AN EMBEDDING OF A SCHWARZSCHILD BLACK HOLE IN TERMS OF ELEMENTARY FUNCTIONS

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We give a six dimensional embedding for the Schwarzschild solution which is described by elementary functions only.

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1. Introduction

The flat embeddings of physically relevant solutions of the Einstein equations are of interest for many reasons. When one succeeds in constructing an universal analytic embedding, then the natural topology of the space-time becomes explicit, and the geometry of the Riemannian space acquires a simple interpretation of the geometry of some hyper-surface in a higher dimensional flat space.

In particular, by studying the six-dimensional embedding of a Schwarzschild black hole, it has been shown ([1] and in much more detail [2]) that the horizon, at $r = 2m$, is not a physical singularity of the solution, establishing in this way a result equivalent to the Kruskal coordinatization [3].

The embedding technique used in [1] and [2], employed the simple Kasner idea [4], which can be summarized in its essence as follows. Consider the 2-dimensional Riemannian space

$$g = \phi^{-2}(x)\left[dx \otimes dx + \epsilon dy \otimes dy\right], \quad \epsilon^2 = 1\,. \tag{1.1}$$

One can construct its 3-dimensional embedding using one of the two following procedures.

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According to the first, one introduces the variables

$$p := \lambda^{-1}\phi^{-1}(x)\cos(\lambda y)\,, \quad q := \lambda^{-1}\phi^{-1}(x)\sin(\lambda y)\,, \tag{1.2}$$

where $\lambda \neq 0$ is a constant. Then, with

$$dp = -\phi^{-1}\sin(\lambda y)dy \ + \lambda^{-1}\cos(\lambda y)[\phi^{-1}(x)]_x dx\,, \tag{1.3a}$$
$$dq = \phi^{-1}\cos(\lambda y)dy + \lambda^{-1}\sin(\lambda y)[\phi^{-1}(x)]_x dx\,, \tag{1.3b}$$

we have that

$$(dp)^2 + (dq)^2 = \phi^{-2}(dy)^2 + (\lambda^{-1}[\phi^{-1}(x)]_x)^2 dx^2 \tag{1.4}$$

so that the metric becomes

$$g = [\phi^{-2} - \epsilon(\lambda^{-1}[\phi^{-1}]_x)^2]\,(dx)^2 + \epsilon((dp)^2 + (dq)^2)\,. \tag{1.5}$$

Therefore, in the regions where the coefficient of $(dx)^2$ has a definite sign, we can introduce the variable

$$r := \int \sqrt{|\phi^{-2}(x) - \epsilon\lambda^{-2}[(\phi^{-1}(x))_x]^2|}\ dx\,. \tag{1.6}$$

Thus we reduce the metric to the 3-dimensional flat form

$$g = \pm dr \otimes dr + \epsilon[dp \otimes dp + dq \otimes dq]\,, \tag{1.7}$$

having at the same time

$$p^2 + q^2 = \lambda^{-2}\phi^{-2}(x)\,. \tag{1.8}$$

Inverting (1.6) for $x = \chi(r)$, we obtain the 3-dimensional embedding

$$\begin{cases} g = \pm dr \otimes dr + \epsilon[dp \otimes dp + dq \otimes dq]\,, \\ p^2 + q^2 = \lambda^{-2}\phi^{-2}[\chi(r)]\,. \end{cases} \tag{1.9}$$

The free constant λ can then be chosen so that the integral (1.6) is as simple as possible, for a given $\phi = \phi(x)$.

The alternative variant of the Kasner method employs hyperbolic functions. It may be reached from the previous treatment in terms of trigonometric functions with the formal replacements

$$\lambda \to i\lambda\,, \quad p \to ip\,, \quad q \to q\,, \quad \phi \to \phi\,.$$

As far as we know, this Kasner method, in its trigonometric or hyperbolic version, is the only known technique which is able to provide us with the *effective* embeddings of the solutions to the Einstein equations.

