GEOMETRIC OBJECTS RELATED TO THE POTENTIAL OF ELECTRIC CHARGES

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Dedicated to the memory of Professor Jan Rzewuski

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We derive explicit formulas for curvature and torsion of a line of the field of n electric charges. These formulas show that in general the torsion of a field line is not zero if $n \geq 3$. We also propose a geometric interpretation of the derived formulas. In the second part of the paper we present an outline of a new description of equipotential surfaces of two and three electric charges. In this description the golden section appears in a natural way when two electric charges are equal. This approach also relates an equipotential surface of three charges to the classic cubic surface containing twenty seven straight lines.

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1. Introductory remarks

This paper consists of two parts. In the first part (Section 2) we follow [1] in deriving explicit formulas for curvature and torsion of a line of the field of n electric charges. In particular, these formulas imply that for two charges the torsion is always zero, ad in general, is not zero when $n \geq 3$. Section 3 gives a geometric interpretation of the formulas derived in Section 2.

The second part (Sections 4 and 5) contains an outline of a new description of equipotential surfaces of two and three electric charges. This description is based on the notion od an auxiliary line L_2 and an auxiliary surface L_3. The auxiliary objects L_2 and L_3 give a geometric insight into a natural parametrization of equipotential surfaces of two and three electric charges. In Section 4 we derive formulas which show that the famous golden section, [2], appears when two charges are equal. In Section 5, making use of the finite Fourier transform $F(3)$, [3], we relate the auxiliary surface L_3 to the classic cubic surface containing twenty seven straight lines, [4–6].

2. Curvature and torsion of a field line of n electric charges

We use the following notation:

q_i $(i = 1, 2, \ldots, n)$ — i-th electric charge,
$\vec{r}(x, y, z)$ — position vector of the point (x, y, z),
$\vec{r}_i = (x_i, y_i, z_i)$ — position vector of q_i,
$R_i = |\vec{R}_i| = |\vec{r} - \vec{r}_i|$ — distance between (x, y, z) and q_i.

Then the potential Φ and the electric field $\vec{E}$ of $q_1, q_2, \ldots, q_n$ are given by the classic formulas

$$\Phi = \sum_{i=1}^{n} q_i R_i^{-1}\,, \tag{2.1}$$

$$\vec{E} = -\operatorname{grad}\Phi\,. \tag{2.2}$$

Let

$$\vec{r}(t) = (x(t), y(t), z(t)) \tag{2.3}$$

be a line parametrized by t. It is well known that the line $\vec{r}(t)$ is by definition a line of the field $\vec{E}$ if $\vec{E}(\vec{r}(t))$ is tangent to the line $\vec{r}(t)$ for any value of the parameter t:

$$\vec{r}\,'(t) = \vec{E}(\vec{r}(t))\,, \tag{2.4}$$

where

$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt}\,. \tag{2.5}$$

The curvature (first curvature) k_1 and torsion (second curvature) k_2 of a line $\vec{r}(t)$ are given by the well known formulas (see, for example, [7, 8]):

$$k_1 = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}\,, \tag{2.6}$$

$$k_2 = -\frac{[\vec{r}\,', \vec{r}\,'', \vec{r}\,''']}{|\vec{r}\,' \times \vec{r}\,''|^2}\,, \tag{2.7}$$

where $[\vec{r}\,', \vec{r}\,'', \vec{r}\,''']$ denotes the mixed product of $\vec{r}\,'$, $\vec{r}\,''$ and $\vec{r}\,'''$:

$$[\vec{r}\,', \vec{r}\,'', \vec{r}\,'''] = (\vec{r}\,' \times \vec{r}\,'')\vec{r}\,'''\,. \tag{2.8}$$

From the above formulas it follows that the curvature k_1 and torsion k_2 can be written in the form

$$k_1 = \frac{|\vec{E} \times \vec{F}|}{|\vec{E}|^3}\,, \tag{2.9}$$

$$k_2 = -\frac{[\vec{E}, \vec{F}, \vec{G}]}{|\vec{E} \times \vec{F}|^2}\,, \tag{2.10}$$

where

$$\vec{F} = \sum_{i=1}^{n} b_i \vec{R}_i \,, \tag{2.11}$$

$$\vec{G} = \sum_{i=1}^{n} c_i \vec{R}_i \,, \tag{2.12}$$

$$a_i = q_i R_i^{-3} \,, \tag{2.13}$$

$$b_i = a_i' = -3R_i^{-2} a_i A_i \,, \tag{2.14}$$

$$c_i = a_i'' = -3R_i^{-2} a_i [(-5R_i^{-2} A_i + A) A_i + B_i + E^2] \,, \tag{2.15}$$

