HYDRODYNAMIC FORM OF THE WEYL EQUATION

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Dedicated to the memory of Professor Jan Rzewuski

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A hydrodynamic form of the wave equation for the neutrino wave function is derived. The formulation is similar to that given by Takabayasi for the nonrelativistic Pauli equation. The hydrodynamic variables comprise one scalar field — the density — and two vector fields — the velocity and momentum. The reduction in the number of variables to four requires a quantization condition — the same as in the nonrelativistic case — that relates the curl of the momentum field to an axial vector built from the velocity field.

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1. Introduction

The same year that Schrödinger discovered the wave equation, Madelung [1] found a hydrodynamic form of wave mechanics. This approach has been fully developed and extended by Takabayasi [2–5] and others [6–8] to nonrelativistic spinning particles described by the Pauli equation. A review of the hydrodynamic formulation of nonrelativistic wave mechanics with a large number of references can be found in [9]. A hydrodynamic formulation of relativistic wave mechanics of the Dirac particle was given by Takabayasi [10]. His paper was based on some earlier studies [11, 12] of bilinear observables in the Dirac theory. This formulation, however, can not be applied to the case of neutrinos since the two scalar invariants, that are used by Takabayasi to rescale all other variables, vanish identically. I shall show that a straightforward extension of the Takabayasi method, developed in [4] to give a hydrodynamic form of nonrelativistic wave mechanics based on the Pauli equation, yields the desired result. However, the resulting hydrodynamic equations are strikingly different in both cases.

In the hydrodynamic formulation of nonrelativistic wave mechanics the flow of probability is described by the hydrodynamic variables: the probability density ρ and the velocity of the probability flow $\vec{v}$. For a spinning particle we need also an additional vector $\vec{s}$ describing the spin. The advantage of a hydrodynamic formulation of wave mechanics is that it enables us to visualize quantum mechanical processes in terms of the familiar variables of classical hydrodynamics. Since the number of hydrodynamic variables always exceeds the number of variables needed to describe the wave function, we must impose an auxiliary condition on the hydrodynamic variables. In other words, not all the states in the hydrodynamic description correspond to the quantum states of the system. A selection principle — *the quantization condition* — is necessary to restrict the hydrodynamic states to only those that can be obtained from a wave function. In the hydrodynamic formulation, the quantization condition specifies the vorticity of the flow.

In the simplest case of a particle moving in a potential field, treated by Madelung, the hydrodynamic variables and the wave function are related through the formulas

$$\rho = |\psi|^2, \qquad m\vec{v} = \vec{p} = \nabla S, \tag{1}$$

where S is identified with the phase of the wave function (R is assumed real)

$$\psi = R\exp(\tfrac{i}{\hbar}S). \tag{2}$$

The requirement that the wave function be singlevalued imposes the following restriction — the quantization condition — on the velocity field

$$\oint_C d\vec{l}\cdot\vec{v} = \frac{2\pi\hbar}{m}n, \tag{3}$$

where C is an arbitrary closed contour. By the Stokes theorem, the quantization condition can also be expressed in terms of a surface integral representing the vorticity flux,

$$\int_S d\vec{S}\cdot(\nabla\times\vec{v}) = \frac{2\pi\hbar}{m}n, \tag{4}$$

where S is any surface spanned by the closed contour C. Thus, the quantization condition states that the motion of the probability fluid is irrotational almost everywhere except possibly at a discrete set of vortex lines whose strength is quantized in units of $2\pi\hbar/m$.

2. Hydrodynamic form of the Pauli equation

Spin one-half particles are described in nonrelativistic wave mechanics by the Pauli equation

$$i\hbar\partial_t\Psi = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - e\vec{A}\right)^2 + e\phi - \mu\vec{B}\cdot\sigma\right]\Psi\,. \tag{5}$$

I have written here the Pauli equation for a general case of electric and magnetic field, but in order to compare the results with those obtained for neutrinos it is sufficient to consider only the field free case.

