent study of the definition of the higher twist operators relevant for studies of deep inelastic structure functions). Our proposal is to subtract these power divergences non-perturbatively, and to use the subtracted operators, which are free of power divergences and renormalon ambiguities, as the basis for the expansion in Eq. (1).

Following the discussion in the preceding sections, we define a subtracted operator $O_{2;S}^{\rm HQET}$ which does not mix with $O_1^{\rm HQET}$,

$$O_{2;S}^{\rm HQET}(\Lambda) = O_2^{\rm HQET}(\Lambda) - c(\Lambda) O_1^{\rm HQET}(\Lambda), \qquad (52)$$

where the dependence of the matrix elements of $O_{2;S}^{\rm HQET}(\Lambda)$ on Λ is at most logarithmic. $c(\Lambda)$ is computed non-perturbatively, whereas the functions C_1 and C_2 are calculated in perturbation theory. Eq. (50) can be rewritten as:

$$O^{\rm QCD}(M) = \left[C_1\left(\frac{m_Q}{M},\frac{\Lambda}{M}\right) + \frac{c(\Lambda)}{m_Q}C_2\left(\frac{m_Q}{M},\frac{\Lambda}{M}\right)\right] O_1^{\rm HQET}(\Lambda) + \frac{1}{m_Q}C_2\left(\frac{m_Q}{M},\frac{\Lambda}{M}\right) O_{2;S}^{\rm HQET}(\Lambda) + O\left(\frac{1}{m_Q^2}\right). \quad (53)$$

By using a combination of perturbative and non-perturbative techniques, we have ensured that the coefficient of $O_1^{\rm HQET}$ in Eq. (53) does not contain terms which diverge linearly with Λ, nor ambiguities due to renormalon singularities at $u = {}^1/_2$. The remaining ambiguities are of $O(\Lambda_{\rm QCD}^2/m_Q^2)$ or less, and are associated with the renormalons at $u = 1, {}^3/_2 \cdots$, or the corresponding power divergences in matrix elements of operators of higher dimension than $O_2^{\rm HQET}$. These can be eliminated by generalizing our procedure to higher orders of the heavy quark expansion, by defining subtracted operators $O_{3;S}^{\rm HQET}$, $O_{4;S}^{\rm HQET}$ $\cdots$ which cannot mix with lower dimensional ones, and using these subtracted operators as the basis of the expansion in Eq. (1). Indeed by using such a basis one eliminates the ambiguities from all coefficient functions, up to the order in $1/m_Q$ for which the subtraction coefficients have been computed. Of course in general the coefficient function of $O_1^{\rm HQET}$ depends logarithmically on Λ, the dependence being given by the anomalous dimension of $O_1^{\rm HQET}$.

Throughout this talk we have been stressing the necessity of performing the subtractions of power divergences non-perturbatively. We now present a specific example demonstrating this explicitly. Consider the evaluation of the subtraction constant $c(\Lambda)$ in the case for which $O_1^{\rm HQET} = \bar{h}h$ and $O_2^{\rm HQET} = \bar{h}\, v\cdot D\, h$ as happens in the expansion of the QCD operator $\bar{Q}Q$, where Q is the field of the heavy quark in the full theory, and in the discussion of $\bar{\Lambda}$ in Section 2.2. The Borel transform of the forward matrix element of $O_2^{\rm HQET}$ between heavy quark states of momentum k ($T = \langle h(k)|\,\bar{h}v\cdot Dh\,|h(k)\rangle$), in the vicinity of $u = {}^1/_2$, has the structure

$$\tilde{T}(u) \propto \frac{1}{(1-2u)}\left[-2v\cdot k\left(\frac{-2v\cdot k}{\Lambda}\right)^{-2u} - \Lambda\right]. \quad (54)$$

The residue of the pole vanishes at $u = {}^1/_2$, in a similar way to that in the coefficient function C_1 in Eq. (51). The two terms in the square brackets in Eq. (54), contribute equal and opposite ambiguous terms of $O(\Lambda_{\rm QCD})$ to

the matrix element of $O_2^{\rm HQET}$ at $v \cdot k = 0$. In perturbation theory however, the first term does not appear at $v \cdot k = 0$, and hence the Borel transform of $c(\Lambda)$ contains a renormalon at $u = {}^1/_2$. It is for this reason that we insist on the non-perturbative determination of the subtraction coefficients.

One can also imagine performing the matching completely non-perturbativel by simulating both the HQET and QCD (with a bottom quark) on the lattice. However the latter requires a very small lattice spacing, $a^{-1} \gg m_b$, which will not be possible for some time to come. Moreover, once one is able to simulate the b-quark directly (and reliably) on the lattice, the necessity of using the HQET is removed. It may still however, be a useful guide to scaling properties and symmetry relations.

5. Conclusions

In this talk we have reviewed the method proposed in Ref. [5] for defining higher dimensional operators of the HQET, in such a way that their matrix elements are free of ambiguities due to (ultra-violet) renormalon singularities, and of power divergences. We have illustrated our approach by proposing a physical definition of the binding energy $\bar{\Lambda}$ $(= m_H - m_Q^S)$ and of the matrix elements of the subtracted kinetic energy operator $(\langle H|\vec{D}_S^2|H\rangle)$. The definition of the higher dimensional operators involves the subtraction of lower dimensional ones with the same quantum numbers. The subtraction coefficients are determined by imposing normalisation conditions on Green functions between quark and gluon states (or in the case of $\bar{\Lambda}$, on the heavy quark propagator). Lattice simulations of the HQET allow for a numerical evaluation of the subtraction coefficients, as well as of the matrix elements of the subtracted operators, and we have presented a determination of $\bar{\Lambda}$ and a bound for λ_1 [6]. The renormalisation procedure proposed above can, however, be applied with any other non-perturbative method for computing matrix elements in effective theories. Our approach can also be extended to other cases, for example to the higher twist contributions to the structure functions of deep inelastic scattering, and to the higher dimensional operators which appear in QCD sum-rules.

Having defined operators $O_{n,\alpha}^{\rm HQET}$ whose matrix elements are free of renormalon ambiguities, we still have to match the HQET operators onto those of QCD, *i.e.* to determine the coefficient functions $C_{n,\alpha}$ of Eq. (1). This is done using perturbation theory, by calculating the matrix elements of $O^{\rm QCD}$ and $O_{n,\alpha}^{\rm HQET}$ between suitable external states, and combining the results with the subtraction coefficients which have been computed non-perturbatively (see Section 4 above).

The subtractions necessary to obtain physical matrix elements appear to be substantial. For the binding energy $\bar{\Lambda}$ the subtraction is of a linearly

divergent term, and we were still able to obtain a result with good precision (see Eq. (32) above). For the kinetic energy we need to subtract a quadratically divergent operator, and with 36 configurations we were only able to obtain the bound in Eq. (49).

The procedures described in this talk allow one to study quantitatively many important physical processes and quantities in heavy quark physics using a systematic expansion in the mass of the heavy quark. These include the leptonic and semi-leptonic decays of heavy mesons and baryons, as well as relations between the masses and lifetimes of heavy hadrons. The "subtracted pole mass" is a suitable parameter for the expansion.

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