LYAPUNOV INSTABILITY OF SQUEEZING (AN APPROACH *via* QUANTUM LYAPUNOV EXPONENTS)*

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It is shown that the known instability of quadrature components of electric field during squeezing corresponds to two kinds of Lyapunov instability. One of them represents an instability of dynamics of *averages* and the other reflects a fundamental instability of the *operator valued* trajectories. The latter instability can be characterized by means of quantum characteristic exponents. The main result of the paper is the derivation of the correct Lyapunov exponent at the level of the Heisenberg picture. This shows that the quantum exponents correctly characterize properties of unstable dynamics of quantum observables.

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1. Introduction

It seems that a standard viewpoint concerning the status of chaos in quantum theories can be summarized as follows.

In classical mechanics there exist systems whose evolution in a phase space is *nonlinear*. For such systems, if additional nonintegrability conditions are satisfied, two phase space trajectories

$$t \mapsto (\vec{q}(t), \vec{p}(t)) \tag{1}$$

that are initially close to each other can diverge in such a way that the distance between them grows exponentially with time, the growth rate being the so-called Lyapunov exponent.

In quantum theories evolution of states is unitary hence linear and conserves scalar products of states. It follows immediately that not only the distance (generated by the norm of a Hilbert space) between two states is conserved by such an evolution but also the dynamics is integrable and quasi-periodic. Accordingly, on very general grounds no chaotic evolution of quantum *states* is possible. This property of quantum theories is the main difficulty in defining the notion of *quantum* chaos [1].

The situation is, in some respect, analogous to the problem of a quantum description of classically nonlinear phenomena like, say, a second harmonic generation, since their classical description involves a nonlinear evolution in a phase space (in symplectic formulation of classical electrodynamics the role of the canonical coordinates can be played by electric and magnetic fields). Still, in spite of this there *does* exist an appropriate quantum description of the phenomenon, but the nonlinearity is transferred from states to observables. (Notice that in classical mechanics canonical coordinates can play the role of both of them and (1) can be regarded also as the trajectory in the space of *observables*.) The Heisenberg equations of motion can be nonlinear operator equations although a solution to them is, of course, a one-parameter family of linear operators. It happens that this is a sufficient condition for a proper quantum treatment of classically nonlinear processes even though the evolution of states in *nonlinear quantum* optics is always linear.

Motivated by this observation two of us proposed in [2, 3] definitions of *quantum Lyapunov exponents* as a possible tool for investigation of stability properties of operator equations. In the present paper we apply in Section 2 these notions to the fixed Hamiltonian system. Then, we apply our analysis to a model describing the evolution of squeezed light in a nonlinear medium.

2. Hamiltonian system

Let us consider the system defined by the Hamiltonian

$$H = \hbar\omega a^\dagger a + i\hbar\frac{\kappa}{2}\left(a^{\dagger 2} - a^2\right), \tag{2}$$

describing a kind of a parametric amplification [4, 5]. Although the equations of motion remain here linear also in the Heisenberg picture, the system will reveal a typical Lyapunov instability. Therefore, it can be utilized for testing the meaning and correctness of the proposed definition of quantum characteristic exponents (see also Section 3).

Time evolution of the annihilation and creation operators in the Heisenberg picture is given by

$$a(t) = \left(\cosh\Omega t - i\frac{\omega}{\Omega}\sinh\Omega t\right)a(0) + \frac{\kappa}{\Omega}\sinh\Omega t\, a^\dagger(0), \tag{3}$$

$$a^\dagger(t) = \left(\cosh\Omega t + i\frac{\omega}{\Omega}\sinh\Omega t\right)a^\dagger(0) + \frac{\kappa}{\Omega}\sinh\Omega t\, a(0), \tag{4}$$

for $\omega^2 < \kappa^2$, where $\Omega = \pm\sqrt{\kappa^2 - \omega^2}$. For $\omega^2 > \kappa^2$ one has

$$a(t) = \left(\cos\Omega' t - i\frac{\omega}{\Omega'}\sin\Omega' t\right)a(0) + \frac{\kappa}{\Omega'}\sin\Omega' t\, a^\dagger(0), \tag{5}$$

$$a^\dagger(t) = \left(\cos\Omega' t + i\frac{\omega}{\Omega'}\sin\Omega' t\right)a^\dagger(0) + \frac{\kappa}{\Omega'}\sin\Omega' t\, a(0), \tag{6}$$

where $\Omega' = \pm\sqrt{\omega^2 - \kappa^2}$.