The hydrodynamic variables are defined in terms of a two-component wave function Ψ through the relations

$$\rho = \Psi^\dagger\Psi,\ \rho\,\vec{s} = \frac{\hbar}{2}\Psi^\dagger\vec{\sigma}\Psi, \quad \rho\,\vec{p} = \frac{\hbar}{2i}\left(\Psi^\dagger\overleftrightarrow{\nabla}\Psi\right), \tag{6}$$

where the double arrow denotes the antisymmetric differentiation, $\overleftrightarrow{\nabla} = \overrightarrow{\nabla} - \overleftarrow{\nabla}$. It can be checked that by construction the spin vector $\vec{s}$ has a fixed length $\hbar/2$ so that the number of hydrodynamic variables is not seven but only six.

The evolution equations in the hydrodynamic formulation can be obtained by evaluating the time derivatives of the hydrodynamic variables (6) with the help of the Pauli equation

$$\partial_t\rho = \partial_t(\Psi^\dagger\Psi) = -\frac{\hbar}{2mi}\partial_k\left(\Psi^\dagger\overleftrightarrow{\partial_k}\Psi\right), \tag{7}$$

$$\partial_t(\rho\,s_i) = \frac{\hbar}{2}\partial_t(\Psi^\dagger\sigma_i\Psi) = -\frac{\hbar^2}{4mi}\partial_k\left(\Psi^\dagger\sigma_i\overleftrightarrow{\partial_k}\Psi\right), \tag{8}$$

$$\partial_t(\rho\,p_i) = \frac{\hbar}{2i}\partial_t(\Psi^\dagger\overleftrightarrow{\partial_i}\Psi) = \frac{\hbar^2}{4m}\partial_k\left(\Psi^\dagger\overleftrightarrow{\partial_i}\,\overleftrightarrow{\partial_k}\Psi\right). \tag{9}$$

On the right hand side of the first equation we recognize the divergence of $\rho\vec{v}$, but in the remaining two equations we need additional relations in order to express the right hand sides in terms of hydrodynamic variables. The existence of such relations is guaranteed by the following property noted already by Takabayasi [4]: every bilinear expression involving the spinors $\Psi^\dagger, \Psi$ and their derivatives can be reduced to the form $\rho \times$ [polynomial in: $\ln\rho, \vec{s}, \vec{u}$].

When this reduction is performed, Eqs. (7)–(9) are transformed into the following evolution equations for the hydrodynamic variables [4, 8, 14]

$$\partial_t \rho + (\vec{v}\cdot\nabla)\rho = -(\partial_k v_k)\rho\,, \tag{10}$$

$$\partial_t s_i + (\vec{v}\cdot\nabla)s_i = \frac{1}{m\rho}\partial_k(\rho\,\varepsilon_{ijl}s_j\partial_k s_l)\,, \tag{11}$$

$$\partial_t v_i + (\vec{v}\cdot\nabla)v_i = \frac{1}{m^2\rho}\partial_k\left[\frac{\hbar^2}{4}\rho\,\partial_i\partial_k\ln\rho + \rho(\partial_i s_n)(\partial_k s_n)\right]. \tag{12}$$

Now, we need the quantization condition which will eliminate, as in the spinless case, two variables because the wave function of a spin-half particle has just two complex components. As was shown in [4–6], this condition involves a contribution from the spin variables. It can be expressed in terms of the polar angle φ and the azimuthal angle ϑ that define the unit vector $\vec{n}$ in the spin direction, $\vec{s} = (\hbar/2)\vec{n}$,

$$n_x = \cos\varphi\,\sin\vartheta, \quad n_y = \sin\varphi\,\sin\vartheta, \quad n_z = \cos\vartheta\,. \tag{13}$$

The condition that the wave function be singlevalued, has now the form

$$\oint_C d\vec{l}\cdot\left(\vec{v} + \frac{\hbar}{2m}\cos\vartheta\nabla\varphi\right) = \frac{2\pi\hbar}{m}n\,. \tag{14}$$

Despite its appearance, the last term in the quantization condition (14) does have a geometric meaning. This can be seen after the transformation of the line integral into the surface integral by the Stokes theorem

$$\int_S d\vec{S}\cdot\left[\nabla\times m\vec{v} - \frac{\hbar}{2m}\sin\vartheta(\nabla\vartheta\times\nabla\varphi)\right] = \frac{2\pi\hbar}{m}n\,. \tag{15}$$

