WHY THE MEAN FIELD APPROACH FAILS IN CASE OF DETERMINISTIC HOMOGENEOUS DYNAMICS ON SQUARE LATTICE OF SPINS?*†

DANUTA MAKOWIEC

Institute of Theoretical Physics and Astrophysics
Gdańsk University,
Wita Stwosza 57, 80-952 Gdańsk, Polska
e-mail fizdm@halina.univ.gda.pl

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The process of adjustment of local microdynamics to the spin relationships occurring from the lattice's architecture of the system in consideration, is of the main interest of the note. Qualitative arguments are formulated to make this process responsible for the violation of the mean field theory approach.

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1. Base for CA theory

One of the problems in the theory of cellular automata is to understand how cellular automata can be meaningfully grouped according to their structure and behavior. The empirical fact coming from computer simulations, such that cellular automata stabilize after many time steps seems to be the best base for the research. However, in general, cellular automata with very similar transition rules may behave quite differently, while cellular automata with very different transition rules may behave identically.

So far, two classification schemes, considered to be quite general have been proposed. Each of these schemes attempts to divide cellular automata into distinct classes.

The first scheme, formulated by Wolfram [1] and further developed by Stauffer *et al.* [2, 3] determines classes via qualitative features of the final

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states of a lattice. Roughly speaking, automata are classified according to their long time effect to a lattice pattern.

The second scheme — the generalized mean field approach, proposed by Gutowitz and Victor [4, 5], provides the classification according to CA action on probability measures. The probability measure $\tau\mu$ resulting from the application of a CA rule τ to a probability measure μ is given

$$\tau\mu(A) = \mu(\tau^{-1}(A)),$$

where A is a μ-measurable open set of configurations. Any measurable open set of configurations can be represented as a union of sets known as cylinder sets. Therefore, the action can be rewritten as an infinite system of the following equations:

$$\mathrm{Prob}^{t+1}(b) = \sum_B \delta(\tau(B,b))\mathrm{Prob}^t(B), \tag{1}$$

where the probability of a finite block b at time $t+1$ is the sum of the probabilities at time t of the possibly smallest blocks B which lead to the block b under the rule τ. Therefore, knowing the probabilities of all B blocks one gets probabilities of all b blocks at the next time step.

So that, the crucial point of the Markov classification lies in the approximation of the measure by, so-called, n-step Markov measures. n-step Markov measure is an assignment of probability by a Markov process to n -element block states (so-called n-step spatial memory).

Moreover, the maximum entropy assumption- which means here that blocks are uniformly distributed all over the lattice, closes the infinite system of equations (1). It opens the possibility to perform the iteration of the map (1) an infinite number of times. Because of it, one can say that the range over which a cell potentially interact becomes infinite. However, the results obtained this way and the computer simulation results do not always overlap as we see in the next sections. There exists microdynamics for which long-range interactions cannot be replaced by the finite-range statistics.

The longtime behavior of CA comes from the simultaneous action of the two kinds of interactions: the microscopic rule which is responsible for the changes of a single automaton state with respect to the states of its neighbors and the lattice architecture, which fixes the location of the cells in space and specifies the neighborhood of each automaton.

The independence of CA simulations of the lattice symmetries is crucial for a certain class of CA applications. It contains, for example, solving the Navier-Stokes equation with CA lattice gases. It has been noted [6], that for such a case the system at the macroscopic scale is isotropic when considered on the triangular lattice. It means that the orientation with respect to the

triangular lattice is irrelevant from the macroscopic point of view, although it plays its role on the microscopic level. This observation is not true for the case of the square lattice. The CA system with a similar microscopic rule but considered on the square lattice does not exhibit isotrophic behavior. Thus, it conserves the lattice symmetries.

Generally, one can say that if the lattice relationships vanish with time, the mean theory is a good enough approximation to a system. However, if the lattice interactions take over the CA dynamics the mean field tools fail.

How can we understand taking over the CA dynamics by lattice interactions?

Let us concentrate on CA defined on a square lattice with deterministic and homogeneously set some dynamical rule. Moreover, let the next time step state of a spin be influenced by the states of its nearest neighbors. It means that the state of four spins located North, West, South and East with respect to a central spin site, as the whole unity, determines the value of the central spin in the next evolution step. The only source of uncertainty is via an initial lattice state prepared at random.