$$A_i = \vec{R}_i \vec{E} \,, \tag{2.16}$$

$$B_i = \vec{R}_i \vec{F} \,, \tag{2.17}$$

$$A = \sum_{i=1}^{n} a_i \,. \tag{2.18}$$

It is easy to verify that the vector $\vec{E} \times \vec{F}$ and the mixed product $[\vec{E}, \vec{F}, \vec{G}]$ can be written in the form

$$\vec{E} \times \vec{F} = \sum_{\substack{i,1=1 \\ (i<j)}} \mu_{ij} \vec{S}_{ij} \,, \tag{2.19}$$

$$[\vec{E}, \vec{F}, \vec{G}] = \sum_{\substack{i,j,k=1 \\ (i<j<k)}}^{n} \sigma_{ijk} V_{ijk} \,, \tag{2.20}$$

where

$$\mu_{ij} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} \,, \tag{2.21}$$

$$\sigma_{ijk} = \begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{vmatrix} \,, \tag{2.22}$$

$$\vec{S}_{ij} = \vec{R}_i \times \vec{R}_j \,, \tag{2.23}$$

$$V_{ijk} = [\vec{R}_i, \vec{R}_j, \vec{R}_k] \,. \tag{2.24}$$

3. A geometric interpretation of the formulas (2.9)–(2.10) as (2.19)–(2.20)

We introduce the following terminology:

$a_i\vec{R}_i$ — the first scaling of $\vec{R}_i$,
$b_i\vec{R}_i$ — the second scaling of $\vec{R}_i$,
$c_i\vec{R}_i$ — the third scaling of $\vec{R}_i$,

$$\left.\begin{aligned} \vec{E} &= \sum_{i=1}^{n} a_i\vec{R}_i \\ \vec{F} &= \sum_{i=1}^{n} b_i\vec{R}_i \\ \vec{G} &= \sum_{i=1}^{n} c_i\vec{R}_i \end{aligned}\right\} \quad \text{Resulting vectors of the first, second and third scaling of } \vec{R}_1, \dots, \vec{R}_n.$$

$P(\vec{A},\vec{B})$ — parallelogram spanned on the vectors $\vec{A}$ and $\vec{B}$,
$P(\vec{A},\vec{B},\vec{C})$ — parallelipiped spanned on the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$,
$\vec{A}\times\vec{B}$ — surface vector of $P(\vec{A},\vec{B})$,
$[\vec{A},\vec{B},\vec{C}]$ — volume of $P(\vec{A},\vec{B},\vec{C})$,
$|\vec{S}_{ij}| = |\vec{R}_i \times \vec{R}_j|$ — elementary surface,
$V_{ijk} = [\vec{R}_i, \vec{R}_j, \vec{R}_k]$ — elementary volume,
$\mu_{ij}\vec{S}_{ij}$ — surface scaling of $\vec{S}_{ij}$,
$\sigma_{ijk}V_{ijk}$ — volume scaling of V_{ijk},
$NES(n) = \binom{n}{2}$ — number of elementary surfaces,
$NEV(n) = \binom{n}{3}$ — number of elementary volumes,

Making use of this terminology we obtain the following geometric interpretation of the formulas (2.9)–(2.10) and (2.19)–(2.20):

(i) The curvature k_1 is equal to the ratio of the surface $|\vec{E}\times\vec{F}|$ to the third power of the length of $\vec{E}$.

(ii) The torsion k_2 is equal to minus the ratio of the volume $[\vec{E},\vec{F},\vec{G}]$ to the square of the surface $|\vec{E}\times\vec{F}|$.

(iii) The volume $[\vec{E},\vec{F},\vec{G}]$ consists of $\binom{n}{3}$ scaled elementary volumes $\sigma_{ijk}V_{ijk}$.

(iv) The surface vector $\vec{E}\times\vec{F}$ consists of $\binom{n}{2}$ scaled elementary surface vectors $\mu_{ij}\vec{S}_{ij}$.

(v) $NEV(n) - NES(n) = \begin{cases} -2 & \text{if} \quad n = 3, 4 \\ 0 & \text{if} \quad n = 5 \\ > 0 & \text{if} \quad n > 5 \end{cases}$

(vi) If $n = 1$ (one electric charge q_1), then $k_1 = k_2 = 0$ because the vectors $\vec{E}$, $\vec{F}$, $\vec{G}$ are then colinear:

$$\vec{E} = a_1 \vec{R}_1 , \quad \vec{F} = b_1 \vec{R}_1 , \quad \vec{G} = c_1 \vec{R}_1 .$$

(vii) If $n = 2$ (two electric charges q_1 and q_2), then in general $k_1 \neq 0$, but $k_2 = 0$ because the vectors $\vec{E}$, $\vec{F}$, $\vec{G}$ are then linear combinations of $\vec{R}_1$ and $\vec{R}_2$, and so are linearly dependent.