It has been shown by Takabayasi [4] that the second term under the integral can be written as a genuine axial vector built from the spin vector and its derivatives. This vector — I shall call it the Takabayasi vector — has the form

$$\vec{T} = \tfrac{1}{2}\varepsilon_{ijk}n_i(\nabla n_j)\times(\nabla n_k)\,. \tag{16}$$

This important result will be rederived in Section 4. With the use of the Takabayasi vector, the quantization condition reads

$$\int_S d\vec{s}\cdot\left[\nabla\times\vec{v} - \frac{\hbar}{2m}\vec{T}\right] = \frac{2\pi\hbar}{m}n\,. \tag{17}$$

Thus, for spinning particles the flow of probability is no longer irrotational in the bulk but its vorticity is equal to $(\hbar/2m)\,\overrightarrow{T}$. In addition, of course, we may also have discrete vortex lines of quantized strength.

3. Hydrodynamic form of the Weyl equation

The time evolution of the wave function for a massless spin one-half particle — the Weyl equation — has the form

$$\partial_t\phi = -c\,\vec{\sigma}\cdot\nabla\phi\,, \tag{18}$$

where ϕ, as in the nonrelativistic case, has two complex components. Notice that the Pauli equation and the Weyl equation are complementary when it comes to the use of fundamental constants. In the first case we have the Planck's constant and the electron mass, whereas in the second case we have only the speed of light. Thus, even for purely dimensional reasons, the hydrodynamic equations in these two cases must have a different form.

My derivation of the hydrodynamic equations for neutrino will follow the method outlined in the preceding section for the nonrelativistic case. The hydrodynamic variables for neutrino will be defined as follows

$$\rho = \phi^\dagger\phi,\;\; \rho\,\vec{v} = c\,\phi^\dagger\vec{\sigma}\phi,\;\; \rho\,\vec{u} = \frac{c}{2i}\phi^\dagger\,\overleftrightarrow{\nabla}\,\phi\,. \tag{19}$$

There are two differences in notation between these formulas and the corresponding ones in the nonrelativistic case. In the second equation I used the symbol $\vec{v}$ rather than $\vec{s}$ and I replaced $\hbar/2$ by c because for massless particles velocity is aligned with spin and therefore the vector $\vec{v}$ should be associated with $\phi^\dagger\vec{\sigma}\phi$ rather than with momentum. This identification will be corroborated later (*cf.* Eqs. (29)–(31) when we will see that it is consistent with the form of the substantial derivative. In the third equation I used a new symbol $\vec{u}$ as compared to the nonrelativistic case since this quantity, due to the absence of the Planck's constant in these equations, can not be assigned the dimension of momentum.

The time derivatives of the bilinear expressions (19) can be calculated by the repeated use of the Weyl equation

$$\partial_t\rho = \partial_t(\phi^\dagger\phi) = -c\,\partial_k(\phi^\dagger\sigma_k\phi)\,, \tag{20}$$

$$\partial_t(\rho\,v_i) = \partial_t(c\,\phi^\dagger\sigma_i\phi) = -c^2\partial_i(\phi^\dagger\phi) - 2c\,\varepsilon_{ijk}t_{jk}\,, \tag{21}$$

$$\partial_t(\rho\,u_i) = \frac{c}{2i}\partial_t\left(\phi^\dagger\,\overleftrightarrow{\partial_i}\phi\right) = -\partial_k t_{ki}\,, \tag{22}$$

where t_{ij} denotes the stress tensor for the neutrino,

$$t_{ij} = \frac{c^2}{2i}\left(\phi^\dagger\sigma_i\,\overleftrightarrow{\partial_j}\phi\right). \tag{23}$$