The complete presentation of experimental data coming from the computer simulations of such CA is presented in [7]. It is noticed there that among these systems one can find a great number of CA such that with probability close to 1 they stabilize with some peculiar patterns. These patterns are distinguishable from the sea of all possible patterns by the following property: at any site of a lattice a rule can be read as the shift of the state of the same located for the whole lattice nearest neighbor. It means that the whole lattice state is conserved in time up to the shift by one lattice step from one of the four lattice directions— South, East, North or West. These states, named **moving structures**, have been noticed firstly in [3]. In [9] there is described the other class of CA corresponding to the moving structure stabilizing class of CA, members of which stabilize with patterns on which the rule is performed as a shift connected with a spin flip (so-called oscillating moving structures).

Concluding, due to the interactions between a CA rule and the lattice relationships, a CA pattern is transferred during the evolution in such a way that all local configurations on which a rule acts differently from some globally chosen shift movement, vanish. It means that these states of neighbors which do not yield the proper nearest neighbor state are not present at final patterns of the moving structure type. Moreover, it was notified that CA can be identified almost uniquely via the type of the absent neighborhood states, [7, 8]. The analyses of the absent neighborhood states can be done, for example, by considering the distribution of neighborhood states on a lattice. Although this is a statistical tool, it is good enough in the case of pointing at not present neighborhood states. Let us call these ab-

sent neighborhood states as **odd neighborhood states** as their action does not agree with the shift direction which is adjusted by the remaining neighborhood states.

Applying this characterization to CA which do not reach final patters as the moving structure, two new CA classes can be named. The first group consists of CA for which distance from their final lattice states to the moving structure is little, if one measures the distance in probability to find odd neighborhood states on the lattice state. Moreover, this distance is often slowly decreasing in time. The second group contains of CA for which the distribution of neighborhood states stabilizes quickly, in less than few steps. However, there is no visible sign that any neighborhood state will vanish.

The set of all neighborhood states consist of 16 elements, named here $\theta_0 \ldots \theta_{15}$ (Fig. 1). These elements can be considered as the input blocks in the Markov approximation. Furthermore, the corresponding action mappings (1) can be formulated and iterated, as follows:

$$\mathrm{Prob}^{t+1}(\theta_i) = \sum_{B=\left(\begin{smallmatrix} & \theta_N & \\ \theta_W & & \theta_E \\ & \theta_S & \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} & \sigma_1 & \\ \sigma_2 & \cdot & \sigma_3 \\ \sigma_4 & \cdot & \sigma_5 \cdot \sigma_6 \\ & \sigma_7 \cdot \sigma_8 & \\ & \sigma_9 & \end{smallmatrix}\right)} \delta(\tau(B, \theta_i))\mathrm{Prob}^t(B) \qquad (2)$$

and according to maximum entropy assumption the following simplification is made:

$$\mathrm{Prob}^t(B) = \mathrm{Prob}^t\left(\begin{smallmatrix} & \theta_N & \\ \theta_W & & \theta_E \\ & \theta_S & \end{smallmatrix}\right) \equiv$$

$$\mathrm{Prob}^t(\theta_E)\ \mathrm{Prob}^t(\theta_N|\ \begin{smallmatrix} \theta_N \\ \theta_E \end{smallmatrix})\ \mathrm{Prob}^t(\theta_W|\ \begin{smallmatrix} \theta_N \\ \theta_W \end{smallmatrix})\ \mathrm{Prob}^t(\theta_S|\ \begin{smallmatrix} \theta_W\ \theta_E \\ \theta_S \end{smallmatrix}) \qquad (3)$$

If the results of the action map on these blocks coincide with the experimental data- one can say that Markov approximation characterizes CA in a satisfactory way. However, there are rules for which the maximum entropy assumption fails very quickly. The lattice dependencies cause that the particular blocks are not randomly distributed.

To see the process of lattice influence, we will analyses CA with rules which are shifts of the South neighbor state in case of all neighborhood states except one. Hence, the set of odd neighborhood states with respect to a considered rule will consist of one element only at each case.

