SELF ORGANIZED CRITICALITY — ANALYTICAL CALCULATIONS AND OPEN PROBLEMS*

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Some analytical calculations and results concerning self organized critical state in the sandpile-like cellular automata defined on the Bethe and square lattices are showed. The possibility of achieving a self organized critical state in nonconservative model systems is discussed.

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1. Introduction

The theory of self organized criticality (SOC) is rather new. Its concept was for the first time introduced in the classical work of Bak, Tang and Wiesenfeld [1] where the final statistically stable state (which they called the self organized critical state) of one special type of cellular automaton was described. Now, the concept of SOC attracts a great attention and provokes a development of a new branch of nonlinear dynamics.

The most interesting was the observation of the above mentioned authors that the self organized critical state is accompanied by the self similar (scale invariant) time and space structures, exhibiting themselves as $1/f$ noise and as fractal space structures. The understanding of the behaviour of the systems with self organized criticality promised therefore to bring more light into the creation of the $1/f$-type noise signals and fractal structures in nature. In the SOC theory this self similar events are explained as a natural consequence of the self organized critical state of a dynamical system.

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The systems which are able to evolve to the SOC state from wide range of the initial conditions are usually spatially extended, consisting of huge amount of elements or particles. These elements "know" only about their nearest neighbours. Such properties are ubiquitous in natural dynamical systems, such as ecosystems, social systems, earth crust *etc.* It is also known from the natural events (such as earthquakes or phase transitions *e.g.*) that a local instability can be propagated through the system by the short range interactions, and often leads to a global effect.

The most familiar dynamical system having all of the above mentioned properties is a pile of sand. It consists of many particles and in accordance with the sand grain size it is spatially extended. A single local instability can cause a global effect (an avalanche). If the pile is in a state with the critical slope (analogous to the SOC state), the simple addition of one grain can cause an avalanche of unpredictable size. One can observe a very small avalanche, hitting only the nearest neighbour grains, as well as a great one, touching the whole system. This fact assures existence of the relaxation effects on all space scales. The pile of sand became a paradigmatic system in the theory of self organized criticality and the basic principles of the theory were formulated in the "sandpile" terminology. A special kind of cellular automata, called sandpile like cellular automata, serves as a model system. In this lecture I shall follow the classical "sandpile" line and I shall present some analytical calculations which were done recently. In Section 2 the model cellular automaton of Bak, Tang and Wiesenfeld (BTW) and its dynamical rules are defined. In Sections 3 and 4 I show the analytical calculations of the properties of the SOC state. In Section 5 one open problem is mentioned.

2. BTW model

One of the most elaborated sandpile models is the critical height undirected sandpile of Bak *et al.* [1]. It is usually defined on two dimensional square lattice and has two step dynamics with two different time scales: input process, which is extremely slow, and relaxation, which is considered to be an instantaneous effect in comparison with the input. These two steps resemble to some extent the dynamics of a pile of sand. Input process is analogous to a slow and careful addition of grains to the pile, which can violate local stability conditions (expressed as a local subcritical slope in realistic sandpile *e.g.*). This starts a quick relaxation process, manifesting itself as an avalanche.

The two dimensional Bak, Tang and Wiesenfeld (BTW) sandpile cellular automaton has $N_2 = N \times N$ sites. In the input process randomly chosen sites on the lattice are seeded by "sand grains" of the unit size. This is

described by the equation

$$\begin{aligned} z_{i,j} &\to z_{i,j} + 1\,, \\ z_{i,j} &< z_c\,, \end{aligned} \tag{1a}$$

where i, j are the integer coordinates of the randomly chosen site and $z_{i,j}$ denotes the height of a sand column at the site (i, j), and z_c is a critical height.

When the local height $z_{i,j}$ reaches a given critical value z_c, the relaxation process starts. That means that from the site (i, j), four grains are distributed to the nearest neighbours, possibly causing some of them to be supercritical too, and so avalanche can spread further and further according to the rules

$$\begin{aligned} z_{i,j} &\to z_{i,j} - 4\,, \\ z_{i,j\pm1} &\to z_{i,j\pm1} + 1\,, \\ z_{i\pm1,j} &\to z_{i\pm1,j} + 1\,. \end{aligned} \tag{1b}$$

Boundary conditions are open; that means, the grains leave the pile at the boundary.