In order to express the stress tensor through the hydrodynamic variables, we shall need the following two spinor identities

$$\phi^\dagger\vec{\sigma}\phi \cdot \left(\phi^\dagger\vec{\sigma}\,\overleftrightarrow{\partial_i}\phi\right) = \phi^\dagger\phi\left(\phi^\dagger\,\overleftrightarrow{\partial_i}\phi\right), \tag{24}$$

$$\phi^\dagger\vec{\sigma}\phi \times \left(\phi^\dagger\vec{\sigma}\,\overleftrightarrow{\partial_i}\phi\right) = i\phi^\dagger\phi\,\partial_i(\phi^\dagger\vec{\sigma}\phi) - i\phi^\dagger\vec{\sigma}\phi\,\partial_i(\phi^\dagger\phi). \tag{25}$$

In terms of the hydrodynamic variables, these identities read

$$v_i t_{ij} = c^2\rho\,u_j\,, \tag{26}$$

$$2\varepsilon_{klm}v_l\,t_{mi} = c^2\rho\,\partial_i v_k\,. \tag{27}$$

The first equation determines the part of the stress tensor that (with respect to the first index) is parallel to $\vec{v}$ and the second equation determines the part that is orthogonal. From both these equations we can determine all components of t_{ij} and we arrive at the formula

$$t_{ij} = \rho\,v_i u_j - \tfrac{1}{2}\rho\,\varepsilon_{ikl}v_k\partial_j v_l\,. \tag{28}$$

Upon substituting this expression into Eqs. (20)–(22), we obtain the following set of evolution equations for the hydrodynamic variables

$$\partial_t\rho + (\vec{v}\cdot\nabla)\rho = -(\nabla\cdot\vec{v})\rho\,, \tag{29}$$

$$\partial_t\vec{v} + (\vec{v}\cdot\nabla)\vec{v} = \tfrac{1}{\rho}\vec{v}\times(\vec{v}\times\nabla\rho) + 2(\vec{u} + \nabla\times\vec{v})\times\vec{v}\,, \tag{30}$$

$$\partial_t\vec{u} + (\vec{v}\cdot\nabla)\vec{u} = \tfrac{1}{2\rho}\partial_i(\rho\,\varepsilon_{ikl}v_k\nabla v_l)\,. \tag{31}$$

These equations have a characteristic hydrodynamic form with substantial derivatives of the variables appearing on the left hand side, as in the nonrelativistic case. Note, that the hydrodynamic variables were chosen in such a way that the speed of light never appears in the evolution equations. It only enters through the choice of initial data, as discussed in the next section.

4. Quantization condition

In order to restrict the solutions of the hydrodynamic equations (29)–(31) to only those that are obtained from a two-component wave function, we must impose constraints on the initial conditions. The evolution equations preserve these constraints at all times. The first constraint is purely algebraic; it determines the length of the velocity vector, $|\vec{v}| = c$. The second constraint — the quantization condition — determines the curl of $\vec{u}$. Since in the relativistic case the kinematics is the same as in the nonrelativistic case, we expect the quantization condition to have again the same form (15). I shall rederive this result now, for completeness, with the use of yet another identity for the spinor field ϕ and its derivatives

$$\varepsilon_{ijk}[(\phi^\dagger\phi)^2\,\partial_j(\phi^\dagger\overleftrightarrow{\partial}_k\phi) - \phi^\dagger\phi\,\partial_j(\phi^\dagger\phi)\,(\phi^\dagger\overleftrightarrow{\partial}_k\phi) - \tfrac{i}{2}\varepsilon_{lmn}\,\phi^\dagger\sigma_l\phi\,\partial_j(\phi^\dagger\sigma_m\phi)\,\partial_k(\phi^\dagger\sigma_n\phi)] = 0\,. \quad (32)$$

In terms of hydrodynamic variables, this identity reads

$$\tfrac{1}{c}\varepsilon_{ijk}\partial_j u_k = \tfrac{1}{4c^3}\varepsilon_{ijk}\varepsilon_{lmn}v_l(\partial_j v_m)(\partial_k v_n) \equiv \tfrac{1}{2}T_i\,. \quad (33)$$