$(\mathbf{A}, \mathbf{A}')$:

$$\{\theta_j : \textstyle\sum_{\sigma_i \in \theta_j} \sigma_i = \pm 4\}$$

$$\begin{pmatrix} & -1 & & & 1 & \\ -1 & \sigma & -1, & 1 & \sigma & 1 \\ & -1 & & & 1 & \end{pmatrix}$$

$$(\theta_0, \theta_{15})$$

$(\mathbf{B}, \mathbf{B}')$:

$$\{\theta_j : \textstyle\sum_{\sigma_i \in \theta_j} \sigma_i = \pm 1\}$$

$$\begin{pmatrix} & -1 & & & 1 & \\ -1 & \sigma & 1, & 1 & \sigma & -1 \\ & -1 & & & 1 & \end{pmatrix} \quad \begin{pmatrix} & 1 & & & -1 & \\ -1 & \sigma & -1, & 1 & \sigma & 1 \\ & -1 & & & 1 & \end{pmatrix} \quad \begin{pmatrix} & -1 & & & 1 & \\ 1 & \sigma & -1, & -1 & \sigma & 1 \\ & -1 & & & 1 & \end{pmatrix} \quad \begin{pmatrix} & -1 & & & 1 & \\ -1 & \sigma & -1, & 1 & \sigma & 1 \\ & 1 & & & -1 & \end{pmatrix}$$

$$(\theta_1, \theta_{14}) \qquad (\theta_2, \theta_{13}) \qquad (\theta_3, \theta_{12}) \qquad (\theta_4, \theta_{11})$$

$(\mathbf{C}, \mathbf{C}')$:

$$\{\theta_j : \textstyle\sum_{\sigma_i \in \theta_j} \sigma_i = 0\}$$

$$\begin{pmatrix} & 1 & & & -1 & \\ -1 & \sigma & 1, & 1 & \sigma & -1 \\ & -1 & & & 1 & \end{pmatrix} \quad \begin{pmatrix} & -1 & & & 1 & \\ 1 & \sigma & 1, & -1 & \sigma & -1 \\ & -1 & & & 1 & \end{pmatrix} \quad \begin{pmatrix} & -1 & & & 1 & \\ -1 & \sigma & 1, & 1 & \sigma & -1 \\ & 1 & & & -1 & \end{pmatrix}$$

$$(\theta_5, \theta_{10}) \qquad (\theta_6, \theta_9) \qquad (\theta_7, \theta_8)$$

Fig. 1. Nearest neighbors configurations on a square lattice. They are listed in up-down symmetric pairs $(\theta_i, \theta_{15-i})$, $i = 0, \ldots, 7$ and in sets $\mathbf{A}, \mathbf{A}', \mathbf{B}, \mathbf{B}', \mathbf{C} \cup \mathbf{C}'$, elements of which have the same total magnetization

TABLE I

Simulations vs. statistical predictions

Odd neigh-borhood	time $\langle T \rangle$	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	D_{13}	D_{14}	D_{15}
θ_8	5.7	.039	.038	.049	.067	.039	.086	.067	.039	0	.048	.067	.077	.086	.067	.115	.116
θ_8	6	.039	.039	.049	.067	.039	.085	.067	.039	0	.049	.067	.078	.085	.067	.113	.117
θ_9	2.0	.113	.066	.110	.067	.098	.055	.043	.066	.054	0	.066	.055	.055	.043	.055	.055
θ_9	2	.113	.066	.109	.066	.098	.055	.043	.066	.055	0	.066	.055	.055	.043	.055	.055
θ_{10}	5.8	.117	.086	.067	.114	.077	.039	.049	.086	.067	.067	0	.039	.039	.049	.067	.038
θ_{10}	6	.117	.085	.067	.113	.078	.039	.049	.085	.067	.067	0	.039	.039	.049	.067	.039
θ_{11}	95.8	.006	.006	.002	.005	.006	.009	.005	.005	.009	.002	.005	0	.007	.004	.007	.924
θ_{11}	156	.010	.010	.008	.010	.010	.032	.010	.010	.025	.008	.010	.048	.032	.010	.025	.742
θ_{12}	6.5	.102	.058	.071	.091	.090	.143	.051	.047	.033	.072	.091	.033	0	.052	.033	.033
θ_{12}	7	.095	.06	.073	.094	.093	.145	.052	.042	.032	.073	.094	.032	0	.052	.032	.032
θ_{13}	4.6	.072	.071	.094	.071	.071	.049	.143	.071	.048	.094	.071	.023	.048	0	.049	.023
θ_{13}	5	.071	.071	.095	.071	.071	.048	.143	.071	.048	.095	.071	.023	.048	0	.048	.023
θ_{14}	6.3	.102	.092	.071	.059	.090	.033	.051	.091	.143	.071	.047	.033	.033	.052	0	.033
θ_{14}	7	.095	.094	.073	.062	.093	.032	.052	.094	.145	.073	.042	.032	.032	.052	0	.032
θ_{15}	2.0	.082	.070	.082	.070	.078	.063	.066	.067	.064	.077	.067	.066	.043	.062	.043	0
θ_{15}	2	.082	.070	.082	.070	.078	.063	.066	.066	.063	.078	.066	.066	.043	.063	.043	0