In the case of the Schwarzschild black hole, as shown in detail in [2], this technique works fairly well. Using the hyperbolic variant, one is able to choose λ so that the apparent singularity related with the horizon $r = 2m$ disappears in the integral (1.6) and the embedding is universal and analytic. However, this embedding involves elliptic functions. This author has (in principle) no aversion towards elliptic functions, which certainly are manageable objects, as in particular has been demonstrated in the present context in [2]. We strongly feel, however, that in general relativity, where we have the freedom of choosing convenient coordinates, any time transcendental functions occur, the reason of this is more related to the unskillful choice of the coordinates, then to the very nature of the things involved.

[For instance, compare the Kinnersely description of the general branch of the D-type solutions [5] of the homogeneous Einstein equations in terms of elliptic functions, with the same result obtained much more simply in terms of the rational functions in [6].]

Guided by this point of view, the main objective of this note is to provide *explicitly* the embedding of the Schwarzschild solution in terms of *elementary* functions. This implies, of course, devicing an embedding technique *different* from the Kasner types. A technique satisfactory for our purposes will be outlined in the following section.

2. Flat embeddings of conformally flat spaces

Consider a conformally flat Riemannian space CV_n (real) with the metric given in terms of the (privileged) coordinates $\{x^\alpha\}$ $(\alpha = 1, 2, \cdots, n)$ by

$$g = \phi^{-2}\left(\frac{x^\mu}{\Lambda}\right)\eta_{\mu\nu}dx^\mu \otimes dx^\nu \tag{2.1}$$

with $\Lambda = \text{const} \neq 0$ having the dimension of length, $\phi = \phi(x^\alpha/\Lambda)$, a dimensionless function [7]. The coordinates x^μ have the dimension of length, and the flat metric $\eta_{\mu\nu}$ is

$$\|\eta_{\mu\nu}\| = \|\text{diag}\,(\underbrace{1, \cdots 1}_{n_+},\ \underbrace{-1, \cdots -1}_{+\ n_-\ =n})\,\|\,. \tag{2.2}$$

There is a natural (canonical) procedure of embedding the metric (2.1) in $n+2$ flat dimensions, which is purely *algebraic*. (See [8] for details; here we shall state only the main ideas.) Indeed, introduce the coordinates

$$\xi^\mu := \phi^{-1}\left(\frac{x^\alpha}{\Lambda}\right)x^\mu\,; \tag{2.3}$$

then

$$\eta_{\mu\nu} d\xi^\mu \otimes d\xi^\nu = \phi^{-2}\eta_{\mu\nu} dx^\mu \otimes dx^\nu + d\phi^{-1} \otimes_s d[\phi^{-1}\eta_{\mu\nu}x^\mu x^\nu]. \quad (2.4)$$

Using this fundamental identity, we have that

$$g = \eta_{\mu\nu} d\xi^\mu \otimes d\xi^\nu - d\phi^{-1} \otimes_s d\big[\phi^{-1}\eta_{\mu\nu}x^\mu x^\nu\big]. \quad (2.5)$$

Introduce now the coordinates ξ^{n+1} and ξ^{n+2} via

$$\begin{aligned} \xi^{n+1} + \xi^{n+2} &:= \Lambda\phi^{-1}, \\ \xi^{n+1} - \xi^{n+2} &:= -\Lambda^{-1}\phi^{-1}\eta_{\mu\nu}x^\mu x^\nu, \end{aligned} \quad (2.6)$$

and the $(n+2)$-dimensional coordinates $\xi^A = (\xi^\mu, \xi^{n+1}, \xi^{n+2})$, (with $A, B, \cdots = 1, 2, \cdots, n+2$), we have now

$$g = \eta_{AB} d\xi^A \otimes d\xi^B \quad (2.7)$$

with

$$\|\eta_{AB}\| = \left\| \begin{array}{c|c} \eta_{\mu\nu} & 0 \\ \hline 0 & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array} \right\|. \quad (2.8)$$

The metric g has thus been embedded into the $(n+2)$-dimensional flat space F_{n+2}, of signature $(n_+ + 1,\ n_- + 1)$. Our conformally flat space CV_n is then a n-dimensional surface in F_{n+2}.