One can check by a direct calculation, using the angular representation (13) of the velocity vector $\vec{v} = c\vec{n}$, that $\vec{T}$ can also be written as

$$\vec{T} = \sin\vartheta\,\nabla\vartheta \times \nabla\varphi\,. \quad (34)$$

This completes the proof of the equivalence of the two forms of the quantization condition. In terms of the hydrodynamic variables $\vec{u}$ and $\vec{v}$, the quantization condition reads

$$\int_S d\vec{S}\cdot[\nabla\times\vec{u} - \tfrac{1}{4c^2}\varepsilon_{ijk}\,v_i\,(\nabla v_j)\times(\nabla v_k)] = 2\pi n c\,. \quad (35)$$

The quantization condition signifies that the curl of $\vec{u}$ in the bulk of the probabilistic fluid is fully determined by the velocity vector $\vec{v}$ and its derivatives. This leaves only the longitudinal part of $\vec{u}$ as an independent variable.

5. Conclusions

The main conclusion of this study is that the flow of probability associated with the Weyl equation can be described in purely classical terms. The probabilistic fluid moves with the speed of light and is endowed with an additional degree of freedom — the longitudinal part of a vector field $\vec{u}$. As compared with relativistic dynamics of a perfect fluid, the flow of the "neutrino fluid " is fairly complex. In addition to the precession of the velocity due to the density gradient, there are is also an additional precession around the sum of vorticity and $\vec{u}$ vectors. The evolution for $\vec{u}$ is linear but it has a complex nonlinear source term. In the face of all these complications, are there any advantages of using the hydrodynamic description? It certainly offers a totally different look at the wave function and its time evolution that involves only observable quantities bilinear in the wave function [13]. Moreover, the hydrodynamic description clearly separates the local dynamical laws and a nonlocal, global quantization condition. This important property has been utilized in the past to derive the quantization condition for the magnetic charge [15] and to clarify the interpretation of the Aharonov–Bohm effect [16].

Finally, I would like to mention a possible connection of the hydrodynamic formulation of wave mechanics of massless fermions to the string theory. In the hydrodynamic formulation an essential role is assigned to quantized vortex lines. Such vortex lines are very similar to relativistic strings. They move with the speed of light, but in contradistinction to free strings they interact mutually, just like vortex lines do in ordinary fluid dynamics. It might be possible to extract some interesting relativistic dynamics of the vortex lines — the strings — from the hydrodynamic equations of the neutrino fluid.

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REFERENCES

[1] E. Madelung, *Z. Phys.* **40**, 322 (1926).
[2] T. Takabayasi, *Prog. Theor. Phys.* **8**, 143 (1952).
[3] T. Takabayasi, *Prog. Theor. Phys.* **9**, 187 (1953).
[4] T. Takabayasi, *Prog. Theor. Phys.* **14**, 283 (1955).
[5] T. Takabayasi, *Prog. Theor. Phys.* **70**, 1 (1983).
[6] D. Bohm, R. Schiller, J. Tiomno, *Suppl. Nuovo Cimento* **1**, 48 (1955).
[7] D. Bohm and R. Schiller, *Suppl. Nuovo Cimento* **1**, 67 (1955).
[8] L. Janossy, M. Ziegler–Naray, *Acta Phys. Hung.* **20**, 23 (1965).
[9] S. K. Ghosh, B.M. Deb, *Phys. Rep.* **92**, 1 (1982).
[10] T. Takabayasi, *Nuovo Cimento* **3**, 233 (1956).
[11] O. Laporte, G.E. Uhlenbeck, *Phys. Rev.* **37**, 1380 (1931).
[12] W. Kofink, *Ann. Phys.* (Leipzig) **38**, 421, 436, 565, 583 (1940).
[13] R. Gurtler, D. Hestens, *J. Math. Phys.* **16**, 573 (1975).
[14] I. Bialynicki–Birula, M. Cieplak, J. Kaminski, *Theory of Quanta*, Oxford University Press, Oxford 1992, p. 105.
[15] I. Bialynicki–Birula, Z. Bialynicka–Birula, *Phys. Rev.* **D3**, 2410 (1971).
[16] T. Takabayasi, *Prog. Theor. Phys.* **69**, 1323 (1983).