Distribution of neighborhood states obtained in computer simulations (upper line of each item) and by iterating formula (2) (lower lines of each item) is presented for CA with dynamics different from the shift of the South neighbor state at one neighborhood state: $\theta_8, \ldots, \theta_{15}$ successively. Computer results are averages over 100 experiments (at CA size L=96). Their STD-errors were not exceeding 5 except $\langle T \rangle$ results- about 20 The iteration was stopped if at the sequential steps changes in distribution of all neighborhood states differed less than 10^{-4}.

Fig. 1 is to introduce the neighborhood states notation as well as to fix two standard ways of grouping them. They are listed there, firstly, in pairs according to the up-down spin-symmetry, and secondly, in mean-field sets which joins neighborhood states for which the total sum of states is the same.

The statistics of the data for distribution of neighborhood states in final lattice states which were obtained in our computer experiments and the results which come from the iteration of a dynamical mapping in the Markovian approximation (2) (3) for all rules having as one odd neighborhood state : $\theta_8, \ldots, \theta_{15}$ are presented in Table I.

The size of the lattice in our simulations was taken as $L = 96$. One can read this size as little. However, thanks to its not large value, the desired effects could be seen quickly. Our earlier works on CA ensure us that properties examined by us are size independent [9]. In particular, to see the independence one may compare the number of steps in CA evolution to reach the moving structure stabilization which represents time of CA evolution, with the lattice size.

2. Local rule versus lattice architecture

Each of the four nearest neighbor configurations, which defines the neighborhood state for a central spin, effects the central spin state in the next time step. This action can be interpreted as overtaking the state of one neighbor by the central spin. Therefore, one can talk about the shift (anti shift, if the flip of a spin state is associated with the action) of a neighboring spin state. Although pointing at one particular direction is impossible in case of a single spin evolution, but spreading this interpretations to two, three, four, ... neighboring spins, one can settle rather restricted set of active shift directions for any dynamics. This is the reason to understand the spin state changes as seeds of lattice regions which are shifted as a whole unities towards one direction. The area of the particular shifted region is larger if more probable are neighborhood states which support via the CA rule this shift. The pressure of the common shift is so high that neighborhood states which act against it (odd neighborhood states) are transferred into suitable ones.

To elucidate the process of decaying of odd neighborhood states let us consider the following extreme situation: only a spin at an ith node is in the odd neighborhood state, while all other spins have the proper neighborhood states. So, all spins but not the ith, take the state of the South nearest neighbor, while the ith spin takes the opposite the South neighbor state, see Fig. 2. In the next time step, $t + 1$, the odd neighborhood state will become the proper one. So, the original odd neighborhood state disappears.

However, the perturbation of the lattice caused by flipping σ_S spin state makes possible reconstruction of the odd neighborhood state, if only the odd neighborhood state has the following property:

(i) If

$$\sigma_E = \sigma_N ,$$

$$\sigma_W = -\sigma_S ,$$

then at the right side of an ith node (marked **R** in Fig. 2), the following neighborhood states: θ_3, θ_4, θ_{11} and θ_{12} can appear.