The equations which characterize the surface CV_n can be easily determined. First, we have

$$\begin{aligned} \eta_{AB}\xi^A\xi^B &= \eta_{\mu\nu}\xi^\mu\xi^\nu + (\xi^{n+1} + \xi^{n+2})(\xi^{n+1} - \xi^{n+2}) \\ &= \phi^{-2}\eta_{\mu\nu}x^\mu x^\nu - \phi^{-2}\eta_{\mu\nu}x^\mu x^\nu = 0. \end{aligned}$$

Since $\xi^\mu/(\xi^{n+1} + \xi^{n+2}) = x^\mu/\Lambda$, we can state the first of (2.6) in the form

$$(\xi^{n+1} + \xi^{n+2})\phi\left[\frac{\xi^\mu}{\xi^{n+1} + \xi^{n+2}}\right] = \Lambda. \quad (2.10)$$

Summarizing, we arrive at the description of our CV_n as the n-dimensional surface in F_{n+2} of signature $(n_+ + 1,\ n_- + 1)$ determined by

$$g = \eta_{AB} d\xi^A \otimes d\xi^B, \quad (2.11a)$$

$$0 = \eta_{AB}\xi^A\xi^B, \quad (2.11b)$$

$$\Lambda = (\xi^{n+1} + \xi^{n+2})\phi\left[\frac{\xi^\mu}{\xi^{n+1} + \xi^{n+2}}\right]. \quad (2.11c)$$

This result is fundamental in the general theory of conformally flat spaces in n dimensions and the theory of the conformal extensions $C_P(n_+, n_-)$ of the Poincaré group $P(n_+, n_-)$ of the flat space of $n = n_+ + n_-$ dimensions, with signature (n_+, n_-).

In the present context, we can specialize these general results to the case $n = 2$, with the function ϕ dependent only on one variable, say u^1.

Consider thus the 2-dimensional conformally flat space

$$g = \phi^{-2}\left(\frac{x^1}{\Lambda}\right)\left[dx^1 \otimes dx^1 + \epsilon dx^2 \otimes dx^2\right], \quad \epsilon = \pm 1, \tag{2.12}$$

where $\phi(z)$ is an arbitrary function [obviously, $\partial/\partial x^2$ is a Killing vector of this space].

We will assume about this function that

$$\frac{d\phi(z)}{dz} \neq 0, \tag{2.13}$$

or that this function is invertible in the sense that

$$\phi = \phi(z) \iff z = \psi(\phi). \tag{2.14}$$

According to our general theory, the space (2.12) can be embedded in 4 dimensions as follows

$$g = d\xi^1 \otimes d\xi^1 + \epsilon d\xi^2 \otimes d\xi^2 + d\xi^3 \otimes d\xi^3 - d\xi^4 \otimes d\xi^4, \tag{2.15a}$$

$$0 = (\xi^1)^2 + \epsilon(\xi^2)^2 + (\xi^3)^2 - (\xi^4)^2, \tag{2.15b}$$

$$\Lambda = (\xi^3 + \xi^4)\phi\left[\frac{\xi^1}{\xi^3 + \xi^4}\right]. \tag{2.15c}$$

These formulae permit us now to construct a specific embedding of the considered space in 3 flat dimensions. Indeed, denote

$$u := \xi^3 + \xi^4. \tag{2.16}$$

Inverting (2.15 c) we have according to (2.14)

$$\xi^1 = u\psi\left(\frac{\Lambda}{u}\right). \tag{2.17}$$

Substituting this into (2.15a) we have

$$g = \epsilon d\xi^2 \otimes d\xi^2 + du \otimes_s \left[d(\xi^3 - \xi^4) + \left\{\frac{d}{du}u\psi\left(\frac{\Lambda}{u}\right)\right\}^2 du\right]. \tag{2.18}$$

