NON-MARKOVIAN DICHOTOMIC NOISES*,**

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Two kinds of stochastic processes are discussed: explicitly non-markovian dichotomic noise with exponential damping of the memory, and implicitly non-markovian composite noise being a (linear and/or nonlinear) combination of several independent markovian dichotomic noises. The description of stochastic flows driven by such noises is given. To illustrate how the non-markovianity changes the behavior of the driven process, the evolution in time of the probability density $P(x,t)$ describing the flow $\dot{X}(t) = \xi(t)$ (the random telegraph process) driven by the non-markovian process $\xi(t)$ is calculated and compared with that driven by markovian $\xi(t)$. Among others, in the non-markovian case oscillations in $P(x,t)$ are found, and the possibility of additional noise-induced transitions is indicated.

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1. Introduction

In applications of stochastic theory to various physical, chemical, biological, *etc.* problems, there is growing interest in the use of colored noises, as more realistic than the widely used Gaussian white noise. Of these, more and more popular [1–15] becomes recently the so-called dichotomic noise (DN), *i.e.* the two-state stochastic process (random telegraph signal). Its main assets are: *(i)* DN is colored, *(ii)* application of DN results in relatively simple calculations, especially when the noise enters kinetic equations in a nonlinear fashion [14] or when dealing with linear multidimensional flows [10, 11], *(iii)* well-defined limiting procedures lead from DN to both

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Gaussian white noise and white shot noise [1–4, 15]. Moreover, existing formulations can be generalized for composite DN's, which seem to be still better representations (models) of real noises and fluctuations.

So far, almost without exception, solely markovian stochastic processes have been used as driving noises. However, in real systems where the noise originates (at least partially) from the averaging out of very many fast variables [16], we may expect that system variables form a kind of hierarchies, in which the "higher-level" variables are driven by "lower-level" ones, *e.g.* according to the (simplified) scheme:

$$\dot{X}_n = f(X_n) + g(X_n, X_{n-1}), \tag{1.1}$$

where variable $X_{n-1}(t)$ acts as a driving stochastic process for the process $\dot{X}_n$ at the level n. Therefore in many cases the markovianity is but an idealization. On the other hand, non-markovian stochastic processes are more difficult to use as the driving forces than markovian ones. This seems to be one of the reasons why in most of applications so far it is the markovian noises which have been used as the driving processes. Only very recently a few papers have been published which deal with non-markovian driving, either explicitly [13] or implicitly [17]. Besides, non-markovian seems to be also the composite noise built of markovian DN and markovian Gaussian white noise [12]. Systematic theory of explicitly non-markovian noises with exponential damping of the memory has been recently proposed by the present author [15]. On the other hand, it is well-known that almost any stochastic flow $\dot{X}(t)$ driven by a colored markovian noise is a non-markovian process by itself. In this sense there is a vast literature on non-markovian stochastic processes, though this fact is mentioned explicitly very rarely (for the explicit discussion of non-markovian effects in driven processes *cf. e.g.* [18, 19] and references therein). What we want to stress here is that, to the best of author's knowledge, almost absent from the literature is the use of non-markovian noises as driving processes.

This lecture will present the general properties of both markovian and non-markovian DN, the master equations for stochastic flows driven by such noise, and some techniques which enable the mentioned above simple applications of markovian DN. The general formulation for markovian DN is based mainly on Refs. [1–3] (with generalizations for composite DN's), for non-markovian DN — on Ref.[15]. The numerical results for non-markovian random telegraph process (Section 4) and the general results for the composite noise (Section 6) are new.

The text consists of three distinct parts. First lists definitions and properties of general DN itself, containing both markovian and exponentially damped explicitly non-markovian components (Sections 2–3). Second part is devoted to the stochastic flows driven by such noise (Section 4) and demonstrates, among others, how the non-markovianity of the driving noise changes the behaviour of the driven process. Third part (Sections 5–6) presents general formulation for stochastic flows driven by markovian DN, and by composite noises.

2. Definitions and basic properties

The asymmetric dichotomic noise $\xi(t)$, called also the random telegraph signal, is the random two-state process with zero mean:

$$\xi(t) \in \{\Delta_1, -\Delta_2\}\,, \quad \xi^2(t) = \Delta^2 + \Delta_0 \xi(t)\,, \quad \langle \xi(t) \rangle = 0\,, \tag{2.1}$$

where $\Delta^2 = \Delta_1 \Delta_2$, $\Delta_0 = \Delta_1 - \Delta_2$. Let λ_1 and λ_2 be the probabilities of switching (per unit time) between states $\xi_1 = \Delta_1$ and $\xi_2 = -\Delta_2$. Therefore, $\tau_i = 1/\lambda_i$ will be mean sojourn times in these states. Denote also $2D = \Delta_1 + \Delta_2$, $\Lambda = \lambda_1 + \lambda_2$.

The condition $\langle \xi(t) \rangle = 0$ can be written explicitly in two equivalent ways as:

$$\sum_i \xi_i \tau_i = 0 \quad \text{and} \quad \sum_i P_1(\xi_i, t)\xi_i = 0\,, \tag{2.2}$$

where $P_1(\xi_i, t)$ is the probability of finding the process in the state ξ_i at time interval $(t, t+dt)$. (2.2) together with the normalization $\sum_i P_1(\xi_i, t) = 1$ gives that

$$\frac{\Delta_1}{\lambda_1} = \frac{\Delta_2}{\lambda_2} = w_0\,, \tag{2.3}$$

$$P_1(\xi, t) = P_{st}(\xi)\,,\ P_{st}(\xi = \xi_i) \equiv P_{st,i} = 1 - \frac{\Delta_i}{2D} = 1 - \frac{\lambda_i}{\Lambda} = \frac{\tau_i}{\tau_1 + \tau_2}\,, \tag{2.4}$$

which means that the zero-mean DN is stationary.

The basic property of DN, *viz.* $\xi(t)^2 = \Delta^2 + \Delta_0 \xi(t)$, enables the linearization of functions of $\xi(t)$ (at least of these which can be expanded into a power series of its arguments):

$$f(\xi) = f_1 + \xi f_2\,. \tag{2.5}$$

Especially,

$$\frac{1}{1 + a\xi} = \frac{1 + a\Delta_0 - a\xi}{(1 + a\Delta_1)(1 - a\Delta_2)}\,, \tag{2.6a}$$

$$e^{a\xi} = e^{a\Delta_0/2}(\alpha + \beta\xi)\,, \quad \beta = \frac{1}{D}\sinh(aD)\,, \quad \alpha = \cosh(aD) - \frac{\Delta_0}{2D}\sinh(aD)\,, \tag{2.6b}$$

$$\delta_{\xi(t),\xi_i} = P_{st}(\xi_i) + \frac{\epsilon_i}{w_0 \Lambda}\xi(t)\,, \tag{2.6c}$$

where $\epsilon_1 = 1$, $\epsilon_2 = -1$. The last identity enables us to write another identity, which will be useful below:

$$\xi(t) = \tfrac{1}{2}\Delta_0 + D\left[\delta_{\xi(t),\Delta_1} - \delta_{\xi(t),-\Delta_2}\right]. \tag{2.6d}$$

Asymmetric dichotomic noise ξ can be expressed by symmetric dichotomic noise ξ_s of non-zero mean:

$$\xi = \xi_s - \langle \xi_s \rangle = \xi_s + \tfrac{1}{2}\Delta_0\,, \quad \xi_s \in \{\pm D\}. \tag{2.7}$$

White noises as limits of the dichotomic noise [1-3,15]

White noises are understood here as sequences of δ-spikes. The limit

$$\lambda_1 \to \infty\,, \quad \Delta_1 \to \infty\,, \quad \Delta_1/\lambda_1 = \Delta_2/\lambda_2 = w_0\,, \tag{2.8}$$

with w_0 kept constant defines the so-called *white shot noise* (WSN), being the sequence of separated positive δ-spikes on negative background. The limit:

$$\lambda_1 = \lambda_2 = \lambda \to \infty,\ \Delta_1 = \Delta_2 = \Delta \to \infty,\ \Delta^2/2\lambda = D_0^2\,, \tag{2.9}$$

defines the *Gaussian white noise* (GWN) as the dense set of positive and negative δ-spikes, which corresponds to the Stratonovich interpretation of the Wiener process. GWN can be obtained also from the WSN as the limit:

$$\lambda_2 \to \infty\,,\ \Delta_2 \to \infty\,,\ w_0 = \Delta_2/\lambda_2 \to 0,\ \lambda_2 w_0^2 = D_0^2\,. \tag{2.9a}$$

In all calculations based on these limits it is implicitly assumed that

$$\lim_{\lambda\to\infty} \lambda^m e^{-\lambda^n t} = \delta(t)\,, \quad m,n > 0\,. \tag{2.10}$$

Probabilities

Basic quantities are the n-point (unconditional) probabilities P_n and conditional probabilities $P_{m|n}$, defined as follows:

$$P_n(\xi_1,t_1;\ldots;\xi_n,t_n) = \langle \delta_{\xi(t_1),\xi_1} \cdots \delta_{\xi(t_n),\xi_n} \rangle\,, \tag{2.11}$$

$$P_{m|n}(1;\cdots;m|m+1;\cdots;m+n) = \frac{P_{m+n}(1;\cdots;m+n)}{P_n(m+1;\cdots;m+n)}\,, \tag{2.12}$$

where $\langle\cdots\rangle$ denotes the averaging over all realizations of the process $\xi(t)$, and n stands for the couple (ξ_n, t_n). (2.12) is just the Bayes rule.