(ii) If

$$\sigma_E = -\sigma_S ,$$

$$\sigma_W = \sigma_N ,$$

then at the left side of an ith node (marked **L** in Fig. 2), the following neighborhood states: θ_1, θ_4, θ_{11} and θ_{14} can appear.

(iii) If

$$\sigma_N = -\sigma_S ,$$

then at the bottom side of an ith node (marked **B** in Fig. 2), the following neighborhood states: θ_2, θ_4, θ_5, θ_7, θ_8, θ_{10}, θ_{11} and θ_{12} can appear.

The list of possible to reconstruct neighborhood states does not contain the following ones: θ_0, θ_6, θ_9 and θ_{15}. These four neighborhood states will vanish with probability close to 1 at the first time step.

Next case to consider is the rule with θ_{11} being the odd neighborhood state. Let us notice that the neighborhood state θ_{11} can survive on a lattice at all listed possibilities. The joint probability to reconstruct this neighborhood state at the first step when states of spins are scattered at random on a lattice, is $^5/_8$ ($\mathrm{Prob}^0\{\sigma_i = 1\} = \mathrm{Prob}^0\{\sigma_i = -1\} = {}^1/_2$). Let us have a look at the properties of the distribution of neighborhood states for this rule listed in table 1. The results of computer simulations and iteration in this case exhibit rather large discrepancy. The statistical predictions provide the large number of spins having θ_{15} neighborhood state and not vanishing number of spins having θ_{11} neighborhood state while the reality, expressed by data from computer experiments, gives the rigid South shifted moving structure in all 100 experiments and almost complete magnetization with all spins up of the each final lattice state (in 79 experiments the magnetization was complete). To explain this fact, let us look again at the process of transferring of neighborhood states. According to the dynamical rule θ_{11} transfers into θ_{15}, however it itself can be found at the left, right or bottom of its old place. Moreover, θ_{15} does not build in any active (=changed by θ_{11}) lattice configuration. It means that θ_{15} cannot be changed by this

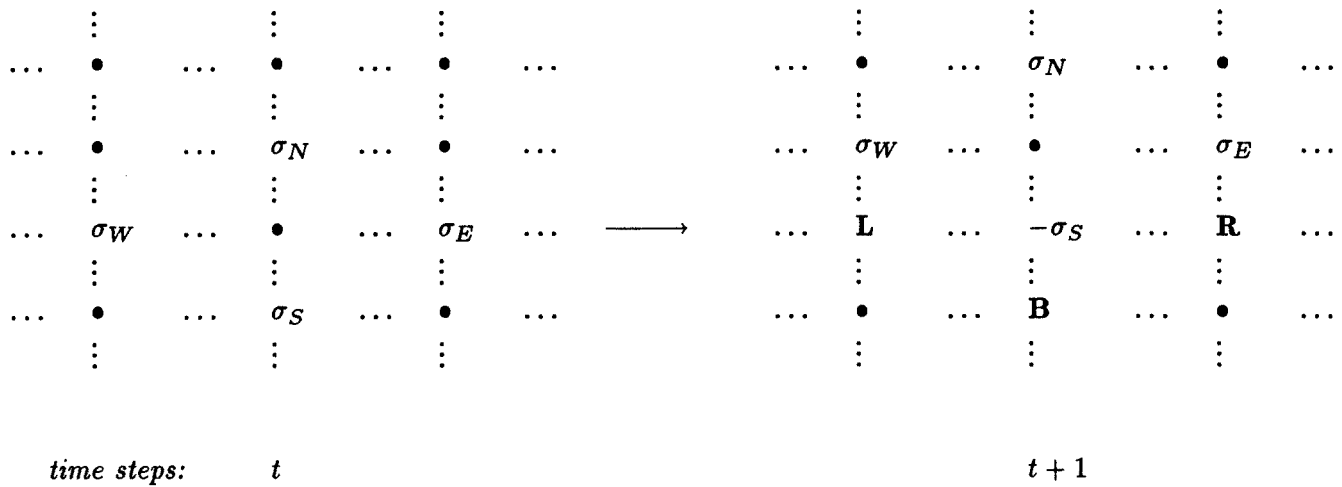

Fig. 2. Lattice relationships. A neighborhood state $(\sigma_E, \sigma_N, \sigma_W, \sigma_S)$ is the only odd state with respect to the shift of the South neighbor state - the common action of all other neighborhood states. Therefore, the change made at one spin state transfers the odd neighborhood $(\sigma_E, \sigma_N, \sigma_W, \sigma_S)$ into the proper one $(\sigma_E, \sigma_N, \sigma_W, -\sigma_S)$ and makes possible recovering the odd neighborhood states at the left (**L**), right (**R**) or bottom (**B**) side of the anti shifted spin.