This model evolves to the SOC state from all initial sandgrain distributions on a square lattice. In this statistically stable state the system is conservative in a sense that the average number of grains on the lattice is conserved in time. The landmark of the SOC state is a power law avalanche size distribution [1, 2, 8]:

$$P(s) = s^{-(b+1)}. \tag{2}$$

In (2) s denotes the size of the avalanche and b is the scaling exponent. Moreover, also the single site and pair probabilities are constant in the SOC state. That means, the probability to find a certain amount of grains on a randomly chosen site, or a certain combination of grains on two sites deep in a lattice (not at the boundaries) is constant.

The initial studies of the dynamics of BTW model, and other models as well, were done numerically [1, 3, 8, 12–14]. But if one defines the BTW model on the Bethe lattice, the avalanches do not make loops, and the model can be solved analytically [2].

It is possible to simplify BTW model significantly and to make it completely deterministic by choosing only one input site, for example the central one (N should be odd, of course, in this case). Such sandpile automaton was discussed by Wiesenfeld *et al.* [3]. This automaton behaves fully deterministically, but in spite of that, the statistical distributions of the avalanche

sizes still maintain a power law character (2) in the self organized critical state [3]. Thanks to its deterministic nature, the dynamics of the finite size cellular automaton is (after some initial transients) fully periodic in the self organized critical state . In the phase space the system is quickly attracted to the periodic attractor. The periodicity makes thus a reliable criterion for indication of the SOC state in this type of models. Period T strongly depends on the lattice size, but doesn't depend at all on the initial sandgrain distribution. Numerically the period was evaluated in Ref. [3] for the square lattices with small number of sites $N^2 = N \times N$, where $N < 11$. Analytically the period was calculated in [4, 5] for a common situation, where the central site is not the only input site, for n-dimensional system and different boundary conditions.

3. Analytical calculations on the Bethe lattice

The major analytical work in the SOC theory was done by Dhar et others and published in a series of beautiful papers [2, 6, 7]. The most complete analytical calculations concerns critical heigh BTW model defined on a Bethe lattice with coordination number 3. The critical sand column height, which starts the relaxation process (avalanche) equals $z_c = 4$. The dynamical rules of the model are similar to that described in the previous Section. The only difference is that during an avalanche, relaxing site distributes only three grains to its nearest neighbours. That is why the SOC state on the Bethe lattice is characterized by the two following conditions:

1) Only the configurations with 3, 2 or 1 grains per site are possible.

2) One grain per site configurations can arise only as a rest after the toppling process. The sites with one grain toppled therefore one step before.

If the number of sites in Bethe lattice equals N, the number of possible configurations in the SOC state is 3^N. But not all of them are allowed, some of them are excluded by the conditions 1) and 2). Examples of disallowed configurations are shown in Fig. 1.

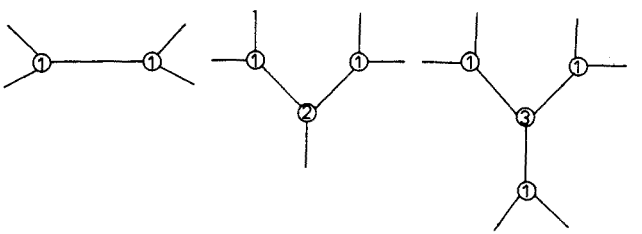

Fig. 1. Some examples of disallowed configuration on the Bethe lattice.

In order to get some insight into the analytical calculations on the Bethe lattice, I shall repeat some Dhar's results here, namely the calculation of the single site probabilities [2].

First, let us derive some useful recursion relations. Let us have a tree T rooted on the vertex a. If we delete this vertex, the tree T breaks into two subtrees T_1 and T_2 with vertices a_1 and a_2. Two types of allowed configurations are defined on the tree T:

1. strongly allowed
2. weakly allowed

Let us imagine that the configuration C on T is allowed and that we connect the vertex a with another site b on which there is only one grain. A new configuration $\tilde{C}$ arises. If $\tilde{C}$ is still allowed, the old configuration C was strongly allowed. If our new configuration $\tilde{C}$ is disallowed, the old configuration C was weekly allowed.