Let

$$w := \xi^2\,, \quad v := \xi^3 - \xi^4 + \int^u \left\{ \frac{d}{du} u\psi\left(\frac{\Lambda}{u}\right)\right\}^2 du\,. \tag{2.19}$$

Then g is in the form of a 3-dimensional flat metric

$$g = \epsilon dw \otimes dw + du \otimes dv\,. \tag{2.20}$$

It remains to work out equation (2.15 b) in terms of the coordinates $\{u, v, w\}$.
We first write this equation in the form

$$\epsilon w^2 + (\xi^1)^2 + u(\xi^3 - \xi^4) = 0\,. \tag{2.21}$$

Thus, substituting for ξ^1 from (2.17) and for $\xi^3 - \xi^4$ from (2.19) we have

$$\epsilon w^2 = \left[u\psi\left(\frac{\Lambda}{u}\right)\right]^2 + u\left[v - \int^u \left\{\frac{d}{du} u\psi\left(\frac{\Lambda}{u}\right)\right\}^2\right] = 0\,. \tag{2.22}$$

Therefore, this equation amounts to

$$\epsilon w^2 + uv = H(u)\,, \tag{2.23}$$

where $H = H(u)$ which determines that

$$H(u) = u \int^u \left\{\frac{d}{du} u\psi\left(\frac{\Lambda}{u}\right)\right\}^2 du - \left[u\psi\left(\frac{\Lambda}{u}\right)\right]^2. \tag{2.24}$$

We can now obtain a nicer expression for this function in terms of $\phi = \phi(z)$. Indeed, from

$$\frac{d}{du}\left[\frac{H(u)}{u}\right] = \frac{\Lambda^2}{u^2}\left[\dot{\psi}\left(\frac{\Lambda}{u}\right)\right]^2, \tag{2.25}$$

it follows that

$$H(u) = u \int^u du \frac{\Lambda^2}{u^2}\left[\dot{\psi}\left(\frac{\Lambda}{u}\right)\right]^2 = -u\Lambda \int^{\Lambda/u} ds \dot{\psi}(s)\dot{\psi}(s)\,. \tag{2.26}$$

Alternatively, remembering (2.14), this result can be stated as

$$H(u) = -u\Lambda \int^{\Lambda/u} d\psi(s) \frac{1}{\frac{ds}{d\psi}} = -u\Lambda \int^{\psi(\Lambda/u)} dz \frac{1}{\frac{d\phi(z)}{dz}}\,. \tag{2.27}$$

We can now summarize the result obtained in the following form. The metric

$$g = \phi^{-2}\left(\frac{x^1}{\Lambda}\right)\left\{dx^1 \otimes dx^1 + \epsilon dx^2 \otimes dx^2\right\}, \quad \epsilon^2 = 1, \tag{2.28}$$

with $\Lambda =$ const. of dimension of length, $\phi = \phi(z)$ dimensionless and arbitrary but such that $(d\phi/dz) \neq 0$, so that $\phi = \phi(z) \iff z = \psi(\phi)$, can be always embedded into the 3-dimensional flat space as

$$g = \epsilon dw \otimes dw + du \otimes dv\,, \tag{2.29}$$

with the embedding variables (u, v, w) submitted to the condition

$$0 = \epsilon w^2 + uv - H(u)\,, \tag{2.30}$$

where the function $H = H(u)$ is constructed form $\phi = \phi(z)$ by the formula which involves a single quadrature

$$H(u) = -\Lambda u \int^{\phi(\Lambda/u)} dz \frac{1}{\frac{d\phi(z)}{dz}}\,. \tag{2.31}$$

Note that the integration constant of the integral given above is without any importance. This constant affects H according to $H \to H + ku$ ($k =$ const), and can be thus absorbed in (2.30) by a translation of v, *i.e.* $v \to v + k$, which of course does not affect the metric (2.29). Therefore, the integration constant in (2.31) can be chosen in an arbitrary manner, *e.g.* making $H(u)$ — with given $\phi = \phi(z)$ — as simple as possible.