These distributions have the following obvious properties, resulting directly from their definitions:

$$\sum_{\xi_0} P_{n+1}(\mathbf{0};\mathbf{1};\cdots;\mathbf{n}) = P_n(\mathbf{1};\cdots\mathbf{n}), \tag{2.13}$$

$$\sum_{\xi_0} P_{1|n}(\mathbf{0}|\mathbf{1};\cdots;\mathbf{n}) = 1 \tag{2.14}$$

(normalization). Definition (2.11) implies that

$$P_{n+1}(\xi_0,t_1;\xi_1,t_1;\cdots;\xi_n,t_n) = \delta_{\xi_0,\xi_1} P_n(\xi_1,t_1;\cdots;\xi_n,t_n), \tag{2.15}$$

and (2.12) with (2.15) that

$$P_{1|n}(\xi_0,t_1|\xi_1,t_1;\cdots;\xi_n,t_n) = \delta_{\xi_0,\xi_1}\,. \tag{2.16}$$

3. Master equations

The specific, non-markovian dichotomic process considered here is defined [15] by the following non-markovian master equation, fulfilled by every probability $P(\xi,t) \equiv P_{n+1}(\xi,t;\xi_1,t_1;...;\xi_n,t_n)$, $t \geq t_0 = \max\{t_1,...,t_n\}$:

$$\dot{P}(\Delta_1,t) = -\dot{P}(-\Delta_2,t) = -\int_{t_0}^{t} dt' K(t-t')[\lambda_1 P(\Delta_1,t') - \lambda_2 P(-\Delta_2,t')] \tag{3.1}$$

(overdot denotes $\partial/\partial t$), with the kernel $K(\tau)$ containing both markovian and non-markovian contributions:

$$K(t-t') = \gamma_0\delta(t-t') + \gamma_1 e^{-\nu(t-t')}, \tag{3.2}$$

where the parameters γ_0 and γ_1 describe the relative contributions of markovian and non-markovian parts, and ν the rate of damping of the non-markovian memory. For $\gamma_1 = 0$, $\gamma_0 = 1$ we recover the formulae for markovian DN [20].

It is easy to check that, therefore, the conditional probabilities $P_{n|m}$ fulfill the same master equation (3.1).

The solution to these equations, valid for $n \geq 1$, $t > t_0 = t_1$, reads [15]:

$$P_{n+1}(\xi,t;\mathbf{1};\ldots;\mathbf{n}) = \left[P_{\rm st}(\xi) + \Lambda^{-1}\psi(t-t_1)\psi_0(\xi)\phi_0(\xi_1)\right]P_n(\mathbf{1};\ldots;\mathbf{n}), \tag{3.3}$$

and thus

$$P_{1|n}(\xi, t|1; ...; n) = P_{st}(\xi) + \Lambda^{-1}\psi_o(\xi)\psi(t - t_1)\phi_0(\xi_1)\,, \tag{3.4}$$

where

$$\phi_0(x) = \phi(t_0) = \lambda_1\delta_{\Delta_1,x} - \lambda_2\delta_{-\Delta_2,x}\,, \quad \psi_0(\xi) = \delta_{\Delta_1,\xi} - \delta_{-\Delta_2,\xi}\,,$$
$$\psi(t) = \Gamma^{-1}\left[(\theta_1 - \nu)e^{-\theta_1 t} - (\theta_2 - \nu)e^{-\theta_2 t}\right],$$
$$\theta_{1,2} = \tfrac{1}{2}\left(\nu + \gamma_0\Lambda \pm \Gamma\right), \quad \Gamma = \sqrt{(\gamma_0\Lambda - \nu)^2 - 4\gamma_1\Lambda}\,. \tag{3.5}$$

Therefore, the time dependence of probability distributions is described by the combination of two exponentials. Moreover, for some combinations of parameters γ_0, γ_1, Λ and ν, this dependence may become damped oscillatory. The physical meaning of these parameters implies that $\nu > 0$ and $\Lambda > 0$. γ_i can be either positive or negative, with limitations imposed by the convergence condition $\theta_i > 0$. Note that for purely markovian process (2.1), $\gamma_1 = 0$, the time dependence is given by $\psi(t) = \exp(-\gamma_0\Lambda t)$. The "phase diagram" of the types of behaviour of probability distributions in the parameter space is shown in Fig.1.

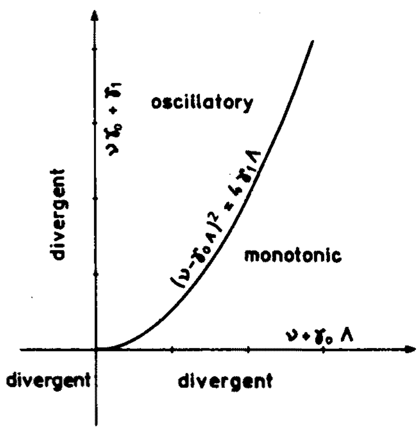

Fig. 1. Regions in the parameter space of divergent, monotonically damped, and damped oscillatory behaviour of P(t)'s.

These distributions have the following property: because

$$\lim_{t\to\infty} \psi(t - t_i) = 0 \quad \text{for } t_i < \infty\,, \tag{3.6}$$

then

$$\lim_{t\to\infty} P_{1|n}(\xi, t|\xi_1, t_1; \ldots \xi_n, t_n) = P_{st}(\xi)\,, \tag{3.7a}$$

$$\lim_{t\to\infty} P_{n+1}(\xi,t;\xi_1,t_1;...\xi_n,t_n) = P_{st}(\xi)P_n(\xi_1,t_1;...\xi_n,t_n)\,, \tag{3.7b}$$

when t_1 remains finite, or rather, more strictly, when $t - t_1 \to \infty$. This means that also for the non-markovian process the events separated by very long times become uncorrelated, and that the considered non-markovian process is irreversible and its stationary distributions do not remember initial state. Moreover, the stationary distributions are the same as for markovian process. These properties are the direct consequence of the assumption of exponential damping of the memory kernel.

The above results give main characteristics of non-markovian dichotomic noise and are valid for ordered time sequences $t \geq t_1 \geq \ldots \geq t_n$ only. It can be shown that choosing initial condition at some time t_0 earlier than at least one of time moments from the set $\{t_1,\ldots t_n\}$ leads to results incompatible with each other. This property is related to the non-markovian character of the process $\xi(t)$.

Averages

Any average both of the functions of the process $\xi(t)$, $\langle F(\xi(t_1))\, G(\xi(t_2))\ldots\rangle$, and of the functionals of $\xi(t)$, $\langle F(...;[\xi(t)])\rangle$, by virtue of (2.1), can be expressed by combinations of averages of the type:

$$\langle\xi(t_1)\ldots\xi(t_n)\rangle = \sum_{\xi_1}\cdots\sum_{\xi_n}\xi_1\ldots\xi_n P_n(\xi_1,t_1,\ldots\xi_n,t_n)\,. \tag{3.8}$$

The latter, in turn, can be calculated from the recurrence formula [15]:

$$\langle\xi(t_1)\ldots\xi(t_n)\rangle = \Delta^2\psi(t_1-t_2)K_{n-1}(t_2,\ldots t_n)\,, \tag{3.9}$$

with

$$K_{n-1}(t_2,\ldots t_n) = \langle\xi(t_3)\ldots\xi(t_n)\rangle + \Delta_0\psi(t_2-t_3)K_{n-2}(t_3,\ldots t_n)\,. \tag{3.10}$$

with $K_1 = \langle\xi(t_1)\rangle = 0$. Especially, the two-point correlation function reads:

$$K_2 = \langle\xi(t_1)\xi(t_2)\rangle \equiv C_2(|t_1-t_2|) = \Delta^2\psi(|t_1-t_2|)\,. \tag{3.11}$$

This simplifies for symmetric DN, $\Delta_0 = 0$:

$$\begin{aligned}\langle\xi(t_1)\ldots\xi(t_{2n})\rangle =&\Delta^2\psi(t_1-t_2)\Delta^2\psi(t_3-t_4)\ldots\Delta^2\psi(t_{2n-1}-t_{2n})\\ =&\langle\xi(t_1)\xi(t_2)\rangle\langle\xi(t_3)\xi(t_4)\rangle\ldots\langle\xi(t_{2n-1})\xi(t_{2n})\rangle, \qquad (3.12a)\\ \langle\xi(t_1)\ldots\xi(t_{2n+1})\rangle =&0. \qquad (3.12b)\end{aligned}$$

