

κ -DEFORMED REALIZATION OF $D = 4$ CONFORMAL ALGEBRA *

M. KLIMEK

Institute of Mathematics and Computer Science
Technical University of Częstochowa
Dąbrowskiego 73, 42-200 Częstochowa, Poland

AND

J. LUKIERSKI^{† ‡}

Department de Physique Theorique, Universite de Geneve
24 quai Ernest-Ansermet, CH-1211 Geneve 4, Switzerland

Dedicated to the memory of Professor Jan Rzewuski

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We describe the generators of κ -conformal transformations, leaving invariant the κ -deformed d'Alembert equation. In such a way one obtains the conformal extension of the off-shell spin zero realization of κ -deformed Poincaré algebra. Finally the algebraic structure of κ -deformed conformal algebra is discussed.

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1. Introduction

Recently the idea of quantum deformations (see *e.g.* [1–3]) has been applied to $D = 4$ Poincaré algebra [4–13] as well as $D = 4$ conformal algebra [6, 14–16]. For the Poincaré algebra one can distinguish two types of deformations

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‡ On leave of absence from the Institute of Theoretical Physics, University of Wrocław, M. Bornha 9, 50-204 Wrocław, Poland.

- a) The q -deformation $\mathcal{U}_q(\mathcal{P}_4)$, with the dimensionless parameter q (see *e.g.* [5]).

Because the conformal transformations describe the symmetry of the world without dimensionfull parameters, the existence of q -deformed conformal algebra $\mathcal{U}_q(O(4, 2))$ should be expected. Indeed, such deformations described by Drinfeld Jimbo scheme were proposed [6, 14–16], with the following inclusion valid in the algebra sector

$$\mathcal{U}_q(\mathcal{P}_4) \subset \mathcal{U}_q(O(4, 2)), \quad (1.1)$$

- b) The κ -deformed $\mathcal{U}_\kappa(\mathcal{P}_4)$, with dimensionfull parameter κ [4, 6–13].

In such a case the problem of finding a κ -deformation of the conformal algebra $\mathcal{U}_\kappa(O(4, 2))$ such that

$$\mathcal{U}_\kappa(\mathcal{P}_4) \subset \mathcal{U}_\kappa(O(4, 2)), \quad (1.2)$$

has not been yet discussed.

In this paper we would like to consider the problem of the inclusion (1.2) on the level of particular representations. The realizations of the κ -deformed Poincaré algebra with arbitrary spin were given in [7, 9]. If (\vec{m}, \vec{l}) is any finite-dimensional realization of the standard Lorentz algebra $(i, j, k = 1, 2, 3)$

$$\begin{aligned} [m_i, m_j] &= i\epsilon_{ijk} m_k, \\ [m_i, l_j] &= i\epsilon_{ijk} l_k, \\ [l_i, l_j] &= -i\epsilon_{ijk} m_k, \end{aligned} \quad (1.3)$$

then the realization on the multicomponent field $\Psi_A(p_\mu)$ ($A = 1 \dots N$; $N = \dim m_i = \dim l_i$) is given by the formulae:

$$M_i = -i\epsilon_{ijk} p_j \partial_k + m_i, \quad P_\mu = p_\mu, \quad (1.4a)$$

$$N_i = i \left(p_i \partial_0 + \kappa \sinh \frac{p_0}{\kappa} \partial_i \right) + e^{\pm \frac{p_0}{2\kappa}} l_i \pm \frac{1}{2\kappa} \epsilon_{ijk} p_j m_k. \quad (1.4b)$$

If $N = 2S + 1$ and the matrices m_i, l_i describe the irreducible representations $(s, 0)$ and $(0, s)$ of the Lorentz algebra, we obtain the classical κ -deformed field realizations with spin s (see [9]; the case $s = \frac{1}{2}$ corresponding to κ -deformed Dirac equation has been considered explicitly also in [17, 18]). For the spinless case ($m_i = l_i = 0$) the boost generators can be written as follows:

$$\begin{aligned} M_i &= -i\epsilon_{ijk} p_j \partial_k, \quad P_\mu = p_\mu, \\ N_i &= i(p_i \hat{\partial}_0 + \hat{p}_0 \partial_i) \cosh \frac{p_0}{2\kappa}, \end{aligned} \quad (1.5)$$