The number of weekly allowed configurations $N_w(T)$ on the tree T is given as a sum of weekly allowed configurations in a case with one, two and three grains on the root site a, namely

$$N_w(T) = \sum_{\omega=1}^{3} N_w(T,\omega). \tag{3a}$$

In the same way the number of all strongly allowed configuration $N_s(T)$ is given as

$$N_s(T) = \sum_{\omega=1}^{3} N_s(T,\omega). \tag{3b}$$

In order to evaluate equations (3a), (3b), we need to know $N_w(T,\omega)$ and $N_s(T,\omega)$. Taking into account the definitions of the strongly and weekly allowed configurations, one immediately recognizes that

$$N_w(T,1) = N_s(T_1)N_s(T_2)\,, \tag{4a}$$
$$N_w(T,2) = N_s(T_1)N_w(T_2) + N_w(T_1)N_s(T_2)\,, \tag{4b}$$
$$N_w(T,3) = N_w(T_1)N_w(T_2)\,, \tag{4c}$$
$$N_s(T,1) = 0\,, \tag{4d}$$
$$N_s(T,2) = N_s(T_1)N_s(T_2)\,, \tag{4e}$$
$$N_s(T,3) = N_s(T_1)N_s(T_2) + N_w(T_1)N_s(T_2) + N_s(T_1)N_w(T_2) \tag{4f}$$

and the sums (3a), (3b) are therefore given as

$$N_w(T) = [N_s(T_1) + N_w(T_1)][N_w(T_2) + N_s(T_2)]\,, \tag{5a}$$
$$N_s(T) = 2N_s(T_1)N_s(T_2) + N_w(T_1)N_s(T_2) + N_s(T_1)N_w(T_2)\,. \tag{5b}$$

Let us also define $X(T)$, the ratio of the number of strongly and weekly allowed configurations on the tree T

$$X(T) = \frac{N_w(T)}{N_s(T)}. \tag{6}$$

When the number of sites goes to infinity, the trees T_1 and T_2 are equivalent and can be taken as a trees of the $n-1^{\text{st}}$ generation, while the tree T is considered to be a tree of the n^{th} generation. From equations (5a), (5b) it is clear that

$$X(T_n) = \frac{1 + X(T_{n-1})}{2}. \tag{7}$$

Whether this sequence tends to some fixed point one can find by solving the fixed point equation

$$X = \frac{1+X}{2}$$

which gives the fixed point

$$X = 1. \tag{8}$$

Deep in a lattice, for $n \to \infty$ the ratio of strongly and weekly allowed configuration tends to 1.

Let us now imagine a point O somewhere deep in a Bethe lattice with a large amount of sites. This point is connected to three trees T_1, T_2, and T_3. Let us calculate the total number of allowed configurations $N(\omega)$ if ω grains are on the point O. From the condition 2) and the Equation (8) one can derive the following expressions

$$N(1) = \prod_{i=1}^{3} N_s(T_i), \tag{9a}$$

$$N(2) = (1 + \sum_{i=1}^{3} X_i) \prod_{i=1}^{3} N_s(T_i), \tag{9b}$$

$$N(3) = (1 + \sum_{i=1}^{3} X_i + \sum_{i<k} X_i X_k) \prod_{i=1}^{3} N_s(T_i), \tag{9c}$$

which means that the total number of the allowed configurations is

$$N_{\text{total}} = \sum_{\omega=1}^{3} N(\omega) \tag{10}$$

and the probability $P(\omega)$ of having sand column of height ω at site O is then given for $\omega = 1, 2, 3$ as

$$P(\omega) = \frac{N(\omega)}{N_{\text{total}}}. \tag{11}$$

If the site O lies deep in the lattice as we assumed, $X_i \to 1$ and from the equations $(9a-c), (10)$ and (11) we get the numerical estimates of $P(\omega)$ for all values of ω; the single site probabilities:

$$P(1) = \tfrac{1}{12}, \qquad P(2) = \tfrac{4}{12}, \qquad P(3) = \tfrac{7}{12}. \tag{12}$$

By the similar but more sophisticated calculations, which are beyond the scope of this lecture, it is also possible to find the numerical estimates of the probabilities $P(\omega, \gamma)$ of having ω sandgrains on the site O_1 and γ grains on the site O_2, in the case that both sites are far from the lattice boundaries. The sites can be nearest neighbours or they can be distant. The important expression (2), together with the scaling exponent b (which equals $1/2$ for the Bethe lattice) was also analytically derived [2].