These results are valid only for ordered time sequences $t_1 \geq t_2 \geq \ldots \geq t_n$. For $t_1 \to \infty$, t_2 remaining finite,

$$\lim_{t_1\to\infty}\langle\xi(t_1)\xi(t_2)\ldots\xi(t_n)\rangle = 0\,. \tag{3.13}$$

One of corollaries of these results reads:

$$\frac{\partial}{\partial t}\langle\xi(t)\xi(t_1)\ldots\xi(t_n)\rangle = -\chi(t-t_1)\langle\xi(t)\xi(t_1)\ldots\xi(t_n)\rangle\,, \tag{3.14}$$

where

$$\chi(t) = -\dot{\psi}(t)/\psi(t) = \frac{\theta_2(\theta_2-\nu)-\theta_1(\theta_1-\nu)e^{-\Gamma t}}{\theta_2-\nu-(\theta_1-\nu)e^{-\Gamma t}}\,. \tag{3.15}$$

In the markovian case $\chi(t) \to \Lambda$, and the result (3.14) leads directly to the Shapiro–Loginov theorem [21] which states that, for any (markovian) exponentially correlated coloured noise $\xi(t)$ of zero mean

$$\frac{d\langle\xi(t)f(t)\rangle}{dt} = -\Lambda\langle\xi(t)f(t)\rangle + \langle\xi(t)\dot{f}(t)\rangle\,. \tag{3.16}$$

This formula is very useful in practical calculations (it will be used in Sections 5 and 6 below. *Cf.* also [10, 11, 14]). Unfortunately, for non-markovian processes $\xi(t)$ the Shapiro–Loginov theorem ceases to be true.

Another corollary is the shape of the frequency spectrum (power spectrum) of the noise:

$$g(\omega) = \int_{-\infty}^{+\infty} d\tau\, e^{i\omega\tau}\, C_2(\tau)\,, \tag{3.17}$$

which reads for the non-markovian DN:

$$g(\omega) = \Delta^2\frac{x\omega^2-\nu(\omega^2-y)}{x^2\omega^2+(\omega^2-y)^2}\,, \tag{3.18}$$

with $x = \nu+\gamma_o\Lambda$, $y = (\nu\gamma_0+\gamma_1)\Lambda$. The shape of $g(\omega)$ is shown in Fig. 2 for a few different values of memory parameter ν. It is seen that the deviations from the Lorentzian shape are the more pronounced, the longer is the non-markovian memory $1/\nu$.

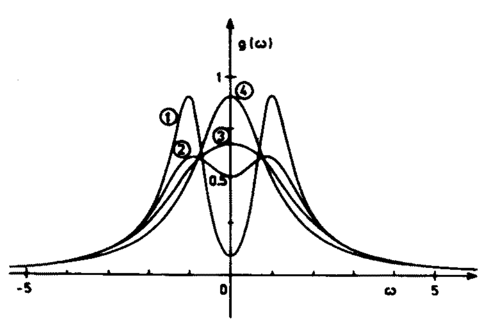

Fig. 2. The frequency spectrum of mixed markovian and non-markovian DNs. Curves differ by the values of ν; 1: $\nu = 0.1$, 2: $\nu = 1.0$, 3: $\nu = 2.0$, 4: $\nu = 10.0$. Curves 1-3 correspond to oscillating $\psi(t)$, curve 4 to monotonically decreasing $\psi(t)$.

Still another corollary, valid for symmetric DNs ($\Delta_0 = 0$) reads (details of calculations are given in the Appendix A):

$$A(t,t_0) = \left\langle \exp\left[\alpha \int_{t_0}^{t} dt'\xi(t')\right]\right\rangle = \sum_{j=1}^{3} \frac{(z_j+\theta_1)(z_j+\theta_2)}{(z_j-z_k)(z_j-z_l)} e^{z_j(t-t_0)},$$

$$= \alpha^2\Delta^2 \sum_{j=1}^{3} \frac{z_j+\nu}{z_j(z_j-z_k)(z_j-z_l)} e^{z_j(t-t_0)}, \qquad (3.19a)$$

$$\left\langle \xi(t) \exp\left[\alpha \int_{t_0}^{t} dt'\xi(t')\right]\right\rangle = \frac{1}{\alpha}\frac{\partial}{\partial t} A(t,t_0), \qquad (3.19b)$$

$$\left\langle \int_{t_0}^{t} dt'\xi(t) \exp\left[\alpha \int_{t_0}^{t'} dt''\xi(t'')\right]\right\rangle = \frac{1}{\alpha}\left[A(t,t_0)-1\right], \qquad (3.19c)$$

where z_j are the solutions of the algebraic equation:

$$z^3 + (\theta_1+\theta_2)z^2 + \left(\theta_1\theta_2 - \alpha^2\Delta^2\right)z - \alpha^2\Delta^2\nu = 0. \qquad (3.20)$$

4. Processes driven by non–markovian DN

We shall consider general one-dimensional stochastic flows of the form:

$$\dot{X} = f(X) + g(X)\xi(t). \qquad (4.1)$$

More general forms of the type of $\dot{X} = F(X, \xi(t))$, can be reduced to (4.1) by virtue of (2.1)[1]. Generalizations for multidimensional flows will be discussed below.

Basic quantities describing the flow (4.1). are the probability density $P(x,t)$ that at time interval $(t, t+dt)$ the value of the process $X(t)$ lies in the interval $(x, x+dx)$ and the joint probability densities $p(x, \xi_i, t)$ that $X(t) \in (x, x+dx)$ and $\xi(t) = \xi_i$, defined:

$$P(t) = P(x,t) = \sum_k p_k \delta(X(t,[\xi]) - x) \equiv \langle \delta(X(t,[\xi]) - x)\rangle, \qquad (4.2)$$

[1] Note that such more general forms are meaningful, in general, for colored noises only. Non-linear functions of white noises are ill-defined: the white noise is equivalent to a series of delta-functions, which means that the square (and higher powers) of white noise is meaningless.

$$p_i(t) = p(x, \xi_i, t) = \langle \delta(X(t, [\xi]) - x) \delta_{\xi(t),\xi_i} \rangle . \tag{4.3}$$

Note that the (Dirac) δ-function $\delta(x - X(t, [\xi])$ and the (Kronecker) δ-function $\delta_{\xi(t),\xi_i}$ are the corresponding probability distributions *for k-th realization of the stochastic process* $\xi(t)$,[2] that at time interval $(t, t + dt)$ the value of the process $X(t)$ lies in the interval $(x, x + dx)$, and that $\xi(t) = \xi_i$, p_k being the probability of the k-th realization. The averaging is over all possible realizations of $\xi(t)$.

The standard method [1-3] leads to the following master equations for $p_i(t)$ [15]:

$$\frac{\partial}{\partial t} p_i(t) = - \frac{\partial}{\partial x} [f(x) + \xi_i g(x)] p_i(t) - \epsilon_i \gamma_0 [\lambda_1 p_1(t) - \lambda_2 p_2(t)] - \epsilon_i \gamma_1 \int_{t_0}^{t} dt' e^{-\nu(t-t')} [\lambda_1 h_1(t;t') - \lambda_2 h_2(t;t')] , \tag{4.4}$$

where $\epsilon_1 = 1$, $\epsilon_2 = -1$, and

$$h_i(t;t') = h(x, t; \xi_i, t') = \langle \delta(X(t, [\xi]) - x) \delta_{\xi(t') \xi_i} \rangle , \quad t \geq t' . \tag{4.5}$$

Equation for $P(x, t)$ results from the obvious relations:

$$P(x, t) = p_1(t) + p_2(t) = h_1(t;t') + h_2(t;t') . \tag{4.6}$$

This means that for the non-markovian case the standard procedure [1–3] does not lead to a closed set of equations describing the probability densities of interest. Indeed, master equation for functions $h_i(t;t')$ contains still higher-order functions:

$$\frac{\partial}{\partial t} h_i(t, t') = - \frac{\partial}{\partial x} \left[f(x) + \frac{\Delta_0}{2} g(x) \right] h_i(t, t') - \frac{D}{2} \frac{\partial}{\partial x} g(x) \left[{}^2h_{1i}(t, t') - {}^2h_{2i}(t, t') \right] , \tag{4.7}$$

where

$${}^2h_{ji}(t, t') = {}^2h(x, \xi_j, t; \xi_i, t') = \langle \delta(X(t, [\xi]) - x) \delta_{\xi(t),\xi_j} \delta_{\xi(t'),\xi_i} \rangle , \tag{4.8}$$

and so on. To obtain a workable scheme of calculation, such *hierarchy* of master equations must be broken by some approximation.