where we choose the new variables

$$\hat{\partial}_0 = \partial_0 \left(\cosh \frac{p_0}{2\kappa} \right)^{-1}, \quad \hat{p}_0 = 2\kappa \sinh \frac{p_0}{2\kappa}, \quad (1.6)$$

and

$$[\hat{\partial}_0, \hat{p}_0] = 1. \quad (1.7)$$

The algebra satisfied by the generators (1.5) is the κ -Poincaré algebra with the condition $\vec{P} \cdot \vec{M} = 0$, *i.e.*

$$[M_i, M_j] = i\epsilon_{ijk} M_k, \quad [M_i, N_j] = i\epsilon_{ijk} N_k, \quad (1.8a)$$

$$[N_i, N_j] = -i\epsilon_{ijk} M_k \left(1 + \frac{\hat{P}_0^2}{2\kappa^2} \right), \quad (1.8b)$$

$$[M_i, P_j] = i\epsilon_{ijk} P_k, \quad [M_i, \hat{P}_0] = 0, \quad (1.8c)$$

$$[N_i, P_j] = i\hat{P}_0 \left(1 + \frac{\hat{P}_0^2}{4\kappa^2} \right)^{\frac{1}{2}} \delta_{ij}, \quad (1.8d)$$

$$[N_i, \hat{P}_0] = iP_i \left(1 + \frac{\hat{P}_0^2}{4\kappa^2} \right)^{\frac{1}{2}}, \quad (1.8e)$$

where we use the modified energy operator

$$\hat{P}_0 = 2\kappa \sinh \frac{P_0}{2\kappa} \quad (1.9)$$

and also the supplementary condition satisfied by spinless κ -deformed boost generators:

$$P_i N_j - P_j N_i = i\epsilon_{ijk} M_k \hat{P}_0 \left(1 + \frac{\hat{P}_0^2}{4\kappa^2} \right)^{\frac{1}{2}}, \quad (1.10)$$

The choice (1.9) of the energy operator transforms the κ -deformed mass-shell condition in standard basis (see [8])

$$\vec{P}^2 - \left(2\kappa \sinh \frac{P_0}{2\kappa} \right)^2 = -M^2, \quad (1.11)$$

into the classical mass-shell condition (see also [13])

$$\vec{P}^2 - (\hat{P}_0)^2 = -M^2. \quad (1.12)$$

Further in Section 2 we describe the extension of the generators (1.5) of the κ -Poincaré algebra to the generators of κ -conformal algebra, by supplementing (1.5) with the dilatation generator D and four conformal generators

K_μ which preserve the massless κ -deformed mass-shell conditions (1.11) (or equivalently (1.12)) with $M = 0$). In such a way we deform the generators of the $D = 4$ conformal transformations leaving invariant classical $D = 4$ wave equation $\square \varphi = 0$ to the ones describing in standard basis [8] the invariance of the κ -deformed $D = 4$ wave equation

$$\left[\Delta - \left(2\kappa \sin \frac{\partial_t}{2\kappa} \right)^2 \right] \varphi(x) = 0. \quad (1.13)$$

The main result of our consideration is the conclusion that the commutators of the generators of the κ -deformed conformal algebra contain not only the nonlinearities in the fourmomentum variable, but also the quadratic terms in the Lorentz and dilatation generators. In Section 3 we present the general discussions, in particular we list some problems and give an outlook.

2. Spinless realizations of the κ -deformed conformal algebra

The classical form (1.12) of the κ -deformed mass-shell condition implies that the additional generators leaving invariant the massless κ -deformed field equation (1.13) can be written in complete analogy to the classical conformal generators. Denoting by

$$\hat{x}_0 = x_0 \frac{1}{\cos \frac{\partial_0}{2\kappa}}, \quad D_0 = 2\kappa \sin \frac{\partial_0}{2\kappa}, \quad (2.1)$$

the relation (1.7) takes the form

$$[D_0, \hat{x}_0] = 1. \quad (2.2)$$

If we introduce

$$\vec{x}^2 = \vec{x} \cdot \vec{x} + \hat{x}_0^2, \quad (2.3)$$

one can supplement (1.5) with the following generators:

$$\begin{aligned} D &= i\vec{x} \cdot \vec{\partial} + i\hat{x}_0 D_0 + i, \\ K_i &= -i\vec{x}^2 \partial_i + 2x_i D, \\ K_0 &= -i\vec{x}^2 D_0 + 2\hat{x}_0 D. \end{aligned} \quad (2.4)$$

It is interesting to observe that the algebra $O(4, 1)$ formed by the generators (M_i, P_i, D, K_i) is classical and the κ -deformed conformal generators (N_i, P_0, K_0) belong to the coset $\mathcal{K} = \frac{O(4, 2)}{O(4, 1)}$. The classical algebra $O(4, 1)$ can be interpreted as the $D = 3$ Euclidean conformal algebra, extending conformally the classical E_3 subalgebra of κ -deformed $D = 4$ Poincaré algebra.