4. Analytical calculations of the deterministic BTW model

For the square lattice, I shall present our analytical method of the attractor period calculations done for the most simple deterministic version of the BTW automaton. In this model only one site serves as a sandgrain input place. Its coordinates are (i^*, j^*). Let us first define the spilling number $x_{i,j}(\tau)$ as a number of topplings of the site i, j during the time interval τ. Then the change of the height of column $z_{i,j}$ at an arbitrary site (i, j) is expressed as:

$$\begin{aligned} z_{i,j}(t) \to & z_{i,j}(t+\tau) \\ = & z_{i,j}(t) + x_{i-1,j}(\tau) + x_{i+1,j}(\tau) + x_{i,j+1}(\tau) + x_{i,j-1}(\tau) - 4x_{i,j}(\tau) \\ & + \tau \delta_{i,i^*} \delta_{j,j^*}. \end{aligned} \tag{13}$$

(One must of course take into account that some terms at the right hand side are zero if the site i, j lies on the boundary: $x_{0,j}(\tau) = x_{i,0}(\tau) = x_{N+1,j}(\tau) = x_{i,N+1}(\tau) = 0$.)

In the SOC state this deterministic BTW model has a periodical dynamics [3], which means that the configurations $\{z_{i,j}\}$ repeat themselves after T time units:

$$z_{i,j}(t) = z_{i,j}(t+T) \quad \text{for all} \quad i, j, t. \tag{14}$$

Substituting $\tau = T$ into the equation (13) we get the following system of N^2 equations ($x_{i,j}(T) = x_{i,j}$), where all $x_{i,j}$ together with T are *integers*:

$$x_{i-1,j} + x_{i+1,j} + x_{i,j+1} + x_{i,j-1} - 4x_{i,j} = -T\delta_{i,i^*}\delta_{j,j^*} . \qquad (15)$$

Thus, the problem of the period calculation is simplified. One needs only to find the *smallest* integer T, such that the system (15) is solvable in integers.

The matrix representation of the system (15) is

$$(\boldsymbol{H}_2 - E)\boldsymbol{x}_2 = \boldsymbol{t}_2 . \qquad (16a)$$

The matrix $\boldsymbol{H}_2$ is well known in the solid state theory [9] as a Hamiltonian describing nearest neighbour hopping on a two dimensional lattice

$$\boldsymbol{H}_2 = \begin{pmatrix} \boldsymbol{H}_1 & \boldsymbol{I} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{I} & \boldsymbol{H}_1 & \boldsymbol{I} & \dots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{I} & \boldsymbol{H}_1 \end{pmatrix} . \qquad (16b)$$

It has the dimension N^2. E in (16a) represents the "energy" and equals 4 for the two dimensional BTW model defined in Section 2. $\boldsymbol{H}_1$ has dimension N and is a Hamiltonian of nearest neighbour hopping on a one dimensional lattice

$$\boldsymbol{H}_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} , \qquad (16c)$$

$\boldsymbol{x}_2$ stands for a spilling vector to be determined. All components of $\boldsymbol{t}_2$ are zero except the $(j^* + (i^* - 1)N)$-th for which equals $(-T)$. Both $\boldsymbol{x}_2$ and $\boldsymbol{t}_2$ are of the dimension N^2, and $\boldsymbol{I}$ is the identity matrix of the dimension N.

The eigenvalues of $\boldsymbol{H}_1$ are given as

$$E_k = 2\cos(\frac{\pi k}{N+1}) \qquad k = 1, 2, ..., N \qquad (17)$$

and the matrix $\boldsymbol{Q}_1$ which diagonalizes $\boldsymbol{H}_1$ has the elements

$$\boldsymbol{Q}_1(r,s) = \Psi_s(r) = \sqrt{\frac{2}{N+1}} \sin\left(\frac{\pi rs}{N+1}\right) . \qquad (18)$$

A diagonalization matrix of the Hamiltonian $\boldsymbol{H}_2$ is easily determined by rewriting the Hamiltonian $\boldsymbol{H}_2$ in terms of tensor multiplications:

$$\boldsymbol{H}_2 = \boldsymbol{H}_1 \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{H}_1 . \qquad (19)$$