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[2] *i.e.*, given definite series of switches between $+\Delta_1$ and $-\Delta_2$ at given specific times $0 < t_1 < t_2 < ... < t_i < ... < t$

In [15] we have shown that a good approximation is given by the *ansatz*, based on the shifting of the time dependence of the auxiliary function h by the function $\psi(t-t')$:

$$h_i(t;t') \approx \Delta^2 \psi(t-t') p_i(t') . \tag{4.9}$$

In this approximation the stationary distribution has the same form as the stationary distribution for the purely markovian case, Eq. (5.14) below, with renormalized parameter Λ in the exponent [15]:

$$\Lambda \to \left[\gamma_0 + \frac{2\gamma_1 \nu}{(\nu+\theta_1)(\nu+\theta_2)}\right]\Lambda . \tag{4.10}$$

Markovian-type approximation $h_i(t;t') \approx p_i(t)$ leads to much poorer results than approximation (4.9), whereas approximation "inverse" to (4.7): $h_i(t;t') \approx [\Delta^2\psi(t-t')]^{-1} p_i(t)$ seems to be completely wrong. Another possible relatively simple approximation — the markovian approximation to $\dot{h}_i(t,t')$ — can be obtained by neglecting last term on the r.h.s. of Eq. (4.7), containing higher-order functions ${}^2h_{ji}$. For $\Delta_0 = 0$, however, this approximation becomes identical with the zero-order approximation $h_i(t,t') = p_i(t')$ (markovian approximation for $h_i(t,t')$ itself), which is rather poor in comparison with (4.9) [15]. Therefore we shall use the approximation (4.9) to show how the non-markovianity of the driving noise influences the driven process.

For this aim let us consider the *random telegraph process*[3]:

$$\dot{X}(t) = \xi(t) , \tag{4.11}$$

i.e. the flow (4.1) with $f=0$, $g=1$. Such flow has this advantage that the observed effects (the behaviour of the driven process) are purely stochastic, and are not obscured by the deterministic part of the flow.

The elimination of functions p_i, h_i leads eventually to the following non-markovian telegrapher's equation for $P(x,t)$:

$$\left(\frac{\partial^2}{\partial t^2} + \Delta_0 \frac{\partial^2}{\partial t \partial x} - \Delta^2 \frac{\partial^2}{\partial x^2} + \gamma_0 \Lambda \frac{\partial}{\partial t}\right) P(x,t)$$
$$= -\gamma_1 \Lambda \Delta^2 \int_{t_0}^{t} dt' e^{-\nu(t-t')} \psi(t-t') \frac{\partial}{\partial t'} P(x,t') . \tag{4.12}$$

The evolution in time of $P(x,t)$ is presented in Figs. 3–10 for the case of symmetric DN ($\Delta_0 = 0$). Figs. 3–7 illustrate the effect of increasing

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[3] this term is used in literature also to denote the dichotomic noise $\xi(t)$ itself. To avoid confusion, the latter is called throughout this paper the *random telegraph signal*, and the term *random telegraph process* is reserved for the flow (4.11)

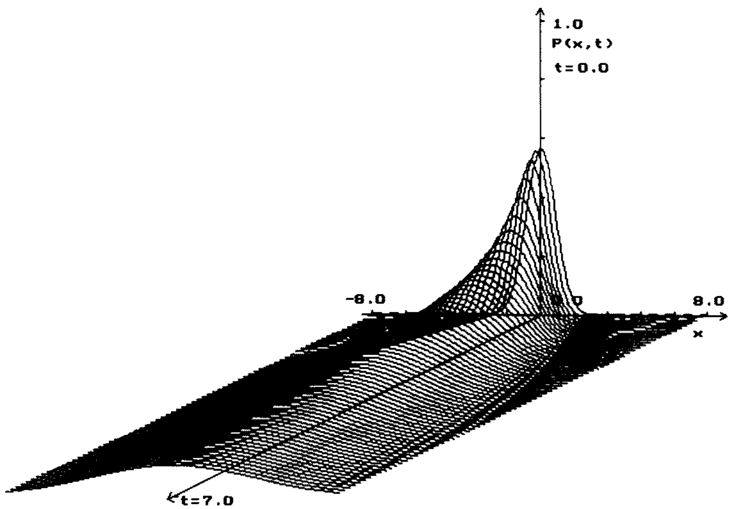

Fig. 3. Evolution in time of the probability density P(x,t) for the random telegraph process (4.11) driven by purely markovian symmetric DN: $\gamma_0 = 1$, $\gamma_1 = 0$, $\Delta^2 = 5$, $\Lambda = 5$.

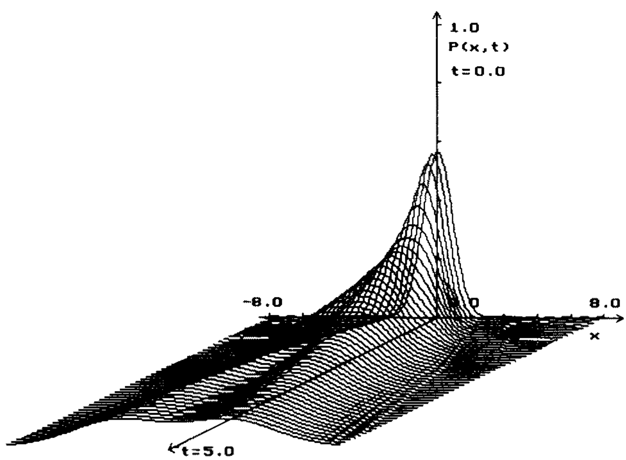

Fig. 4. The same as in Fig. 3, with mixed driving: $\gamma_0 = 0.5$, $\gamma_1 = 0.5$, $\Delta^2 = 5$, $\Lambda = 5$, $\nu = 0.05$.

non-markovianity, from purely markovian-driven process in Fig. 3 ($\gamma_0 = 1$, $\gamma_1 = 0$), to purely non-markovian-driven one in Fig. 7 ($\gamma_0 = 0$, $\gamma_1 = 1$), with three mixed-driven intermediate cases in Figs. 4–6 ($0 < \gamma_i < 1$).

The temporal characteristics of the noise: noise correlation time $2/\Lambda = 0.4$, and memory $1/\nu = 20$ are chosen so as to augment the effects of non-markovianity. The role of these parameters is shown in Figs. 8

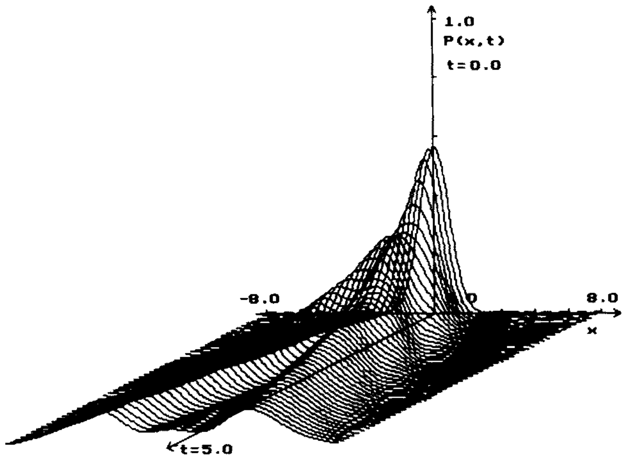

Fig. 5. The same as in Fig. 4, with $\gamma_0 = 0.2$, $\gamma_1 = 0.8$, $\Delta^2 = 5$, $\Lambda = 5$, $\nu = 0.05$.

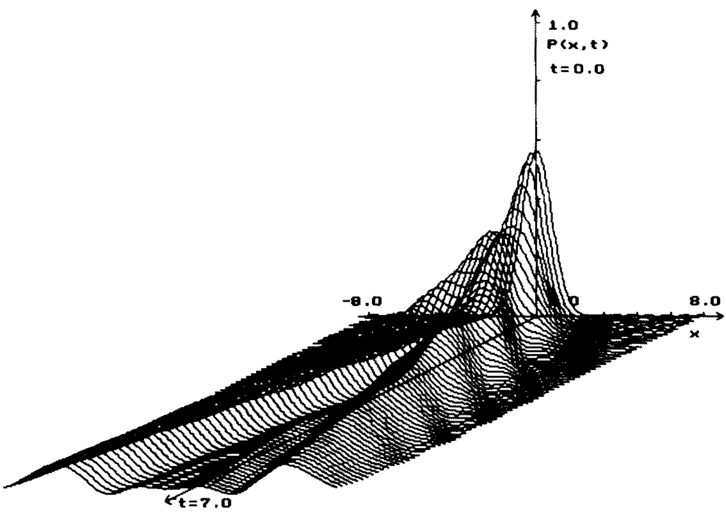

Fig. 6. The same as in Fig. 4, with $\gamma_0 = 0.1$, $\gamma_1 = 0.9$, $\Delta^2 = 5$, $\Lambda = 5$, $\nu = 0.05$.

(for the markovian case) and 9 (for the non-markovian case). The shape of $P(x,t)$ for the purely non-markovian case with $2/\Lambda = 40$ and $1/\nu = 0.2$ is very similar to that shown in Fig. 8 for purely markovian driving with the same Λ. This implies that the strongly coloured (long correlation time) non-markovian DN with very short memory differs but little from the markovian DN (with otherwise the same characteristics) in its effect on the driven stochastic flow.