We shall describe now algebraically the κ -deformed realizations of conformal algebra, given by (1.5) and (2.4). One gets the following set of relations:

a) The classical $O(4, 1)$ algebra with the generators (M_i, P_i, D, K_i)

$$\begin{aligned} [M_i, M_j] &= i\epsilon_{ijk} M_k, & [M_i, P_j] &= i\epsilon_{ijk} P_k, \\ [M_i, D] &= 0, & [P_i, P_j] &= 0, \\ [M_i, K_j] &= i\epsilon_{ijk} K_k, & [D, P_i] &= -iP_i, \\ [D, K_i] &= iK_i, & [K_i, K_j] &= 0, \\ [P_i, K_j] &= -2i\delta_{ij} D - 2i\epsilon_{ijk} M_k. \end{aligned} \quad (2.5)$$

b) The $O(4, 1)$ covariance relations of the coset generators (N_i, K_0, \hat{P}_0) :

$$\begin{aligned} [M_i, N_j] &= i\epsilon_{ijk} N_k, & [M_i, \hat{P}_0] &= [M_i, K_0] = 0 \\ [P_i, N_j] &= -i\delta_{ij} \hat{P}_0 \left(1 + \frac{\hat{P}_0^2}{4\kappa^2}\right)^{\frac{1}{2}}, & [P_i, \hat{P}_0] &= 0 \\ [P_i, K_0] &= -2iN_i \left(1 + \frac{\hat{P}_0^2}{4\kappa^2}\right)^{-\frac{1}{2}}, \\ [D, N_i] &= -N_i \frac{\hat{P}_0^2}{4\kappa^2} \left(1 + \frac{\hat{P}_0^2}{4\kappa^2}\right)^{-1}, \\ [D, \hat{P}_0] &= -i\hat{P}_0, & [D, K_0] &= iK_0, \\ [K_i, N_j] &= -i\delta_{ij} K_0 \left(1 + \frac{\hat{P}_0^2}{4\kappa^2}\right)^{\frac{1}{2}} - iN_j N_i \frac{\hat{P}_0}{2\kappa^2} \left(1 + \frac{\hat{P}_0^2}{4\kappa^2}\right)^{-\frac{3}{2}} \\ &\quad + \frac{1}{4\kappa^2} N_j P_i \left(1 - \frac{\hat{P}_0^2}{2\kappa^2}\right) \left(1 + \frac{\hat{P}_0^2}{4\kappa^2}\right)^{-2} \\ [K_i, \hat{P}_0] &= -2iN_i \left(1 + \frac{\hat{P}_0^2}{4\kappa^2}\right)^{-\frac{1}{2}}. \end{aligned} \quad (2.6)$$

c) The relations for the coset generators

$$\begin{aligned} [N_i, N_j] &= -i\epsilon_{ijk} M_k \left(1 + \frac{\hat{P}_0^2}{2\kappa^2}\right), \\ [N_i, \hat{P}_0] &= -iP_i \left(1 + \frac{\hat{P}_0^2}{4\kappa^2}\right)^{\frac{1}{2}}, \\ [N_i, K_0] &= iK_i \left(1 + \frac{\hat{P}_0^2}{4\kappa^2}\right)^{\frac{1}{2}} + iN_i D \frac{\hat{P}_0}{2\kappa^2} \left(1 + \frac{\hat{P}_0^2}{4\kappa^2}\right)^{-1} \\ &\quad - \frac{1}{4\kappa^2} N_i \hat{P}_0 \left(1 - \frac{\hat{P}_0^2}{2\kappa^2}\right) \left(1 + \frac{\hat{P}_0^2}{4\kappa^2}\right)^{-2}, \\ [\hat{P}_0, K_0] &= 2iD. \end{aligned} \quad (2.7)$$

We see from the relations (2.6)–(2.7) that the commutators of the boosts N_i with the conformal generators $K_\mu = (K_i, K_0)$ contains the bilinear terms in boosts and the dilatation generators. One can conclude therefore that the algebra with (2.5)–(2.7) describes the quadratic algebra with the energy-dependent structure constants, which in the limit $\kappa \rightarrow \infty$ provides the $D = 4$ classical conformal algebra.