The embedding of metrics of the form (2.28) is essentially *different* from the Kasnerian technique which employs either trigonometric or hyperbolic functions. In the next section we shall illustrate the virtues of our method in the concrete example of the Schwarzschild solution.

Strangely enough in the case considered it leads, as we shall see, to integral (2.31) in terms of elementary functions and the embedding in terms of *elementary functions*!

3. An embedding of the Schwarzschild solution

Consider the Schwarzschild solution given in its canonical form by

$$g = -\left(1 - \frac{2m}{r}\right)dt \otimes dt + \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr$$
$$+ r^2\left[d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi\right] \quad m > 0\,. \tag{3.1}$$

We want to construct an universal analytic embedding of this metric in a 6-dimensional real flat space by using the method described in the previous section. In the first step, we introduce

$$x^1 := r\sin\theta\sin\phi\,, \quad x^2 = r\sin\theta\cos\phi\,, \quad x^3 = r\cos\theta\,. \tag{3.2}$$

The metric is then

$$g = -\left(1-\frac{2m}{r}\right)dt\otimes dt + \left[\frac{1}{1-\frac{2m}{r}}-1\right]dr\otimes dr \\ + dx^1\otimes dx^1 + dx^2\otimes dx^2 + dx^3\otimes dx^3\,. \tag{3.3}$$

Therefore it may be written in the form

$$g = g' + g''\,, \tag{3.4}$$

where

$$g'' := dx^1\otimes dx^1 + dx^2\otimes dx^2 + dx^3\otimes dx^3 \tag{3.5}$$

and

$$g' := -\left(1-\frac{2m}{r}\right)dt\otimes dt + \frac{\frac{2m}{r}}{1-\frac{2m}{r}}dr\otimes dr \tag{3.6}$$

with

$$r = \sqrt{(x^1)^2+(x^2)^2+(x^3)^2}\,. \tag{3.7}$$

The problem consists now in embedding g' into a 3-dimensional flat space. For this purpose we write g' as

$$g' = \left(1-\frac{2m}{r}\right)\left[\frac{\frac{2m}{r}}{(1-\frac{2m}{r})^2}dr\otimes dr - dt\otimes dt\right]. \tag{3.8}$$

Therefore, defining

$$\rho := \int^r \sqrt{\frac{2m}{r}}\,\frac{dr}{1-\frac{2m}{r}} \tag{3.9}$$

we can write

$$g' = \phi^{-2}\left(\frac{\rho}{2m}\right)\left[d\rho\otimes d\rho - dt\otimes dt\right]\,, \tag{3.10}$$

where

$$\phi = \left(1-\frac{2m}{r(\rho)}\right)^{-1/2} \equiv \phi\left(\frac{\rho}{2m}\right)\,. \tag{3.11}$$

This is exactly the canonical form of the 2-dimensional metric (2.28), with $\Lambda \sim 2m$, $x^1 \sim \rho$ and $x^2 \sim t$ in the case $\epsilon = -1$. We have now to compute the integral (2.31),

$$H(u) = -2mu \int^{\psi(2m/u)} dz \frac{1}{\frac{d\phi(z)}{dz}} . \tag{3.12}$$

According to the definition of $\phi = \phi(z)$ and of its inverse function $z = \psi(\phi)$, the upper limit of the integral is $\phi = \frac{2m}{u}$, *i.e.*, we have the equation

$$\left(1 - \frac{2m}{r}\right)^{-1/2} = \frac{2m}{u} \tag{3.13}$$

or

$$r = \frac{2m}{1 - \left(\frac{u}{2m}\right)^2} = \frac{m}{1 + \frac{u}{2m}} + \frac{m}{1 - \frac{u}{2m}} . \tag{3.14}$$

Clearly, $r = 2m$ corresponds to $u = 0$, and $u = \pm\infty$, correspond both to $r = 0$. Positive r corresponds to $2m > \mu - 2m$. This restriction we will find later to be unessential. [For the moment, we can proceed with the *complexified* Schwarzschild solution, considering the original coordinates (r, φ, θ, t) and the constant m as *complex*, as well as all the corresponding embedding variables. Then the problem of the range of r is unessential.] Notice also that $u \to \pm 2m$ yields $r = \infty$.