Consider now the quadrature operators

$$P_\alpha(t) = \frac{1}{2}\left(\mathrm{e}^{i\alpha}\, a(t) + \mathrm{e}^{-i\alpha}\, a^\dagger(t)\right), \tag{7}$$

$$Q_\alpha(t) = \frac{1}{2i}\left(\mathrm{e}^{i\alpha}\, a(t) - \mathrm{e}^{-i\alpha}\, a^\dagger(t)\right), \tag{8}$$

satisfying the "Hamilton equations"

$$\frac{d}{d\alpha}P_\alpha(t) = -Q_\alpha(t), \tag{9}$$

$$\frac{d}{d\alpha}Q_\alpha(t) = P_\alpha(t). \tag{10}$$

Let us consider a coherent state $a(0)|w\rangle = w|w\rangle$ and denote $w = |w|\mathrm{e}^{i\theta}$. From now on let us consider only the hyperbolic case (squeezing) $\omega^2 < \kappa^2$ (see also Section 3). We find

$$\langle w|P_\alpha(t)|w\rangle = \frac{|w|}{2}\Bigg(e^{\Omega t}\Big(\cos(\alpha+\theta)+\frac{\omega}{\Omega}\sin(\alpha+\theta)+\frac{\kappa}{\Omega}\cos(\alpha-\theta)\Big)$$
$$+e^{-\Omega t}\Big(\cos(\alpha+\theta)-\frac{\omega}{\Omega}\sin(\alpha+\theta)-\frac{\kappa}{\Omega}\cos(\alpha-\theta)\Big)\Bigg), \quad (11)$$

$$\langle w|Q_\alpha(t)|w\rangle = \frac{|w|}{2}\Bigg(e^{\Omega t}\Big(\sin(\alpha+\theta)-\frac{\omega}{\Omega}\cos(\alpha+\theta)+\frac{\kappa}{\Omega}\sin(\alpha-\theta)\Big)$$
$$+e^{-\Omega t}\Big(\sin(\alpha+\theta)+\frac{\omega}{\Omega}\cos(\alpha+\theta)-\frac{\kappa}{\Omega}\sin(\alpha-\theta)\Big)\Bigg). \quad (12)$$

These formulas show that, depending on the sign of Ω, one part of the averages exponentially grows whereas the other asymptotically vanishes. For example, the exponential growth of $\langle w|P_\alpha(t)|w\rangle$ does not occur for $\Omega > 0$ if and only if

$$\cos(\alpha+\theta)+\frac{\omega}{\Omega}\sin(\alpha+\theta)+\frac{\kappa}{\Omega}\cos(\alpha-\theta)=0. \quad (13)$$

A small change of either α or θ leads to exponentially divergent trajectories. Thus even without calculating any characteristic exponents we find here a typical Lyapunov instability.

A standard method of describing the instability of squeezing quadratures [7] is to replace the annihilation operators by their eigenvalues corresponding to some coherent state. In this particular context this is justified since the equations are linear and, as we have seen, the averages are evidently unstable. This, semiclassical in nature, method can, of course, fail if the Heisenberg equations of motion will be *nonlinear operator equations*, as is often the case in nonlinear quantum optics. In such a case this kind of substitution destroys important *quantum* characteristics encoded in the algebraic relations between the annihilation and creation operators. It follows that from this perspective some operator criterion for instability of *operator trajectories* is essential. This role, as we shall demonstrate below, is played by quantum Lyapunov exponents.

At this stage we shall consider three kinds of exponents, defined as

$$\lambda_{\alpha,w} = \lim_{t\to\infty}\frac{1}{t}\ln\Big|\langle w|\frac{dQ_\alpha(t)}{d\alpha}|w\rangle\Big| = \lim_{t\to\infty}\frac{1}{t}\ln\Big|\frac{d}{d\alpha}\langle w|Q_\alpha(t)|w\rangle\Big|, \quad (14)$$

$$\lambda_\alpha = \lim_{t\to\infty}\frac{1}{t}\ln\Big\|\frac{dQ_\alpha(t)}{d\alpha}\Big\|, \quad (15)$$

$$\lambda_\theta = \lim_{t\to\infty}\frac{1}{t}\ln\Big|\frac{d}{d\theta}\langle w|Q_\alpha(t)|w\rangle\Big|. \quad (16)$$

Exponents of the form (14) and (15) were proposed in [2]; equation (16) is just a classical Lyapunov exponent applied to the trajectory

$$t \mapsto \langle w | Q_\alpha(t) | w \rangle . \tag{17}$$

The three exponents have different meanings. Quantities $\lambda_{\alpha,w}$ and λ_α are notions characterizing the stability of solutions of the Heisenberg equations of motion with respect to changes of parameters in observables. Moreover, since $|w\rangle$ is α-independent the exponent $\lambda_{\alpha,w}$ is simultaneously a classical exponent characterizing the stability with respect to α of the dynamics (17). Exponent λ_α is not capable of detecting any instability with respect to modifications of states but is a purely operator criterion on existence of trajectories that are unstable with respect to changes of initial conditions in the Heisenberg equations of motion. Indeed, since the norm of an operator is physically a maximal average of the observable represented by this operator it follows that a positivity of λ_α indicates an existence of a state whose corresponding trajectory (17) is unstable. The third exponent λ_θ measures the stability of the trajectory (17) with respect to modifications of states.