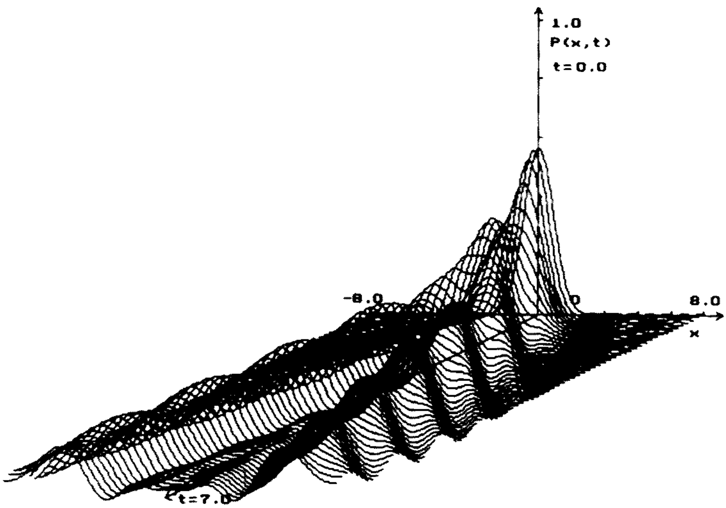

Fig. 7. The same as in Fig. 4, with purely non-markovian driving: $\gamma_0 = 0.0$, $\gamma_1 = 1.0$, $\Delta^2 = 5$, $\Lambda = 5$, $\nu = 0.05$.

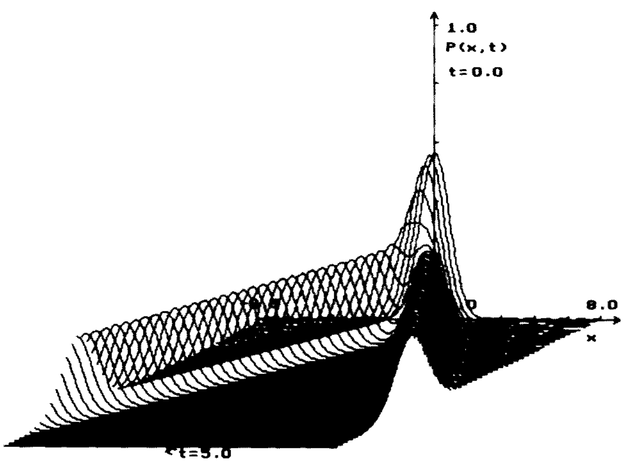

Fig. 8. The same as in Fig. 3 (purely markovian driving): $\gamma_0 = 1.0$, $\gamma_1 = 0.0$, $\Delta^2 = 5$, $\Lambda = 0.05$.

Fig. 10 illustrates the effect of the reversal of the sign of the markovian component ($\gamma_0 = -1$, $\gamma_1 = 1$). Characteristic in this case is the reappearance of the (diffused) central peak of $P(x,t)$ in the course of evolution. Here the values of Λ and ν are very close (long both correlation time and memory), which is forced by the convergence conditions (*cf.* Fig. 1 above). For both these characteristic times short ($\Lambda = 5$, $\nu = 5.1$), $P(x,t)$ becomes similar to that from Fig. 3.

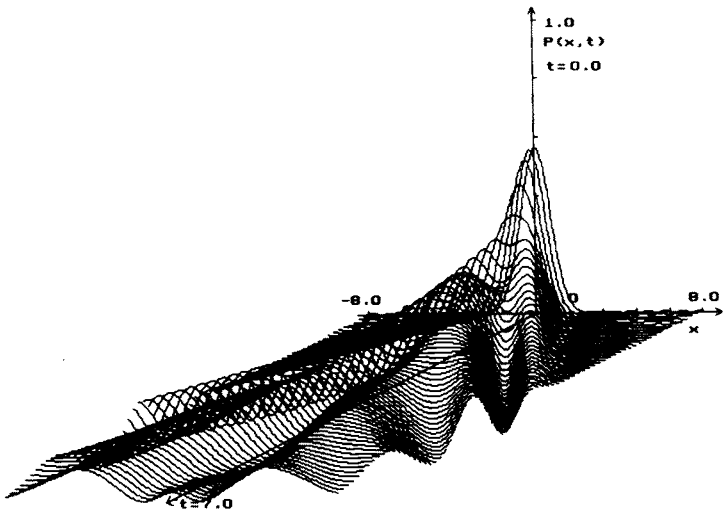

Fig. 9. The same as in Fig. 7 (purely non-markovian driving): $\gamma_0 = 0.0$, $\gamma_1 = 1.0$, $\Delta^2 = 5$, $\Lambda = 0.1$, $\nu = 0.05$.

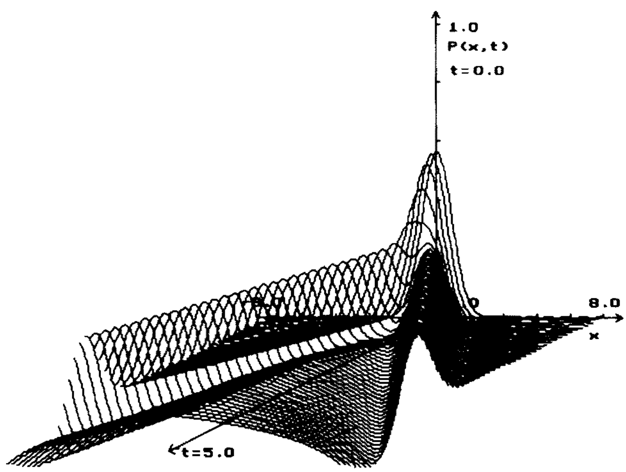

Fig. 10. The same as in Figs. 4–6 (mixed driving), with reversed markovian component: $\gamma_0 = -1.0$, $\gamma_1 = 1.0$, $\Delta^2 = 5$, $\Lambda = 0.1$, $\nu = 0.11$.

All non-markovian cases shown above correspond to oscillating $\psi(t)$. For noise parameters resulting in non-oscillating $\psi(t)$, $P(x,t)$ is qualitatively similar to corresponding markovian distributions.

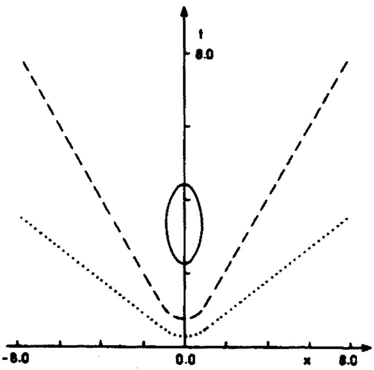

Fig. 11. Trace of location of maxima of $P(x,t)$ at the (x,t) plane, for the random telegraph process (4.11) driven by purely markovian symmetric DN: $\gamma_0 = 1, \gamma_1 = 0$. Continuous line: $\Delta^2 = 5$, $\Lambda = 5$, dotted line: $\Delta^2 = 5$, $\Lambda = 0.05$, dashed line: $\Delta^2 = 1$, $\Lambda = 0.05$.

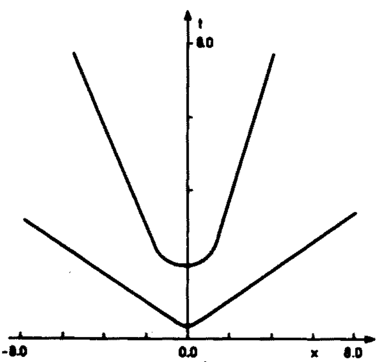

Fig. 12. The same as in Fig. 11, for mixed driving process: $\gamma_0 = 1.0$, $\gamma_1 = 1.0$, $\Delta^2 = 5$, $\Lambda = 1$, $\nu = 0.05$.

The most striking non-markovian effects visible in the above results are: the presence of oscillations in the time evolution, and the appearance in the course of time of several additional peak splittings (it is well-known — and well-visible in Figs. 8 and 11 — that strong enough markovian DN is able to force one such splitting). According to some interpretations [1, 4], appearance of additional peaks in $P(x,t)$ means the appearance of the noise-induced transitions between macroscopic states having no deterministic counterpart. Assuming this philosophy to be true, the non-markovianity

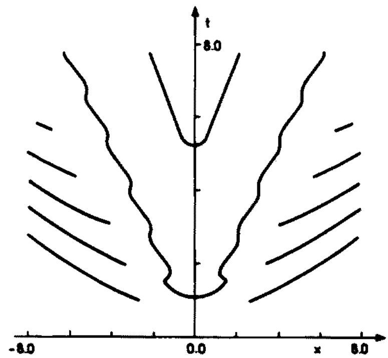

Fig. 13. The same as in Fig. 11, for purely non-markovian driving process: $\gamma_0 = 0.0$, $\gamma_1 = 1.0$, $\Delta^2 = 5$, $\Lambda = 5$, $\nu \doteq 0.05$.

may lead to a multitude of such transitions: more and more new transient, locally stable states (local maxima of probability density) appear in the course of the random telegraph process driven by non-markovian DN. This point is illustrated in Figs. 11–13, where the traces of maxima of $P(x,t)$ are drawn on the (x,t) plane. Fig. 11 illustrates the splitting of $P(x,t)$ by markovian driving DN, in dependence on the noise strength Δ^2 and noise correlation time $2/\Lambda$. Figs. 12 and 13 show the effect of non-markovianity of the driving DN. Fig. 13 corresponds to $P(x,t)$ from Fig. 7.

