Vol. 26(1995) *ACTA PHYSICA POLONICA B* No 4

QUANTUM MECHANICS OF THE $b\bar{b}$*†

L. MOTYKA AND K. ZALEWSKI**

Institute of Physics, Jagiellonian University
Reymonta 4, 30-059 Kraków, Poland

(Received February 27, 1995)

We study the properties of nonrelativistic models of $b\bar{b}$ quarkonia. We want to check if the present data can be reproduced by such a model within the experimental errors. A general formula including as special cases many existing models is introduced. The properties of a family of models are investigated. We find a new potential in the form of $a\sqrt{r} - b/r + A$, which is agreed with data.

PACS numbers: 12.39. Jh

1. Introduction

There is no entirely satisfying theory of heavy mesons. Methods like the lattice QCD or QCD sum rules give only approximate results. One can also look at heavy quarkonia by means of nonrelativistic quantum mechanics. This approach is very simple and appears to be successful with predictions of mass spectrum and decay widths.

Models proposed recently by some authors, make it possible to fit the mass spectrum of bottomonium within the precision of about 2.3 MeV per level [1, 2]. It is much better than expected from estimates of the necessary relativistic and field theory corrections, which are of order of some tens of MeV. The standard explanation of this fact, is the absorptions of the corrections by redefinition of physical parameters of the system and using an effective Hamiltonian. In the infinite quark mass limit, the effective potential should be identical with the physical static interquark potential.

---

* Presented at the Cracow Epiphany Conference on Heavy Quarks, held in honour of the 60th birthday of Kacper Zalewski, Kraków, Poland, January 5–6, 1995.

† Supported in part by the KBN grant 2P30207607.

** Also at the Institute of Nuclear Physics, Kraków, Poland.

For heavy mesons this equivalence is also present, but only approximately. Thus, we believe, that good understanding of the problem, will let us learn something about the relativistic theory. Especially, we are interested, if the present set of observables may be fitted consistently by a nonrelativistic model, within its experimental errors. If a general failure of different models happens, one could extract from the results some hints on how to construct the relativistic model.

2. The observables

This paper is based on a set of experimental data for bottomonia [3], which can be easily expressed in terms of quantum mechanics. We know with good precision, the masses of five $b\bar{b}$ states below the $B\overline{B}$ threshold. It is hard to describe the physics over this threshold in the language of the QM, because of strong coupled channel effects and of the related mixing the states with the $B\overline{B}$ continuum (Ref. *e.g.* [4]). The P states have well separated fine structures. Their splitting is a purely relativistic effect. To get rid of it, we use the centre-of-gravity masses.

Other observables, which can be fitted, are electric dipole transition widths (between S and P states), associated with dipole matrix elements. They also should be averaged over the fine structure. From leptonic decay widths of the three S states we can extract the absolute values of the wave functions at the origin. We use here the Van Royen–Weisskopf formula [5] with the first order QCD correction. The correction factor to the zero-order value of the leptonic decay width is $1 - (16\alpha_s/3\pi)$, which is about 0.7 at the m_b scale [3]. It is significantly different from 1, so we can expect that unknown, higher order corrections are also important. This theoretical uncertainty introduces a common systematic error, for the extracted S-wave functions at the origin, which is about 10% .

3. The model

The nonrelativistic system of two interacting bodies of mass m_b each can be described by the three dimensional Schrödinger equation:

$$-\frac{\nabla^2}{m_b}\Psi(\vec{r}) + V(r)\Psi(\vec{r}) = E\Psi(\vec{r})\,. \tag{1}$$

Now, we may scale the separation vector $\vec{r}$, and use a dimensionless vector $\vec{x} = \lambda\vec{r}$ instead. We decompose also the potential in a following way: $V(x) = Cv(x) + A$, where C and A are constants, and $v(x)$ is a dimensionless function. The proper choice of λ brings the potential $v(x)$

to its simplest, canonical form. In the case of a central potential we can separate the angular degrees of freedom and make the substitution:

$$\Psi_{nlm}(\vec{x}) = \frac{\varphi_{nl}(x)}{x} Y_{lm}(\Omega), \tag{2}$$

to obtain a one dimensional, reduced equation for the eigenfunctions and the eigenvalues:

$$-\frac{d^2}{dx^2}\varphi_{nl}(x) + \left[cv(x) + \frac{l(l+1)}{x^2}\right]\varphi_{nl}(x) = e_{nl}\varphi_{nl}(x). \tag{3}$$