A simple calculation shows that

$$\lambda_{\alpha,w} = \lambda_\theta = \Omega \tag{18}$$

for $\Omega > 0$ and

$$\lambda_{\alpha,w} = \lambda_\theta = -\Omega \tag{19}$$

for $\Omega < 0$ unless $\lambda_{\alpha,w} = \lambda_\theta = -\infty$ which may occur if the conditions of the form (13) are satisfied. The unboundedness of $Q_\alpha(t)$ and $P_\alpha(t)$ makes however $\lambda_\alpha = \infty$, which shows that a straightforward application of the norm version of the exponent does not lead here to a proper characterization of the instability (this difficulty will be always present if a derivative of an operator is unbounded). In this case we have to consider a modification of λ_α. We shall consider two possibilities.

In the first place we can use the freedom present in the definition of the quantum Lyapunov exponent given in [2] where one considers a *directional derivative* in some direction in a Banach space of linear operators. Since

$$\begin{aligned} Q_\alpha(t) = & \Big(\cosh \Omega t - \frac{\kappa}{\Omega} \cos 2\alpha \, \sinh \Omega t\Big) Q_\alpha(0) \\ & + \Big(-\frac{\omega}{\Omega} \sinh \Omega t + \frac{\kappa}{\Omega} \sin 2\alpha \, \sinh \Omega t\Big) P_\alpha(0) \\ = & Q_\alpha\big(t, Q_\alpha(0), P_\alpha(0)\big) , \end{aligned} \tag{20}$$

we can differentiate $Q_\alpha\big(t, Q_\alpha(0), P_\alpha(0)\big)$ in a direction $E\,Q_\alpha(0)\,E$, where E is some projector cutting off the spectrum of $Q_\alpha(0)$ and making $E\,Q_\alpha(0)\,E$

bounded. The derivative is defined as

$$D_{E\,Q_\alpha(0)\,E}Q_\alpha(t) = \lim_{s\to 0}\frac{1}{s}\Big(Q_\alpha\big(t, Q_\alpha(0)+s\,E\,Q_\alpha(0)\,E, P_\alpha(0)\big) - Q_\alpha\big(t, Q_\alpha(0), P_\alpha(0)\big)\Big)\,. \tag{21}$$

We find that

$$D_{E\,Q_\alpha(0)\,E}Q_\alpha(t) = \Big(\cosh\Omega t - \frac{\kappa}{\Omega}\cos 2\alpha\,\sinh\Omega t\Big)\,E\,Q_\alpha(0)\,E\,. \tag{22}$$

Now, since the derivative is a bounded operator the difficulties encountered for λ_α disappear and we can calculate the modified exponent

$$\lambda'_\alpha = \lim_{t\to\infty}\frac{1}{t}\ln\Big\|D_{E\,Q_\alpha(0)\,E}Q_\alpha(t)\Big\| \tag{23}$$

equal again to Ω for $\Omega > 0$ and $-\Omega$ for $\Omega < 0$, which shows that not only the exponents are independent of the cutoff but also yield the correct characterization of the instability. A physical meaning of this result is that in order to characterize a stability of operator equations it is sufficient to consider those trajectories in the space of observables that differ initially by a bounded operator.

Another way of circumventing the difficulties with the unboundedness of amplitudes is the following. Consider an operator

$$I_n = \frac{1}{\pi}\int_{\mathcal{O}_n} d^2w|w\rangle\langle w|\,, \tag{24}$$

where $\{\mathcal{O}_n\}$ is an increasing family of compact subsets of the complex plane $\mathbf{C}$, $I_n \nearrow \mathbf{1}$ and $\mathcal{O}_n \nearrow \mathbf{C}$. The cut off operators $I_nP_\alpha I_n := P^n_\alpha$ and $I_nQ_\alpha I_n := Q^n_\alpha$ are bounded and satisfy the "Hamilton equations". The exponents

$$\lambda^n_\alpha = \lim_{t\to\infty}\frac{1}{t}\ln\Big\|\frac{dQ^n_\alpha(t)}{d\alpha}\Big\|\,, \tag{25}$$

are equal to $\lambda'_\alpha = \lambda_{\alpha,w} = \lambda_\theta$ which suggests another possible definition of quantum Lyapunov exponents