5. Processes driven by markovian DN

Exact master equations describing the time dependence of probability densities related to such flows can be obtained only for the markovian DN's [1–3]. We have seen that in the non-markovian case one must resort to approximations [15]. Therefore we shall discuss in this section the markovian DN. The markovian case is presented below *(i)* for the sake of completeness, *(ii)* to introduce a technique of deriving master equations used below for the flows driven by composite noises.

In addition to probability densities $P(x,t)$ and $p(x,\xi_i,t)$ defined by Eqs. (4.2)–(4.3) we introduce the auxiliary density $Q(x,t)$, defined:

$$Q(t) = Q(x,t) = \langle \delta(X(t,[\xi]) - x)\xi(t) \rangle \,, \tag{5.1}$$

which implies

$$Q(t) = \Delta_1 p_1(t) - \Delta_2 p_2(t) \,. \tag{5.2}$$

The standard method [1-3] leads to the master equations (4.4) for $p_i(t)$ which in the markovian case do not contain last term of the r.h.s. $(\gamma_1 = 0)$

(therefore, we shall put $\gamma_0 = 1$ in the remainder of this Section and in Section 6).

Appropriate linear combinations of these equations lead to master equations for $P(t)$ and $Q(t)$. The latter can be obtained also in a different way, which is more convenient (in the markovian case), because it can be readily generalized for the composite DNs (considered in the subsequent section). Therefore, for the sake of completeness, we present here also this second method of derivation. Its main elements, none especially new, are: *(i)* the basic property of DN, Eq. (2.1), *(ii)* the Shapiro–Loginov theorem (3.16) and *(iii)* Haken's method [22] of derivation of evolution equations for probability density[4]. The latter, suitably adjusted for present situation, is as follows. Differentiation of the definition (4.2) gives:

$$\begin{aligned}\frac{\partial}{\partial t}P(x,t) &= \left\langle \frac{\partial}{\partial X(t)}\delta(X(t)-x)\dot{X}(t)\right\rangle \\ &= -\frac{\partial}{\partial x}\langle\delta(X(t)-x)\left[f(X(t))+g(X(t))\xi(t)\right]\rangle \\ &= -\frac{\partial}{\partial x}\left[f(x)\,P(x,t)+g(x)\,Q(x,t)\right], \end{aligned} \tag{5.3}$$

where the well-known properties of δ-function have been used. Note that in the last line of Eq. (5.8) the functions $f(x)$ and $g(x)$ are included in the action of the differential operator[5] [22].

The equation for the auxiliary function $Q(x,t)$ is obtained in the same way, with the use of the Shapiro–Loginov theorem, Eq. (3.14), and the property (2.1):

$$\frac{\partial}{\partial t}Q(x,t) = -\Lambda Q(x,t) - \frac{\partial}{\partial x}\left[h(x)Q(x,t)+\Delta^2 g(x)P(x,t)\right]. \tag{5.4}$$

where $h(x) = f(x) + \Delta_0 g(x)$.

Eqs. (5.3) and (5.4) form closed set of two linear partial differential equations for probability densities P and Q. As we have mentioned, identical equations result from Eqs. (4.4).

It is easy to find stationary solution of Eqs. (5.3)–(5.4) — for time-independent case they are equivalent to:

$$f(x)P_{st}(x) + g(x)Q_{st}(x) = C_1,$$

[4] constructed in principle there for the derivation of the Fokker–Planck equation

[5] This follows from the properties of δ-function: it is easy to check that the distribution $\left[\frac{\partial}{\partial x}\delta(X(t)-x)f(x)\right]$ is equivalent to the distribution $f(X(t))\left[\frac{\partial}{\partial x}\delta(X(t)-x)\right]$: multiply both distributions by a trial function $q(x)$ and integrate (by parts) over a small finite interval around $x = X(t)$; in both cases the result is $-f(X)[dq(X)/dX]$ which proves the equivalence of both distributions.

$$\frac{d}{dx}\left[h(x)Q_{st}(x) + \Delta^2 g(x)P_{st}(x)\right] = -\Lambda Q_{st}(x)\,, \tag{5.5}$$

i.e.,

$$P_{st}(x) = \frac{g(x)}{D_{\text{eff}}(x)} e^{\Lambda \int^{x} m(x')dx'} \left[C_2 - C_1 \int^{x} dx' k(x') e^{-\Lambda \int^{x'} m(x'')dx''}\right], \tag{5.6}$$

where

$$m(x) = f(x)/D_{\text{eff}}(x)\,, \quad k(x) = \frac{\Lambda}{g(x)} + \frac{d}{dx}\frac{f(x)}{g(x)}\,, \tag{5.7}$$

$$D_{\text{eff}}(x) = \left[\Delta_1 g(x) + f(x)\right]\left[\Delta_2 g(x) - f(x)\right], \tag{5.8}$$

and where C_1, C_2 are to be determined from boundary conditions imposed on probability flow on boundaries of the domain $\mathcal{D}_x$ of x and from

$$\int_{\mathcal{D}_x} P(x,t)dx = 1, \qquad P(x,t) \geq 0\,, \;\; \forall\, t \geq 0,\; \forall\, x \in \mathcal{D}_x. \tag{5.9}$$

The use of the so-called *natural boundary conditions* (standard assumption for one-dimensional flows, which gives $C_1 = 0$) leads to the well-known formula [1–3]:

$$P_{st}(x) = \mathcal{N}^{-1} \frac{|g(x)|}{D_{\text{eff}}(x)} \exp\left[\Lambda \int^{x} dx \frac{f(x)}{D_{\text{eff}}(x)}\right] \Theta\left(D_{\text{eff}}(x)\right), \tag{5.10}$$

where $\mathcal{N}$ is the normalization constant, and $\Theta(x)$ is the Heaviside step function, "expressing that the probability is zero in the 'unstable' region of negative D" [3].

The limiting procedures (2.8), (2.9) enable us to obtain corresponding equations and formulas for stochastic flows driven by (asymmetric) white shot noise (WSN) with exponentially distributed weights, and for Gaussian white noise (GWN).

Thus, in WSN limit we have

$$\begin{aligned} \frac{\partial}{\partial t}P(x,t) &= -\frac{\partial}{\partial x}\left[f(x)P(x,t) + g(x)Q(x,t)\right], \\ Q(x,t) &= -w_0 \frac{\partial}{\partial x} g(x)\left[Q(x,t) + w_0 \lambda_2 P(x,t)\right], \end{aligned} \tag{5.11}$$

which, after formal elimination of the auxiliary correlation density $Q(x,t)$ gives the known [2–3] equation:

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\left\{f(x) + w_0^2 \lambda_2 g(x) \frac{\partial}{\partial x} g(x)\left[1 + w_0 \frac{\partial}{\partial x} g(x)\right]^{-1}\right\} P(x,t)\,. \tag{5.12}$$

The stationary solution of (5.11) reads [2–3]:

$$P_{st}(x) = \frac{\mathcal{N}^{-1}}{|\Delta_2 g(x) - f(x)|} \exp\left\{ \int^{x} \frac{f(x)dx}{w_0 g(x)[\Delta_2 g(x) - f(x)]} \right\}. \tag{5.13}$$

In the GWN limit we get simply the appropriate Fokker–Planck equation (in Stratonovich interpretation):

$$\frac{\partial}{\partial t} P(x,t) = \frac{\partial}{\partial x}\left[-f(x) + D_0^2 g(x) \frac{\partial}{\partial x} g(x)\right] P(x,t), \tag{5.14}$$

together with

$$Q(x,t) = -D_0^2 \frac{\partial}{\partial x} g(x) P(x,t), \tag{5.15}$$

and with well-known stationary solution:

$$P_{st}(x) = \frac{\mathcal{N}^{-1}}{|g(x)|} \exp\left\{ \int^{x} \frac{f(x)dx}{D_0^2 g(x)^2} \right\}. \tag{5.16}$$

Formal generalization of the above formulas for multidimensional stochastic flows is trivial — it is sufficient to substitute the "scalars" $X(t)$, x, f, g by "vectors" (column matrices) $\boldsymbol{X}(t)$, $\boldsymbol{x}$, $\boldsymbol{f}$, $\boldsymbol{g}$, respectively. However, the multidimensional stationary solution cannot be written down readily. The stationary N-dimensional equation for $P_{st}(\boldsymbol{x})$ reads:

$$\frac{\partial}{\partial \boldsymbol{x}} \cdot \left[\boldsymbol{G}(\boldsymbol{x}) P_{st}(\boldsymbol{x})\right] = \Lambda B(\boldsymbol{x}) P_{st}(\boldsymbol{x}) - C(\boldsymbol{x}), \tag{5.17}$$

where

$$\boldsymbol{G}(\boldsymbol{x}) = \Delta^2 \boldsymbol{g}(\boldsymbol{x}) - B(\boldsymbol{x}) \boldsymbol{h}(\boldsymbol{x}),$$

$$C(\boldsymbol{x}) = \Lambda A(\boldsymbol{x}) + \frac{\partial}{\partial \boldsymbol{x}} \cdot \left[A(\boldsymbol{x}) \boldsymbol{h}(\boldsymbol{x})\right],$$

$$A(\boldsymbol{x}) = \frac{[\boldsymbol{H}(\boldsymbol{x}) \cdot \boldsymbol{C}_1]}{[\boldsymbol{H}(\boldsymbol{x}) \cdot \boldsymbol{g}(\boldsymbol{x})]},$$

$$B(\boldsymbol{x}) = \frac{[\boldsymbol{H}(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x})]}{[\boldsymbol{H}(\boldsymbol{x}) \cdot \boldsymbol{g}(\boldsymbol{x})]},$$

$\boldsymbol{C}_1$ is the column of integration constants (unfortunately, for multidimensional flows, one usually cannot put all $C_{1,i}$'s equal to zero), and $\boldsymbol{H}(\boldsymbol{x})$ is an arbitrary row matrix, whose elements can be, in general, functions of $\boldsymbol{x}$. The safest choice is $\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})$, so that the denominators above are positive-definite.