From (19), one immediately realizes that $\boldsymbol{H}_2$ is diagonalized by a matrix $\boldsymbol{Q}_2$ which has a simple form

$$\boldsymbol{Q}_2 = \boldsymbol{Q}_1 \otimes \boldsymbol{Q}_1 \,. \tag{20}$$

The solution of the equations (15, 16) is written with a help of the Green function as

$$\boldsymbol{x}_2 = \boldsymbol{G}_2(E)\boldsymbol{t}_2 \,, \tag{21}$$

where

$$\boldsymbol{G}_2 = \boldsymbol{Q}_2 \boldsymbol{\Lambda}^{-1} \boldsymbol{Q}_2^{-1} \,, \tag{22}$$

is the Green function and

$$\boldsymbol{\Lambda} = \boldsymbol{Q}_2^{-1}(\boldsymbol{H}_2 - E)\boldsymbol{Q}_2$$

is a diagonal matrix.

With a help of tensor algebra, $\boldsymbol{H}_2$ is easily diagonalized

$$\boldsymbol{Q}_2^{-1}\boldsymbol{H}_2\boldsymbol{Q}_2 = (\boldsymbol{Q}_1 \otimes \boldsymbol{Q}_1)^{-1}(\boldsymbol{I} \otimes \boldsymbol{H}_1 + \boldsymbol{H}_1 \otimes \boldsymbol{I})(\boldsymbol{Q}_1 \otimes \boldsymbol{Q}_1) \,. \tag{23}$$

The solutions x_k in (21), $k = j + (i-1) \times N$, are easily calculated from the expressions $(21 - 23)$, and the final decomposed formula connecting the period T and the spilling number of the site (i, j) has the form:

$$x_k = T \sum_{p=1}^{N} \sum_{q=1}^{N} \Psi_p(i)\Psi_q(j) \frac{1}{4 - E_p - E_q} \Psi_p(i^*)\Psi_q(j^*). \tag{24}$$

As the elements of $\boldsymbol{H}_2$ and $\boldsymbol{t}$ in (16) are integers, the right hand side of the equation (24) must have the following form

$$x_k = \frac{P_k}{Q_k} T, \tag{25}$$

with P_k, Q_k being *incommensurable* integers. Evaluating the double sum in (24) for all sites and keeping its rational character, a series of integer numbers T_k is easily found. We simply put $T_k = Q_k$ which guarantees that x_k is also integer, as needed. The period (T) is associated with the *smallest* integer divisible by *all* denominators Q_k.

The n-dimensional problem is just a direct generalization of the two dimensional case. All we need are a n-dimensional versions of (19) and (20), which we get with a help of one dimensional Hamiltonian $\boldsymbol{H}_1$ and its diagonalization matrix $\boldsymbol{Q}_1$ using the tensor multiplication rules [4]. The generalization of the period calculations to the periodic boundary conditions in n-dimensional sandpile cellular automaton is also straightforward [5].

5. Nonconservative models and self organized criticality

In this Section we discuss one of the main problems of the SOC theory, namely whether the SOC state is a consequence of the conservative dynamics of the system. In 1991 Feder and Feder [10] introduced a nonconservative sandpile automaton as a model system for the stick–slip processes. Their motivation was to model the dynamics leading to the earthquakes. It is known from seismological studies [11], that the stick–slip processes are decisive for the dynamics of the earth crust and prepare the conditions for future earthquakes.

The Feders' model describes the propagation of stress across the square lattice, but I will stay in the sandpile terminology. The basic property of the Feders' model lies in a fact that during an avalanche process not all "material" is distributed to the nearest neighbours. Some amount of matter disappears (dissipates). Thus the model is not conservative anymore in the previously defined sense. The second significant property is that the model is driven continuously. The insufficiency of the Feders' model is given by the fact that the amount of dissipated material cannot be controlled. This disadvantage was removed in a model of Christiansen and Olami [12]. The dynamics of the model was studied numerically by Jánosi and Kertész [13] and Grassberger [13].