To compute the integral (3.12) the most convenient thing is to use simply r as integration variable. This we do in the two steps

$$H(u) = -2mu \int \left(\frac{dz}{dr}\right)^2 \frac{dr}{\frac{d\phi}{dr}} = -\frac{u}{2m} \int \left(\frac{d\rho}{dr}\right)^2 \frac{dr}{\frac{d\phi}{dr}} . \tag{3.15}$$

The upper limit of the integral, according to (3.13), is then $2m/\left(1-(u/2m)^2\right)$. For $\left(d\rho/dr\right)^2$, according to (3.9) we have

$$\left(\frac{d\rho}{dr}\right)^2 = \frac{\frac{2m}{r}}{(1 - \frac{2m}{r})^2} \tag{3.16}$$

and for ϕ we have $\phi = (1-(2m/r))^{-1/2}$. Therefore, our integral amounts to

$$H(u)=\frac{-u}{2m}\int^{2m/(1-(u/2m)^2)}\frac{\frac{2m}{r}}{(1-\frac{2m}{r})^2}\frac{dr}{\frac{d}{dr}(1-\frac{2m}{r})^{-1/2}}$$

$$=\frac{u}{m}\int^{2m/(1-(u/2m)^2)}\frac{rdr}{\sqrt{1-\frac{2m}{r}}}. \tag{3.17}$$

The main point of our embedding procedure consists in the fact that this integral can be easily computed in terms of *elementary* functions. For completeness, we give below the details of the integration.

We have

$$H(u)=\frac{u}{m}\int^{2m/(1-(u/2m)^2)}\frac{r^{3/2}dr}{\sqrt{r-2m}} \qquad \text{(introducing } r=s+2m)$$

$$=\frac{u}{m}\int^{2m(u/2m)^2/[1-(u/2m)^2]}\frac{ds}{\sqrt{s}}(s+2m)^{3/2}$$

$$=\frac{2u}{m}\int^{\cdots}d\sqrt{s}(s+2m)^{3/2} \qquad \text{(introducing } \sqrt{s}=\sqrt{2m}\omega)$$

$$=\frac{2u}{m}(2m)^2\int^{(u/2m)/\sqrt{1-(u/2m)^2}}d\omega(1+\omega^2)^{3/2}. \tag{3.18}$$

At this point, introduce as integration variable

$$\omega=\sinh\eta\,. \tag{3.19}$$

The upper limit will then correspond to η_0 defined by

$$\sinh\eta_0=\frac{\frac{u}{2m}}{\sqrt{1-(\frac{u}{2m})^2}} \tag{3.20}$$

which implies

$$\eta_0=\frac{1}{2}ln\left(\frac{1+\frac{u}{2m}}{1-\frac{u}{2m}}\right).$$

Therefore, after the substitution (3.19) our integral amounts to

$$H(u) = (2m)^2 \frac{2u}{m} \int^{\eta_0} \cosh^4 \eta d\eta . \tag{3.21}$$

This we can easily integrate

$$\frac{H(u)}{(2m)^2} = \frac{u}{m}\frac{1}{8}\left[\frac{e^{4\eta_0} - e^{-4\eta_0}}{4} + 2(e^{2\eta_0} - e^{-2\eta_0}) + 6\eta_0\right] . \tag{3.22}$$

At this point, we have selected the constant of integration in such a manner that for $u = 0 \to \eta_0 = 0$ the integral vanishes, *i.e.*, that $H(u)$ has at least a double root at $u = 0$.