This equation is much simpler, than the Schrödinger equation, we started with. It contains only one parameter c, instead of the four parameters in equation (1). This reduction does not disturb testing of the agreement of the results with experimental data. This is so, because the observables which we take into account depend on their dimensionless equivalents in a trivial way. So our strategy is to use such quantities built of observables, that they can be calculated from the reduced equation (3). We choose

$$\begin{aligned} b_1 &= \frac{M_{2\mathrm{S}} - M_{1\mathrm{S}}}{M_{3\mathrm{S}} - M_{1\mathrm{S}}}, \qquad b_2 = \frac{M_{3\mathrm{S}} - M_{2\mathrm{P}}}{M_{2\mathrm{S}} - M_{1\mathrm{P}}}, \qquad b_3 = \frac{M_{2\mathrm{S}} - M_{1\mathrm{P}}}{M_{2\mathrm{S}} - M_{1\mathrm{S}}}, \\ b_4 &= \frac{|\Psi_{2\mathrm{S}}(0)|^2}{|\Psi_{1\mathrm{S}}(0)|^2}, \qquad b_5 = \frac{|\Psi_{3\mathrm{S}}(0)|^2}{|\Psi_{1\mathrm{S}}(0)|^2}, \\ b_6 &= |\Psi_{1\mathrm{S}}(0)|^{2/3}\langle 2P|R|3S\rangle, \qquad b_7 = |\Psi_{1\mathrm{S}}(0)|^{2/3}\langle 1P|R|2S\rangle, \\ b_8 &= \frac{\langle 2S|R|2P\rangle}{\langle 1S|R|2P\rangle}. \end{aligned} \tag{4}$$

We also define a χ^2 measure of the quality of the fit

$$\chi^2 = \sum_{i=1}^{8}\left[\frac{b_i^{\mathrm{th}} - b_i^{\mathrm{exp}}}{\sigma(b_i)}\right]^2, \tag{5}$$

where $\sigma(b_i)$ is the experimental error of the parameter b_i. This has two important features — different types of quantities are involved here, and it can be calculated from the reduced equation. So, we can optimize the agreement of the model with experiment, using only the reduced set of parameters, with significant time gain.

There are a few types of potentials used in literature. All of them are monotonically increasing with increasing r and are concave. Most of them are singular at the origin. Some of them are QCD motivated, like the Cornell potential ($v(x) = x - 1/x$) [6], the "Indiana" potential

($v(x) = (x-1)^2/x\ln(x)$) [7] or the famous Richardson potential defined in momentum space [8]. Other, purely phenomenological potentials have usually the form of the difference of two power functions: one increasing and the other decreasing ($v(x) = x^{\alpha} - x^{-\beta}$). Some examples are models proposed by: Lichtenberg *et al.* ($\alpha = 0.75$, $\beta = 0.75$) [9], Song and Lin ($\alpha = 0.5$, $\beta = 0.5$) [10], Martin ($\alpha = 0.1$, $\beta = 0$) [11], Heikkilä *et al.* ($\alpha = 2/3$, $\beta = 1$) [4]. Note that also the Cornell potential belongs to this family ($\alpha = 1$, $\beta = 1$).

We confirm the result of Buchmüller and Tye [12], that the various realistic potentials for quarkonia, become very close to each other in the physically important region. We obtained evidence that QCD motivated potentials (Richardson, "Indiana"), which have not the explicit form of the difference of two power functions, can be successfully approximated by a function of this form (of course only for the intermediate interquark distances). Thus, we focus our attention on our general potential and expect, that it includes a sufficiently wide class of models.