6. Processes driven by composite DN

The dichotomic noise is but an idealization of the real noises. It seems that the reality will be better approximated by linear and/or nonlinear combinations of several such noises, or of DN with other noises. Consider therefore the flow:

$$\dot{X}(t) = f(X) + \sum_{j=1}^{M} g_j(X)\xi_j(t) \tag{6.1}$$

(generalizations to many-dimensional flows are obvious), with $\xi_i(t)$ — independent (uncorrelated) DN's:

$$\langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\Delta_j^2 e^{-|t-t'|/\tau_{c,j}}, \quad \langle \xi_i^n \xi_j^m \rangle = \langle \xi_i^n \rangle \langle \xi_j^m \rangle, \tag{6.2}$$

and with relations (2.1)–(2.4) fulfilled by each ξ_j separately. Note that both WSN and GWN can be obtained as appropriate limits of some of these DN's, therefore the formulae below may serve also for composite noises built of white noises, as *e.g.* the "interrupted diffusion processes" of Refs. [12].

In this case we shall need more auxiliary functions. In general, there are needed 2^M functions (including P). Let

$$Q_{j,k,l,\ldots} = \langle \delta(x - X(t))\xi_j(t)\xi_k(t)\xi_l(t)\ldots\rangle \tag{6.3}$$

($Q_{j,k,l,\ldots}$ are invariant with respect to permutations of indices). The scheme used in Section 5 leads now to the system of 2^M linear partial differential equations:

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial x}\left[f(x)P + \sum_{j=1}^{M} g_j(x)Q_j\right], \tag{6.4a}$$

$$\frac{\partial}{\partial t}Q_j = -\Lambda_j Q_j - \frac{\partial}{\partial x}\left\{[f(x)+\Delta_{o,j}g_j(x)]Q_j + \sum_{k\neq j}^{M} g_k(x)Q_{j,k} + \Delta_j^2 g_j(x)P\right\}, \tag{6.4b}$$

$$\frac{\partial}{\partial t}Q_{j,k} = -(\Lambda_j + \Lambda_k)Q_{j,k} - \frac{\partial}{\partial x}f(x)P - \frac{\partial}{\partial x}\left\{[f(x)+\Delta_{o,j}g_j(x)+\Delta_{o,k}g_k(x)]Q_{j,k} + \sum_{l\neq j,k}^{M} g_l(t)Q_{j,k,l} + \Delta_j^2 g_j(x)Q_k + \Delta_k^2 g_k(x)Q_j\right\}, \tag{6.4c}$$

etc.

In the same way one can handle the flow:

$$\dot{X}(t) = f(X) + g(X)\xi_1(t)\cdots\xi_M(t), \tag{6.5}$$

or flows with linear combinations of such right-hand sides. For the flow (6.5) one gets:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left(fP + g\,Q_{1..M}\right), \tag{6.6a}$$

$$\begin{aligned}\frac{\partial}{\partial t}Q_{1..n} = &-(\Lambda_1+\cdots+\Lambda_n)Q_{1..n} - \frac{\partial}{\partial x}f(x)Q_{1..n}\\ &-\frac{\partial}{\partial x}g(x)\Big\{\Big(\prod_{s=1}^{n}\Delta_s^2\Big)Q_{n+1,..,M} + \sum_{j=1}^{n}\Big(\prod_{s\neq j}^{n}\Delta_s^2\Big)\Delta_{o,j}Q_{j,n+1,..,M}\\ &+\sum_{\{j,k\}}\Big(\prod_{s\neq j,k}^{n}\Delta_s^2\Big)\Delta_{o,j}\Delta_{o,k}Q_{jk,n+1,..,M}+\cdots+\Big(\prod_{s}^{n}\Delta_{o,s}\Big)Q_{1,..,M}\end{aligned} \tag{6.6b}$$

where $n \leq M$, and summations over $\{j, k, \ldots\}$ are over all pairs, triples, etc. For $n = M$, $Q_{n+1,..,M} \to P$.

Note that the case (6.5) with *symmetric* DN's (all $\Delta_{o,j} = 0$) is uninteresting, as the product of several symmetric DN's is just another symmetric DN. However, the product of several asymmetric DN's is no longer a DN, and its WSN limit (taken for all DN's composing it) will be a WSN with non-exponential weight distribution. Such noises are usually non-markovian.

Most general form of the flow is:

$$\begin{aligned}\dot{\boldsymbol{X}} = \boldsymbol{f}(\boldsymbol{X}) + \sum_p g_p(\boldsymbol{X})\xi_p(t) + \sum_{\{p,q\}} g_{pq}(\boldsymbol{X})\xi_p(t)\xi_q(t)\\ + \sum_{\{p,q,r\}} g_{pqr}(\boldsymbol{X})\xi_p(t)\xi_q(t)\xi_r(t) + \cdots,\end{aligned} \tag{6.7}$$

where $\boldsymbol{X}$ may be either one- or multidimensional. This gives the set of equations for probability distribution and for auxiliary functions in the form:

$$\begin{aligned}\frac{\partial P(\boldsymbol{x},t)}{\partial t} = -\frac{\partial}{\partial \boldsymbol{x}}\cdot\Big[(\boldsymbol{x})\mathbf{P}(\boldsymbol{x},\mathbf{t}) + \sum_{\mathbf{p}} g_{\mathbf{p}}(\boldsymbol{x})\mathbf{Q}_{\mathbf{p}}(\boldsymbol{x},\mathbf{t}) + \sum_{\{\mathbf{p},\mathbf{q}\}} g_{\mathbf{pq}}(\boldsymbol{x})\mathbf{Q}_{\mathbf{pq}}(\boldsymbol{x},\mathbf{t})\\ + \sum_{\{p,q,r\}} g_{pqr}(\boldsymbol{x})Q_{pqr}(\boldsymbol{x},t) + \cdots\end{aligned} \tag{6.8a}$$

$$\frac{\partial}{\partial t}Q_{jk..mn}(\boldsymbol{x},t) = -(\Lambda_j+\cdots+\Lambda_n)Q_{jk..mn}(\boldsymbol{x},t) - \frac{\partial}{\partial \boldsymbol{x}}\cdot\boldsymbol{F}_{jk..mn}(\{Q\}), \tag{6.8b}$$

where $\boldsymbol{F}_{jk..mn}(\{Q\})$ is defined by:

$$\begin{aligned}\mathbf{F}_{jk..mn}(\{Q\}) = \boldsymbol{f}(\boldsymbol{x})Q_{jk...mn}(\boldsymbol{x},t) + \sum_p g_p(\boldsymbol{x})Q_{jk...mnp}(\boldsymbol{x},t)\\ + \sum_{\{p,q\}} g_{pq}(\boldsymbol{x})Q_{jk..mnpq}(\boldsymbol{x},t) + \sum_{\{p,q,r\}} g_{pqr}(\boldsymbol{x})Q_{jk..mnpqr} + \cdots\end{aligned} \tag{6.8c}$$

together with the prescription that auxiliary functions $Q_{jk..mnpq..r}$ with repeating indices are to be substituted by appropriate combinations of lower-order auxiliary functions, *e.g.*:

$$Q_{jk..mnpq..r} = \Delta_n^2 Q_{jk..mq..r} + \Delta_{o,n} Q_{jk..mnq..r} \qquad \text{for} \quad n = p. \tag{6.8d}$$