It remains to express this result in terms of the variable u. This is easily done

$$\frac{H(u)}{(2m)^2} = \frac{u}{2m}\frac{1}{4}\left\{\frac{1}{4}\left(\left[\frac{1+\frac{u}{2m}}{1-\frac{u}{2m}}\right]^2 - \left[\frac{1-\frac{u}{2m}}{1+\frac{u}{2m}}\right]^2\right) + 2\left(\frac{1+\frac{u}{2m}}{1-\frac{u}{2m}} - \frac{1-\frac{u}{2m}}{1+\frac{u}{2m}}\right) + 3\ln\frac{1+\frac{u}{2m}}{1-\frac{u}{2m}}\right\}. \tag{3.23}$$

Perhaps the most reasonable policy in giving the final form to this result consist in exhibiting this function in terms of the denominators singular at $u = \pm 2m$. Because

$$\frac{1+\frac{u}{2m}}{1-\frac{u}{2m}} = \frac{2}{1-\frac{u}{2m}} - 1 ; \quad \frac{1-\frac{u}{2m}}{1+\frac{u}{2m}} = \frac{2}{1+\frac{u}{2m}} - 1 \tag{3.24}$$

our function amounts to

$$\begin{aligned}\frac{H(u)}{(2m)^2} =& \frac{u}{2m}\frac{1}{4}\left\{\frac{1}{4}\left[\frac{4}{(1-\frac{u}{2m})^2}\frac{-4}{1-\frac{u}{2m}} + 1\frac{-4}{(1+\frac{u}{2m})^2} - 1\right]\right. \\ &\left. + 2\left[\frac{2}{1-\frac{u}{2m}} - 1 - \frac{2}{1+\frac{u}{2m}} + 1\right] + 3\ln\frac{1+\frac{u}{2m}}{1-\frac{u}{2m}}\right\} \\ =& \frac{u}{2m}\frac{1}{4}\left\{\frac{1}{(1-\frac{u}{2m})^2} - \frac{1}{(1+\frac{u}{2m})^2} + \frac{3}{1+\frac{u}{2m}}\right. \\ &\left. - \frac{3}{1+\frac{u}{2m}} + 3\ln\frac{1+\frac{u}{2m}}{1-\frac{u}{2m}}\right\}.\end{aligned}$$

This is already a fairly satisfactory form of the structural function, which can be still more plausibly stated if we introduce the function

$$h(z) := \tfrac{1}{4}\left\{ \frac{1}{(1-z)^2} + \frac{3}{1-z} - 3\ln(1-z) \right\}; \tag{3.25}$$

as

$$\frac{H(u)}{(2m)^2} = \frac{u}{2m}\left[h\left(\frac{u}{2m}\right) - h\left(\frac{-u}{2m}\right) \right]. \tag{3.26}$$

Having integrated our function H in terms of elementary functions, we can now employ the results of Sec. 2. The part g' of the Schwarzschild metric can be now represented in terms of the coordinates (u, v, w) — all having the dimension of length — in the form

$$g' = -dw \otimes dw + du \otimes_s dv \tag{3.27}$$

with these variables fulfilling the condition

$$-\left(\frac{w}{2m}\right)^2 + \frac{u}{2m}\frac{v}{2m} = \frac{u}{2m}\left[h\left(\frac{u}{2m}\right) - h\left(\frac{-u}{2m}\right) \right], \tag{3.28}$$

where the function $h(z)$ is given by (3.25).

The part g'' of the metric has the simple form of (3.5) and the flat embedding coordinates (x^1, x^2, x^3) according to (3.7) and (3.14) are related to the variable u by the condition

$$\left(\frac{r}{m}\right)^2 \equiv \left(\frac{x^1}{m}\right)^2 + \left(\frac{x^2}{m}\right)^2 + \left(\frac{x^3}{m}\right)^2 = \left(\frac{1}{1-\frac{u}{2m}} + \frac{1}{1+\frac{u}{2m}}\right)^2. \tag{3.29}$$