4. The golden section and an equipotential surface of two electric charges

In this Section we consider the surface of a fixed potential λ (the λ-potential surface) of two electric charges q_1 and q_2. Then the formula (2.1) takes the form

$$\lambda R_1 R_2 = q_1 R_2 + q_2 R_1 . \tag{4.1}$$

We choose the position vectors $\vec{r}_1$ and $\vec{r}_2$ in the following way

$$\begin{cases} \vec{r}_1 = (0, 0, a) , \\ \vec{r}_2 = (0, 0, 0) . \end{cases}$$

Then the λ-potential surface is invariant under any rotation round the z-axis, and so we can restrict ourselves, for example, to the plane $x = 0$. For this choice

$$\begin{aligned} R_1 &= s = (w^2 - 2az + a^2)^{1/2} , \\ R_2 &= w = (y^2 + z^2)^{1/2} , \end{aligned} \tag{4.3}$$

and the equation (4.1) can be rewritten in the form

$$uv = D_1 v + D_2 u , \tag{4.4}$$

where

$$u = \frac{s}{a} , \qquad v = \frac{w}{a} , \tag{4.5}$$

$$D_i = \frac{q_i}{a\lambda} \qquad (i = 1, 2) . \tag{4.6}$$

Introducing new variables

$$\begin{aligned} \xi &= \tfrac{1}{2}(u + v) , \\ \eta &= \tfrac{1}{2}(u - v) , \end{aligned} \tag{4.7}$$

and notation

$$\begin{aligned} E_1 &= \tfrac{1}{2}(D_1 + D_2), \\ E_2 &= \tfrac{1}{2}(D_1 - D_2), \end{aligned} \tag{4.8}$$

we obtain from (4.4) the following equation of hyperbola

$$(\xi - E_1)^2 - (\eta - E_2)^2 = D_1 D_2 . \tag{4.9}$$

We call this hyperbola the auxiliary line L_2 of the λ-equipotential surface of q_1 and q_2. Let

$$P^2 = \begin{cases} D_1 D_2 & \text{if} \quad q_1 q_2 > 0, \\ -D_1 D_2 & \text{if} \quad q_1 q_2 < 0. \end{cases} \tag{4.10}$$

Then the equation (4.9) can be rewritten in the following two forms

$$(\xi - E_1)^2 - (\eta - E_2)^2 = P^2 \quad \text{if} \quad q_1 q_2 > 0 , \tag{4.11}$$

$$(\eta - E_2)^2 - (\xi - E_1)^2 = P^2 \quad \text{if} \quad q_1 q_2 < 0 . \tag{4.12}$$

Here we restrict ourselves to the first case ($q_1 q_2 > 0$). The other case ($q_1 q_2 < 0$) is discussed in [9]. Let us note that if $q_1 q_2 > 0$, then D_1 and D_2 are always positive. Indeed, by the formula (4.1)

$$q_1 > 0 , \quad q_2 > 0 \;\Rightarrow\; \lambda > 0 , \tag{4.13}$$

$$q_1 < 0 , \quad q_2 < 0 \;\Rightarrow\; \lambda < 0 , \tag{4.14}$$

and so, by (4.6), $D_i > 0$ ($i = 1,2$).

The formula (4.11) implies that the auxiliary line L_2 can be parametrized in terms of hyperbolic functions $\cosh\gamma$ and $\sinh\gamma$:

$$\begin{cases} \xi - E_1 = P\cosh\gamma , \\ \eta - E_2 = P\sinh\gamma , \\ P = (D_1 D_2)^{1/2} . \end{cases} \tag{4.15}$$

Taking into account the formulas (4.7), (4.8) and (4.15) we can write the old variables u and v in the form

$$\begin{cases} u = D_1 + Pe^{\gamma} , \\ v = D_2 + Pe^{-\gamma} . \end{cases} \tag{4.16}$$

The formula (4.16) and the geometric meaning of u and v imply that

$$\begin{cases} \gamma_1 \le \gamma \le \gamma_2 , \\ D_1 + D_2 + 2P \ge 1 . \end{cases} \tag{4.17}$$