The nonconservative continuously driven sandpile model of Christiansen and Olami is defined on the square lattice and exhibits the two step dynamics:
Input process — continuous drive:

$$z_i(t + \delta t) = z_i(t) + v\delta t\,. \tag{26a}$$

$$z_i < z_c\,,$$

$$z_c = v = 1\,.$$

Avalanche process:

$$z_{i+nn} \to z_{i+nn} + \alpha z_i\,,$$

$$z_i \to 0\,. \tag{26b}$$

In Equations (26) nn denotes all nearest neighbours, α controlls the amount of material which is distributed to the other sites, and v is the velocity. The other variables denotes the same as in the BTW model. During the second step (an avalanche) the time is kept fixed, so again we have two different time scales: the input time scale is long as compared to the relaxation time.

The main and the most important result achieved by the numerical simulations of the nonconservative model of Christiansen and Olami is that

in a case of nonperiodic boundary conditions, the scale invariant avalanches were observed and thus SOC numerically proved.

This was very surprising, because there exists theoretical arguments that SOC state and scale invariancy connected with it, are connected to the conservativeness of the model. Hwa and Kardar [15] together with Grinstein [16] studied the steady states of the dissipative systems with local interactions in a presence of uncorrelated external noise. Such systems have a coarse grained description given by the Langevine equation

$$\frac{\partial z(\vec{x},t)}{\partial t} = -\Gamma f_x(z) + \eta(\vec{x},t)\,. \tag{27}$$

In (27) z is the one component order parameter (analogous to the sand column height), Γ is a kinetic coefficient which sets the scale for the relaxation, and f_x represents an analytical function of $z(\vec{x},t)$ and its spatial derivatives. The external noise $\eta(\vec{x},t)$ is uncorrelated and Gaussian with the strength D and $\langle\eta(\vec{x},t)\rangle = 0$

$$\langle\eta(\vec{x},t),\eta(\vec{x}\,',t)\rangle = D\delta(\vec{x}-\vec{x}\,')\delta(t-t')\,. \tag{28}$$

If the deterministic part of the Langevine equation is conservative, the power series expansionof f_x must start with the gradient term. In the simplest case Γf_x is given as $\nabla^2 z(\vec{x},t)$. Then the deterministic part of the equation (27) is nothing but the continuity equation

$$\frac{\partial z(\vec{x},t)}{\partial t} = \mathrm{div}\vec{j}, \tag{29}$$

$$\vec{j} = \nabla z(\vec{x},t)\,.$$

If the function f_x has terms proportional to $z(\vec{x},t)$, the solution has terms proportional to $e^{-\xi t}$. They incorporate the characteristic time scale ξ into the dynamics. It was shown in [15] by renormalization group calculations that if f_x in (27) has only gradient (conservative) terms, no characteristic time scale appears. Thus the time scale invariancy is a direct consequence of a conservative dynamics of the system.

The space scale invariancy is also a consequence of the conservativeness. Indeed, the zero component of the Fourier transform $z(k=0,t)$ is given by the random walk equation

$$\frac{\partial z(k=0,t)}{\partial t} = \eta(k=0,t)\,,$$

and thus

$$z(k=0,t) \sim t^{1/2}\,. \tag{30}$$

for large t. That means that $z(k=0,t)$ diverges for t going to infinity and also the static wave number susceptibility

$$\chi(\vec{k}) = \lim_{t\to\infty} \langle\mid z^2(\vec{k},t) \mid\rangle \tag{31}$$

diverges for $t \to \infty$. This can only be the case if the correlation function $G(r)$ falls off more slowly than x^{-d} in a d-dimensional system [15, 16]. This way the algebraic decay of the space correlations is guaranteed for the conservative systems and fluctuations of z on all scales are possible. The existance of a SOC state in conservative systems is thus quite a natural property. In contrast to the results of numerical simulations, there is no theoretical reason to suppose this for nonconservative systems. This discrepancy was not completely explained up to now.

6. Conclusion

In this lecture I showed some recent results and hot problems in the theory of self organized criticality. The main trend in this branch of the nonlinear theory is to explain or support a great amount of numerical results by reasonable analytical arguments. This trend follows the analytical studies of Dhar and others [2, 4–7]. For the nonconservative models, the most pronounced question is to find a relation between nonconservative sandpile-like systems and dissipative noisy systems with local interactions, described by the Langevine equation.

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