Of course, $u = 0$, corresponds in our coordinatization to the critical horizon value, $r = 2m$. If we take the second embedding equation in this "quadratic" form (3.29), the value $r = -2m$ is also permitted. However, it should be noted that with

$$u \to 0 \quad \text{also} \quad h\left(\frac{u}{2m}\right) - h\left(\frac{-u}{2m}\right) \to 0,$$

and the first embedding equation (3.28) implies $w = 0$. Therefore, in terms of our embedding variables, the horizons correspond to the set of points

$$u = 0 = w, v \quad \text{arbitrary}, \quad (x^1)^2 + (x^2)^2 + (x^3)^2 = (2m)^2, \tag{3.30}$$

i.e. to a set of points of dimension 3, as it should be. Clearly nothing special happens to our analytic embedding along this set of points;

our 4-dimensional surface in the flat space with the cartesian coordinates (u, v, w, x^1, x^2, x^3) defined by the conditions (3.28) and (3.29) is analytic.

It is also clear that our embedding covers both the "exterior" and "interior" Schwartzschild solution, from both sides of the horizon. Indeed, $u = 2m$ and $u = -2m$ both correspond to $r^2 = \infty$, while $u = +\infty$ and $u = -\infty$ yield $r^2 = 0$. These "exterior" and "interior" regions, within our embedding are however in a sense disjoint — because of the singularity of the 4-surface at $u = \pm 2m$. Perhaps if we constrain ourselves to the real ranges of the embedding variables, the topology of the surface (4-dimensional) in the flat space (6-dimensional) which corresponds to the Schwarzschild solution remains still somewhat obscure. Within our treatment it seems that the two disjoint sets of points, which according to the Kruskal treatment or á la Kasner embedding represent the distinct horizons, have become "glued" in one set of points. But if we consider the original metric (3.1) as complexified, *i.e.*, a complex solution to the (analytic) Einstein equations over a complex manifold there is no doubt that by our treatment we have produced an analytic embedding of this metric in terms of elementary functions.

For the convenience of the reader, we will now summarize concisely the result obtained. In the 6-dimensional flat space with the coordinates (u, v, w, x^1, x^2, x^3) and with the metric given by

$$g = -dw \otimes dw + du \otimes_s dv + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 \qquad (3.31)$$

the 4-dimensional sub-manifold defined by the analytic conditions

$$\left(\frac{r}{m}\right)^2 \equiv \left(\frac{x^1}{m}\right)^2 + \left(\frac{x^2}{m}\right)^2 + \left(\frac{x^3}{m}\right)^2 = \left(\frac{1}{1-\frac{u}{2m}} + \frac{1}{1+\frac{u}{2m}}\right)^2 \qquad (3.32)$$

and

$$-\left(\frac{w}{2m}\right)^2 + \frac{u}{2m}\frac{v}{2m} = \frac{u}{2m}\left[h\left(\frac{u}{2m}\right) - h\left(\frac{-u}{2m}\right)\right], \qquad (3.33)$$

where

$$h(z) := \tfrac{1}{4}\left\{\frac{1}{(1-z)^2} + \frac{3}{1-z} - 3\ln(1-z)\right\}.$$

For $m = \text{const}$, this space has the intrinsic geometry of the Schwarzschild solution to the Einstein equation. If m and the embedding coordinates are considered real, then the signature of the 6-dimensional space is obviously $(++++--)$, [The "á la Kasner" embedding of [1] and [2] employs the signature $(+++++-)$!] Observe that when we keep m and the variables real, the presence of the term $\ln\{[1+(u/2m)]/[1-(u/2m)]\}$ in the equation (3.33), forces the range of u to be restricted to

$$2m \geq u \geq -2m \qquad (3.34)$$

excluding $u = \pm\infty$ and therefore covering only the values of $r : \infty \geq r \geq 2m$. Thus, so understood, our analytic embedding covers only the Schwarzschild exterior solution.

It may be also observed that the solution understood as internal and external Schwarzschild solution exhibits strange duality. This can be seen from the general formulae when it meets apparent singularity at $r = 2m$.

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Of course, the aim of this paper was to show the existence of embedding formulae which contain the elementary functions. On the other hand it is to be stressed, that the author of this paper certainly agrees with Kruskalization as the fundamental embedding.

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