rule. So, at each time step new spins from the lattice have θ_{15} neighborhood state. Moreover, θ_{15} states crates in this process side by side making the increase of clusters with all spins up. Therefore, after few steps the lattice state looks like islands of all spins up surrounded by spins with remain neighborhood states. At each time step these islands are enlarged if only any spin with θ_{11} state is at their borders. The increase of θ_{15} regions makes squeeze of the region for other possibilities. So that, the assumption on random distribution of neighborhood states is broken. The probability that θ_{11} is involved in configuration with $+1$ around is bigger than it comes from the statistics. Therefore, configurations providing the increase of θ_{11} states, (as it happens while iterating) are less probable on the real lattice and spins with θ_{11} neighborhood state are not present.

The remaining neighborhood states, depending on to which "recovery case" *(i)*, *(ii)* or *(iii)* they belong to, could be reconstructed on the lattice. However, the probability to find them after first time step when the lattice is prepared at random is $1/4$ in both cases *(i)* and *(ii)*, and $1/8$ when *(iii)* is considered.

Generally, all these rules provide the experimental results overlapping the iteration predictions (within the STD-error interval), (see Table I). Moreover, all CA with these rules reach the moving structure stabilization in few steps. During such a short period, the assumption on random scattered blocks (maximum entropy assumption) is fulfilled and the statistical predictions characterize the system faithfully. Especially, existing if the geometrical relation between particular rules - $\pm\frac{\pi}{2}$ disturbance to the South shift, in case of dynamics with θ_8 and θ_{10} as the odd neighborhood state, as well as θ_{12} and θ_{14} as the odd neighborhood states, can be exactly found by comparing the distribution of corresponding neighborhood states, (see Table I for the neighborhood correspondence). The case of CA patterns obtained after the evolution governed by rules with θ_8 or θ_{10} odd neighborhood state is additionally interested by the fact that it involves up-down symmetry, also. This extra symmetry origins from the global rule preference for up spin state in case of the first rule and down spin state when the second rule is considered. This global feature is expressed via the number of neighborhood states resulting up. The rule with θ_8 as the odd state has 7 of 16 neighborhood states giving up spin state while the rule with θ_{10} as the odd state, provides 9. Since the number of neighborhood states resulting up is the same when rules with θ_{12} and θ_{14} as the odd neighborhood states are considered, and equals to 7, the equivalence of neighborhood states in this case does not mix states belonging to up-down symmetric domains.

3. Closing remarks

If the rule considered on CA does not point at the shift direction as evidently as it occurred in the examples of the previous section, (two, three,... odd neighborhood states) the problem of choosing the common shift direction complicates. Therefore, while closing we want to some give some hints about the mechanism of designation of the global shift direction for general CA and for the complete explanation of this problem one can search in [7]. The main role in stating the global shift is played by properties of a local rule while its action is considered with respect to the neighborhood states belonging to the mean-field sets **B** or **B'** (see Fig. 1) of neighborhood states.

Roughly speaking, if a rule restricted to any of these set of states means a shift, then the more popular shift direction, either **B** or **B'** direction, becomes the global shift. The support to the level of popularity comes first of all from the global voting property of a local rule. It means that if a rule yields more *up* spin states than *down*, then the **B'** shift direction is leading and reverse. In case, when there is no preference by a rule in a spin state, then the less number of odd neighborhood states makes the popularity of a shift higher. However, if there is still "equivalence" between shifts, a lattice reaches the state satisfying both shifts.

In case when a rule neither on **B** nor **B'** states is a shift, then there are more than two lattice simple shifts that must be taken into account. As the result of variety of movements these rules could produce very regular patterns consisting of one, two or three neighborhood states only.

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