3. Discussion and outlook

In the presented paper we consider the κ -deformation of the spinless representations of the $D = 4$ conformal algebra. This is only the first step in the programme of description of κ -deformation of $D = 4$ conformal algebra. The next step consists in description of the representations of the $D = 4$ conformal algebra with arbitrary spin, containing in the κ -Poincaré sector the realization (1.4a)–(1.4b). At present we know only how to add to (1.4a)–(1.4b) the κ -deformed dilatation generator; the form of the κ -deformed conformal generators K_μ for arbitrary spin are not known yet. The κ -deformation of the conformal representations with arbitrary spin would permit to generalize the algebra (2.5)–(2.7), without the condition $\vec{P} \cdot \vec{M} = 0$ and (1.10).

Let us recall that the κ -deformed Poincaré algebra takes the form:

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\tau}] &= f_{\mu\nu, \rho\tau}^{(\kappa)}{}^{\alpha\beta}(\vec{P}, P_0) M_{\alpha\beta}, \\ [M_{\mu\nu}, P_\rho] &= f_{\mu\nu, \rho}^{(\kappa)}{}^\tau(P_0) P_\tau, \\ [P_\mu, P_\nu] &= 0, \end{aligned} \quad (3.1)$$

where the “soft” structure constants can be easily reproduced from the formulae for the κ -Poincaré algebra.

The spinless case is characterized by the lack of dependence of the “soft” structure constant $f_{\mu\nu, \rho\tau}^{(\kappa)}{}^{\alpha\beta}$ on the three-momentum \vec{P} , *i.e.*

$$f_{\mu\nu, \rho\tau}^{(\kappa)}{}^{\alpha\beta}(\vec{P}, P_0) \xrightarrow{\text{spin } 0} f_{\mu\nu, \rho\tau}^{(\kappa)}{}^{\alpha\beta}(P_0). \quad (3.2)$$

In the case of $D = 4$ conformal algebra the modification due to the κ -deformation is stronger — instead of inhomogeneous “soft” Lie algebra¹ one obtains the quadratic algebra with the fourmomentum-dependent structure constants. Denoting by H_i the $O(4, 1)$ generators with the algebra given by

¹ The notion of “soft” Lie algebra has been introduced in [19, 20] and used extensively in the supergroup manifold approach to supergravity.

(2.5) and by K_α the $\frac{O(4,2)}{O(4,1)}$ coset generators (see (2.6)–(2.7)) the algebraic relations can be written as follows

$$\begin{aligned} [H_i, H_j] &= c_{ij}^k H_k, \\ [H_i, K_\alpha] &= c^{(\kappa)}_{i\alpha}{}^\beta(P_0) K_\beta + c^{(\kappa)}_{i\alpha}{}^{\beta\gamma}(P_0) K_\beta K_\gamma + c^{(\kappa)}_{i\alpha}{}^{\beta j}(P_0) K_\beta H_j, \\ [K_\alpha, K_\beta] &= c^{(\kappa)}_{\alpha\beta}{}^i(P_0) H_i + c^{(\kappa)}_{\alpha\beta}{}^\gamma(P_0) K_\gamma + c^{(\kappa)}_{\alpha\beta}{}^{\gamma i}(P_0) K_\gamma H_i. \end{aligned} \quad (3.3)$$

We expect that in the case with noncommuting spin at least some “soft” structure constants in (3.3) might depend on the three-momenta P_i .

Having only scalar representations of κ -conformal algebra at present we are not able to describe its tensor products, which involves the representations with any spin. In other words, it is now too early to discuss the Hopf algebra structure of κ -deformed conformal algebra. The question whether the Hopf algebra structure of κ -deformed conformal algebra exists is an open problem. In the case of positive answer it would be interesting to put the κ -deformed algebraic structure described by the relations (3.3) into the framework of bicrossproduct Hopf algebra [21, 22].

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