The lowest value γ_1 and the highest value γ_2 of γ can be determined from the following two conditions

$$u_1 + 1 = v_1 \,, \tag{4.18}$$

$$u_2 = v_2 + 1 \,, \tag{4.19}$$

where

$$\begin{cases} u_i = & u|_{\gamma=\gamma_i} \,, \\ v_i = & v|_{\gamma=\gamma_i} \,, \\ i = & 1, 2 \,. \end{cases} \tag{4.20}$$

Then γ_i, u_i and v_i ($i = 1, 2$) are given by the formulas

$$\begin{aligned} e^{\gamma_1} &= \frac{1}{2P}\{D_2 - D_1 - 1 + [(D_2 - D_1 - 1)^2 + 4P^2]^{1/2}\} \,, \\ e^{\gamma_2} &= \frac{1}{2P}\{D_2 - D_1 + 1 + [(D_2 - D_1 + 1)^2 + 4P^2]^{1/2}\} \,, \end{aligned} \tag{4.21}$$

$$\begin{aligned} u_1 &= \tfrac{1}{2}\{D_1 + D_2 - 1 + [(D_2 - D_1 - 1)^2 + 4P^2]^{1/2}\} \,, \\ v_1 &= \tfrac{1}{2}\{D_1 + D_2 + 1 + [(D_2 - D_1 - 1)^2 + 4P^2]^{1/2}\} \,, \\ u_2 &= \tfrac{1}{2}\{D_1 + D_2 + 1 + [(D_2 - D_1 + 1)^2 + 4P^2]^{1/2}\} \,, \\ v_2 &= \tfrac{1}{2}\{D_1 + D_2 - 1 + [(D_2 - D_1 + 1)^2 + 4P^2]^{1/2}\} \,. \end{aligned} \tag{4.22}$$

Let us denote that au_1 is the minimal value s_{min} of s, and av_2 is the minimal value w_{min} of w.

Let $D_1 = D_2 = 1$. Then the formula (4.22) implies the following proportions of the golden section, [2]:

$$\frac{s_{\mathrm{min}}}{a} = \frac{w_{\mathrm{min}}}{a} = \tfrac{1}{2}(1 + \sqrt{5}) \,. \tag{4.23}$$

5. Twenty seven straight lines and the auxiliary surface L_3 of three electric charges

Let $C(3)$ denote the following cubic surface (in affine coordinates η_1, η_2, η_3):

$$\eta_1^3 + \eta_2^3 + \eta_3^3 + f = 0 \,. \tag{5.1}$$

The surface $C(3)$ has remarkable properties. In particular $C(3)$ contains 27 straight lines, [4–6]. The configuration of the 27 straight lines of $C(3)$ is invariant with respect to the group G which is isomorphic to the Weyl group of the exceptional Lie algebra E_6, *i.e.* G has 51840 elements and contains a simple group of order 25920, [10].

According to the formula (2.1), the λ-potential surface (equipotential surface of a fixed potential λ) of three electric charges q_1, q_2 and q_3 is given by the equation

$$\lambda R_1 R_2 R_3 = q_1 R_2 R_3 + q_2 R_1 R_3 + q_3 R_1 R_2 \,. \tag{5.2}$$

The auxiliary surface L_3 is by definition the surface obtained from (5.2) by its complexification (R_1, R_2 and R_3 are then complex variables). In [9] we show that

(i) L_3 gives a geometric insight into a natural parametrization of the real λ-surface of q_1, q_2 and q_3.

(ii) L_3 can be related to $C(3)$ via the following change of variables

$$\begin{pmatrix} R_1 - d_1 \\ R_2 - d_2 \\ R_3 - d_3 \end{pmatrix} = \frac{\sqrt{3}}{(d_1 d_2 d_3)^{\frac{1}{3}}} \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 \varepsilon_3 & 0 \\ 0 & 0 & d_3 \varepsilon_3^2 \end{pmatrix} F(3) \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}, \tag{5.3}$$

where

$$F(3) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \varepsilon_3 & \varepsilon^2 \\ 1 & \varepsilon_3^2 & \varepsilon_3 \end{pmatrix}, \tag{5.4}$$

$$d_j = \frac{q_j}{\lambda} \qquad (j = 1, 2, 3), \tag{5.5}$$

$$\varepsilon_3 = \mathrm{e}^{2\pi i/3} \,. \tag{5.6}$$

(iii) L_3, written in the new variables η_1, η_2, η_3 (defined by (5.3)) intersects $C(3)$ along a hyperbolic helix.

(iv) The 27 straight lines of $C(3)$ intersect L_3 forming an "acupuncture" of L_3.

(v) There is no invertible inhomogeneous linear transformation which transforms L_3 onto $C(3)$.

Finally, let us note that the matrix $F(3)$, given by (5.4), is the so called finite Fourier transform which has remarkable properties collected in [3].

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