It is possible to give also the general prescription in the case when one or more of the noises $\xi_p(t)$ are either GWN or WSN[6]. To be definite, assume that $\xi_j(t) \to$ GWN or WSN. Dividing Eqs. (6.2b) by Λ_j, and using prescriptions (2.1) or (2.2), we get:

$$\left(1 + \sigma_1 \frac{\partial}{\partial \boldsymbol{x}} g_j(\boldsymbol{x})\right) Q_{jk..nm} = -\frac{\partial}{\partial \boldsymbol{x}} \cdot \boldsymbol{H}_{jk..mn}(\{Q\}), \tag{6.9a}$$

with

$$\begin{aligned} \boldsymbol{H}_{jk..mn}(\{Q\}) =& \sigma_2^2 g_j(\boldsymbol{x}) Q_{k..mn}(\boldsymbol{x}, t) + \sum_{p \neq j} g_p(\boldsymbol{x}) \left[\sigma_2^2 Q_{k..mnp}(\boldsymbol{x}, t) + \sigma_1 Q_{jk..mnp}\right] \\ &+ \sum_{\{p,q\} \neq j} g_{pq}(\boldsymbol{x}) \left[\sigma_2^2 Q_{k..mnpq}(\boldsymbol{x}, t) + \sigma_1 Q_{jk..mnpq}(\boldsymbol{x}, t)\right] + \cdots \end{aligned} \tag{6.9b}$$

together with the same prescription as that for $\boldsymbol{F}_{jk..mn}(\{Q\})$, where $\sigma_1 = w_{0,j}$, $\sigma_2^2 = w_{0,j}^2 \lambda_{2,j}$ for WSN, and $\sigma_1 = 0$, $\sigma_2^2 = D_{0,j}^2$ for GWN. This enables the elimination of all auxiliary functions related to white noise ξ_j (containing the index j among its indices), *i.e.*, reduction of the number of independent equations. Instead, GWN produces just second-order partial differential equations; WSN introduces either infinite-order equations through the inverse operator

$$\left(1 + w_{0,j} \frac{\partial}{\partial \boldsymbol{x}} g_j(\boldsymbol{x})\right)^{-1}, \tag{6.10}$$

or integro-differential equations through the formal solution of Eq. (6.3a). The latter reads for one-dimensional flows ($\boldsymbol{x} \to x$):

$$Q_{jk..mn} = \frac{1}{g_j(x)} e^{-x/w_{0,j}} \left[C_{jk..mn} - \int^x dy\, e^{y/w_{0,j}} \frac{\partial}{\partial y} H_{jk..mn}(\{Q\})\right]. \tag{6.11}$$

[6] more than one of ξ's can be white only when they enter as linear combinations with each other: products of two white noises usually make no sense and cannot be dealt with by methods used in this paper. Of course, white noise can be multiplied by a colored one, especially by DN

It is to be noted that auxiliary functions with two or more indices belonging to white noises do not enter into equations for P and for relevant remaining auxiliary functions.

So far, we have considered the different noises to be uncorrelated. When noises are correlated, *i.e.*, when

$$\langle \xi_k(t)\xi_j(t)\rangle \neq 0 \textit{ for } k \neq j\,, \tag{6.12}$$

we may use the following trick [7, 18]: the original correlated noises $\xi_k(t)$ are written as linear combinations of new, uncorrelated noises $\chi_j(t)$ of otherwise similar characteristics, and the new noises are substituted into the considered flow. Next the methods described above can be used. In some situations such substitution may even diminish the number of independent, now uncorrelated noises, and thus simplify the calculations.

The advantage of the composite noise as a non-markovian one is that for finite number of component markovian DNs or WNs there is closed (finite) set of equations for $P(x,t)$ (finite number of auxiliary functions). Also composite noise seems to be better representation of real noises than any DN process, be it markovian or non-markovian. The disadvantage is that its non-markovian characteristics: the type and range of memory, the amount of markovianity, etc. are not given explicitly by explicit parameters.

Appendix A

The formulae (3.19) can be obtained as follows. For *symmetric* DN $\Delta_0 = 0$):

$$\begin{aligned} A(t,t_0) &= \Big\langle \exp\Big[\alpha\int_{t_0}^{t} dt'\xi(t')\Big]\Big\rangle = \Big\langle \exp\Big[\alpha\int_{0}^{\tau} dt'\xi(t'+\tau)\Big]\Big\rangle \\ &= 1+\sum_{n=2}^{\infty}\frac{\alpha^n}{n!}\int_0^{\tau} dt_1\ldots\int_0^{\tau} dt_n\langle\xi(t_1+\tau)\ldots\xi(t_n+\tau)\rangle \\ &= 1+\sum_{n=2}^{\infty}\alpha^n\int_0^{\tau} dt_1\int_0^{t_1} dt_2\ldots\int_0^{t_{n-1}} dt_n\langle\xi(t_1+\tau)\xi(t_2+\tau)\ldots\xi(t_n+\tau)\rangle \\ &= 1+\sum_{m=1}^{\infty}\alpha^{2m}\Delta^{2m}\int_0^{\tau} ds_1\int_0^{s_1} dt_1\psi(s_1-t_1)\ldots\int_0^{t_{m-1}} ds_m\int_{t_0}^{s_m} dt_m\psi(s_m-t_m)\,. \end{aligned} \tag{A.1}$$

Using the Laplace transform and its well-known properties [24]:

$$\hat{f}(z) = \int_0^\infty d\tau\, e^{-z\tau} f(\tau)\,, \quad \int_0^\infty d\tau\, e^{-z\tau} \int_0^\tau d\tau' f(\tau') = \frac{1}{z}\hat{f}(z)\,,$$

$$\int_0^\infty d\tau\, e^{-z\tau} \int_0^\tau d\tau' f_1(\tau') f_2(\tau - \tau') = \hat{f}_1(z)\hat{f}_2(z)\,, \tag{A.2}$$

we get, subsequently:

$$\hat{\phi}_1(z) = \int_0^\infty d\tau e^{-z\tau} \int_0^\tau dt_1 \int_0^{t_1} dt_2 \psi(t_1 - t_2) = \frac{1}{z^2}\hat{\psi}(z)\,, \tag{A.3}$$

$$\int_0^\infty d\tau e^{-z\tau} \int_0^\tau ds_1 \int_0^{s_1} dt_1 \psi(s_1 - t_1) \int_0^{t_1} ds_2 \int_0^{s_2} dt_2 \psi(s_2 - t_2)$$
$$= \int_0^\infty d\tau e^{-z\tau} \frac{1}{z} \int_0^\tau dt_1 \psi(\tau - t_1)\phi_1(t_1) = \frac{1}{z^3}\left[\hat{\psi}(z)\right]^2 \tag{A.4}$$

etc., which leads eventually to:

$$\hat{A}(z) = \frac{1}{z} + \frac{1}{z}\sum_{m=1}^{\infty} (\alpha^2\Delta^2)^m \left[\hat{\psi}(z)/z\right]^m = \left[z - \alpha^2\Delta^2\hat{\psi}(z)\right]^{-1}, \tag{A.5}$$

$$\hat{\psi}(z) = \frac{z + \nu}{(z + \theta_1)(z + \theta_2)}\,, \tag{A.6}$$

$$\hat{A}(z) = \frac{(z + \theta_1)(z + \theta_2)}{z(z + \theta_1)(z + \theta_2) - \alpha^2\Delta^2(z + \nu)} = \frac{(z + \theta_1)(z + \theta_2)}{(z - z_1)(z - z_2)(z - z_3)}\,, \tag{A.7}$$

where z_j are the solutions of Eq. (3.20).

The inverse Laplace transform of (A.7) gives the formula (3.19a), and the differentiation and integration of (3.19a) leads to (3.19b) and (3.19c).

In the markovian case, $\gamma_0 = 1$, $\gamma_1 = 0$, we get $z_3 = -\nu$, $z_{1,2} = -\lambda \pm g$, $g = \sqrt{\lambda^2 + \Delta^2}$, and

$$A(t) = e^{-\lambda t}\left(\cosh gt + (\lambda/g)\sinh gt\right). \tag{A.8}$$

In the WN limit,

$$\Delta^2\psi(t) \longrightarrow (D_0^2/\gamma_0)\left[\delta(t) - \mu e^{-(\mu+\nu)t}\right], \quad \Delta^2\psi(t) \longrightarrow \frac{D_0^2(z+\nu)}{\gamma_0(z+\nu+\mu)}, \quad \mu = \gamma_1/\gamma_0,$$

$$\hat{A}(z) \longrightarrow \frac{z+\nu+\mu}{z(z+\nu+\mu)-\beta(z+\nu)} = \frac{z+\nu+\mu}{(z-z_1)(z-z_2)}, \quad \beta = D_0^2\alpha^2/\gamma_0,$$

$$z_{1,2} = -\tilde{\lambda} \pm \tilde{g}, \quad \tilde{\lambda} = \nu + \mu - \beta, \quad ,\tilde{g} = \sqrt{\tilde{\lambda}^2 + 4\nu\beta},$$

$$A(t) \longrightarrow \frac{1}{2\tilde{g}} e^{-\tilde{\lambda}t}\left[(\tilde{g}+\beta)e^{\tilde{g}t}(\tilde{g}-\beta) + e^{-\tilde{g}t}\right]. \tag{A.9}$$

In the WN limit $\psi(t)$ is not well-defined when $\gamma_0 = 0$. Nevertheless, in this case $A(t)$ remains well-defined:

$$\hat{A}(z) = \frac{\gamma_1}{(\gamma_1 - D_0^2\alpha^2)z - \nu D_0^2\alpha^2},$$

$$A(t) = \frac{\gamma_1}{\gamma_1 - D_0^2\alpha^2}\exp\left(\frac{\nu D_0^2\alpha^2}{\gamma_1 - D_0^2\alpha^2}t\right). \tag{A.10}$$

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