$$\lambda''_\alpha = \lim_{n\to\infty}\lambda^n_\alpha = |\Omega|\,. \tag{26}$$

3. Example: squeezing in $\chi^{(2)}$ nonlinear medium

An analysis of the squeezed light in a nonlinear medium having the second-order susceptibility $\chi^{(2)}$ leads to the following equations (see [6], Section 11):

$$\frac{d}{dz}a(z) = ka^{\dagger}(z), \tag{27}$$

$$\frac{d}{dz}a^{\dagger}(z) = k^{*}a(z), \tag{28}$$

where $a(z)$ is the annihilation operator, k is the coupling constant and k^* stands for the complex conjugation of k. The dependence of a on z is a result of the one dimensional propagation of the electric field along the z-axis. Obviously, the equations (27), (28) lead to

$$\frac{d^2}{dz^2}a(z) = |k|^2 a(z). \tag{29}$$

It is easy to observe that the same equation can be derived from the Hamiltonian (2), where now $|k|$ is equal to Ω and $\omega = 0$ (*cf.* (3)). Such a Hamiltonian would appear if we included in (2) explicitly an oscillatory time dependence of a, and then eliminate the free evolution by the interaction picture. The solution to equations (27), (28) is

$$a(z) = \cosh(|k|z)a(0) + \frac{k}{|k|}\sinh(|k|z)a^{\dagger}(0). \tag{30}$$

Now it is clear that we can apply the results of Section 2 to analysis of the propagating squeezed light. In particular, we have

$$\begin{aligned} P_{\alpha}(z) = \tfrac{1}{2}\Big(&[e^{|k|z}\sin\alpha + e^{-|k|z}\cos\alpha]\,a(0) \\ &+ [-e^{|k|z}\sin\alpha + e^{-|k|z}\cos\alpha]\,a^{\dagger}(t)\Big), \end{aligned} \tag{31}$$

$$\lambda''_{\alpha} = \lim_{n\to\infty}\lim_{z\to\infty}\frac{1}{z}\ln\left\|\frac{dQ^n_{\alpha}(z)}{d\alpha}\right\| = |k| \geq 0, \tag{32}$$

where we put for simplicity $\frac{k}{|k|} = -1$. The last assumption is discussed in [6].

4. Concluding remarks

The quadrature operators P_α and Q_α of squeezed light exhibit typical Lyapunov instabilities. The instability of averages follows from the more fundamental instability of *observables*, and the latter can be characterized at the level of the operator equations by means of the quantum Lyapunov exponents. The fact that the values of quantum exponents are the same as those of classical exponents applied to averages suggests, that the quantum exponents can play a role of a universal tool applicable to more complicated nonlinear operator equations, where no semiclassical replacements of operators by averages would be justified.

On the other hand, it was shown [3] that a chaotic behavior of a two level system interacting with a single mode of the electromagnetic field that appears after *semiclassical approximations* does not correspond to positivity of quantum Lyapunov exponents calculated without such approximations. This suggests that a semiclassical treatment of a quantum model can change important quantum features of the model.

Finally, let us remark that the exponent $\lambda_{\alpha,w}$ can be also given a purely operator interpretation: It measures a divergence of operator trajectories $t \mapsto Q_\alpha(t)$ with the distance being given by a "metric" induced by a state, thus the "metric" weaker than the one induced by the norm. Again, since for general reasons no instability occurs for trajectories in the space of states, the instability of averages must have a purely operator meaning.

The fact that some instabilities can occur even for quantum systems is not new [3, 9, 8] and is not in itself particularly important here. What is important, is the fact that the quantum exponents form a tool that *works* and hence can be applied for a broader class of nonlinear quantum phenomena.

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REFERENCES

[1] F. Haake, *Quantum Sigantures of Chaos*, Springer, Berlin 1991.
[2] W.A. Majewski, M. Kuna, *J. Math. Phys.* **34**, 5007 (1993).
[3] M. Kuna, W.A. Majewski, *Rep. Math. Phys.* **33**, 111 (1993).
[4] G. Grynberg, *Photons and Atoms: Introduction to Quantum Electrodynamics*, Wiley, New York 1989.
[5] H.P. Yuen, *Phys. Rev.* **A13**, 2226 (1976).
[6] B. Yurke, *Squeezed Light*, University of Rochester, Rochester 1989.
[7] D.F. Walls, G.J. Milburn, *Quantum Optics*, Springer, Berlin 1994.
[8] M. Hossein Partovi, *Phys. Rev.* **A55**, 555 (1992).
[9] R.L. Ingraham, G.L. Acosta, *Phys. Lett.* **A181**, 450 (1993).