4. Results and discussion

We calculated our dimensionless observables from the experimental data. For each type of potential, we varied the c parameter in our reduced equation, to find the minimum of the χ^2. In particular, we applied this procedure to the general form of the potential with fixed exponents α and β, to obtain the minimal χ^2, as a function of these exponents. The χ^2 map gives us reasons to propose a new potential. It appears, that there exists a small region on our map, where $\chi^2 < 7$. Seven is the number of degrees of freedom of the fit — eight observables (4) minus one free parameter. We have chosen one of the potentials from this region, because of its simplicity. It is

$$v(x) = \sqrt{x} - \frac{1}{x}. \tag{6}$$

TABLE I

The observables from data and models

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	χ^2
exp.	0.6290	0.7738	0.2187	0.49	0.43	2.31	1.59	0.110	
error	0.0005	0.0057	0.0009	0.11	0.07	0.16	0.15	0.009	
This paper	0.6292	0.7744	0.2191	0.49	0.36	2.26	1.37	0.124	6.5
Indiana	0.6299	0.7829	0.2202	0.48	0.36	2.18	1.33	0.124	15.5
Lichtenberg	0.6283	0.7950	0.2172	0.48	0.36	2.19	1.34	0.126	26.3
Richardson	0.6276	0.8106	0.2150	0.47	0.36	2.18	1.34	0.127	74
Song-Lin	0.6382	0.7246	0.2375	0.49	0.35	2.04	1.21	0.112	850
Cornell	0.6128	0.8951	0.1946	0.47	0.37	2.42	1.55	0.142	2220
Martin	0.6363	0.6891	0.2707	0.54	0.38	1.81	1.06	0.032	3720

Having found the best model from comparison with the data, we can go back and calculate the dimensional parameters, by comparing the dimensional observables with the results of reduced equation. From $|\Psi_{1S}(0)|$, $M_{3S} - M_{1S}$ and M_{1S}, we can simply calculate the values of λ, m_b and A. The best potential in real space ($\vec{r}$ in GeV^{-1}, V in GeV) is

$$V(r) = 0.70585(\sqrt{r} - \frac{0.46121}{r}) + 8.81724 \tag{7}$$

and the corresponding mass is

$$m_b = 4.79333 \text{ GeV}. \tag{8}$$

In Table I the predictions of 7 models are compared with the experimental data. As seen from the quoted values of χ^2, only potential (7) yields results consistent with the data.

5. Conclusions

The proposed nonrelativistic potentials for $b\bar{b}$ quarkonia, become very similar to each other, if they fit the data well. The reasonable potentials can be described or successfully approximated by the difference of two power functions. Thanks to trivial scaling properties of all observables, it is possible to reduce the number of necessary parameters of a model, with no loss of generality. Using the χ^2 measure, we studied the quality of the fit in the parameter plane, and we have found the new, simple potential (7) with significantly better χ^2 than the other proposals. We have proved, that an effective, nonrelativistic model can describe the bottomonium system within the experimental errors, although the deviations of the b_7 and b_8 are rather large in all models including ours and may well become a problem when data improves. It is encouraging that studying a purely phenomenological model, without assuming anything about the exponent of the short range part of the potential, we obtained the Coulomb-like behaviour of the potential near the origin. Such a behaviour corresponds to one-gluon exchange. The shape of the long distance potential is not linear, but square root like. This is probably due to the fact that the wave functions test only the region below 1 fm and the linear confinement is not seen there.

REFERENCES

[1] D. Besson, T. Skwarnicki, *Ann. Rev. Nucl. Part. Sci.* **43**, 333 (1993).
[2] L.P. Fulcher, *Phys. Rev.* **D42**, 2337 (1990).
[3] Particle Data Group, L.S. Brown *et al.*, *Phys. Rev.* **D50**, 1173 (1994).

[4] K. Heikkilä, N.A. Törnqvist, S. Ono, *Phys. Rev.* **D29**, 110 (1984); *Z. Phys* **C23**, 59 (1984).
[5] R. Van Royen, V. Weisskopf, *Nuovo Cimento* **A50**, 617 (1967).
[6] F. Eichten, K. Gottfried, T. Kinoshita, K.D. Lane, T.M. Yan, *Phys. Rev.* **D21**, 203 (1980).
[7] G. Fogelman, D.B. Lichtenberg, J.G. Wills, *Lett. Nuovo Cimento* **26**, 369 (1979).
[8] J.L. Richardson, *Phys. Lett.* **82B**, 272 (1979).
[9] D.B. Lichtenberg *et al.*, *Z. Phys.* **C41**, 651 (1989).
[10] X.T. Song, H. Lin, *Z. Phys.* **C34**, 223 (1987).
[11] A. Martin, *Phys. Lett.* **B100**, 511 (1981); **B93**, 338 (1980).
[12] W. Buchmüller, S.-H.H. Tye, *Phys. Rev.* **D24**, 132 (1981).