DERIVATIONS IN BRAIDED GEOMETRY*

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Dedicated to the memory of Professor Jan Rzewuski

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Let R be $\Bbbk$-algebra. This paper is a preliminary step to study biderivations of braided Hopf R-algebras. We describe a free or cofree graded Hopf R-algebras as braid dependent deformations of bifree graded Hopf R-algebra. We introduce braid dependent $\Bbbk$-derivations of R-bimodules (and of R-algebras) and consider an application of derivations of zero grade for Dirac theory.

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1. Toward biderivations of Hopf algebras

Throughout this paper $\mathbb{k}$ is an associative unital commutative ring and R is an associative unital $\mathbb{k}$-algebra. Let a natural transformation $B \in \mathrm{Nat}(\otimes, \otimes^{opp})$ be a braiding in a strict monoidal category of R-bimodules.

A principal aim of this paper is to introduce B-dependent $\mathbb{k}$-derivations ($B\mathrm{der}_{\mathbb{k}}$) of R-modules and of R-algebras. This is a modest part of long-term project to study biderivations (*i.e.* derivations which are coderivations) of braided graded biassociative Hopf R-algebras. A braided Hopf algebras ($\equiv$ braided groups) has been introduced by Majid [1991-1993]. Here we construct two examples of braided graded biassociative Hopf algebras as braid dependent deformations of bifree (*i.e.* free and cofree) Hopf algebras: one deformation is free Hopf algebra (product is not deformed), another deformation is cofree Hopf algebra (coproduct is not deformed). Then we consider a Hopf algebra homomorphism $\mathcal{W}$ between these deformed Hopf algebras (as braid dependent deformation of the identity map) and we are showing that this homomorphism of deformed Hopf algebras coincide with braid dependent 'symmetrizer/alternator' introduced by Woronowicz in 1989. The present paper is the first step to develop theory of biderivations of braided biassociative graded Hopf algebras. We belive that theory of biderivations will allows elegant construction of a braided differential geometry initiated by Woronowicz [1989]. A braided differential geometry shall include the classical differential geometry as a particular case if a braiding B is proportional to the switch s, see (3.5). An example of derivations is considered for a simple model of a braiding B.

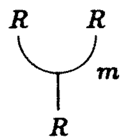
Rota, Sagan and Stein [1980] introduced cyclic derivations (of grade -1) as alternative definition of derivations of noncommutative algebra. Our derivations for $B = s$ coincide with the Hausdorff derivations and not with cyclic derivations as defined by Rota, Sagan and Stein.

Another aim of the present paper is an application of derivations of zero grade for the Dirac theory related to the Królikowski model [1993]. Let R-space (R-bimodule) S be a Clifford (left) module (a spinor space) and TS be a tensor algebra. We consider a $\mathbb{k}$-space of $\mathbb{k}$-derivations $\mathrm{der}_{\mathbb{k}}TS$ and point out that zero grade derivations considered by Królikowski does not factors to derivations of an exterior algebra $\mathrm{der}_{\mathbb{k}}S^{\wedge}$.

Appendix A contains short summary of some recent research by Professor Jan Rzewuski. Other appendices contain side remarks loosely related to the main text.

2. Notations

$\Bbbk$	is an associative unital commutative ring;
R	is an associative unital $\Bbbk$-algebra with a multiplication:



$\in \mathrm{lin}_{\Bbbk}(R \otimes_{\Bbbk} R, R)$;

R-**bimod**	a category of R-bimodules;
R-**alg**	a category of associative unital R-algebras;
$T : R$-**bimod** $\to R$-**alg**	the tensor algebra functor (not additive on morphisms);
$F : R$-**alg** $\to R$-**bimod**	the forgetful functor;
$\otimes$ and $\otimes^{opp}$	bifunctors of tensor products: R-**bimod** $\times$ R-**bimod** $\to$ R-**bimod**.
$\otimes$	means $\otimes_R$ (if not otherwise stated);
$\mathrm{lin} \equiv \mathrm{lin}_R$, $\mathrm{End} \equiv \mathrm{End}_R$	are both sided R-linear bifunctors;
M	$\in$ ob R-**bimod**, is an R-bimodule with $m_l \in \mathrm{lin}_{\Bbbk}(R \otimes_{\Bbbk} M, M)$, $m_r \in \mathrm{lin}(M \otimes_{\Bbbk} R, M)$;
$M^{\otimes} \equiv FTM$	a graded R-bimodule of M-words;
$f^k FTM$	$\equiv \bigoplus_{i \leq k} M^{\otimes i}$, a filtration of a graded R-bimodule;
$\mathrm{id}^{\otimes n}$	$\equiv (\mathrm{id}_M)^{\otimes n} = \mathrm{id}_{M^{\otimes n}}$.

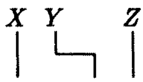
By definition functors T and F are adjoint [Kan 1958]: bifunctors $\mathrm{lin}_R(\cdot, F\cdot)$ and $\mathrm{alg}_R(T\cdot, \cdot)$ are naturally equivalent. This means that exists a natural bijection

$$\forall\, M \times A \in R\text{-}\mathbf{bimod} \times R\text{-}\mathbf{alg},$$
$$\mathrm{lin}_R(M, FA) \ni \ell \longleftrightarrow \tilde{\ell} \in \mathrm{alg}_R(TM, A), \quad \tilde{\ell}|M \equiv \ell\,. \tag{2.1}$$

3. Braided monoidal category, tangles and braids groups

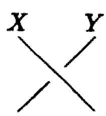
Monoidal categories were introduced by Mac Lane in 1963 under the name *categories with monoidal multiplication* [Mac Lane 1971]. A category R-**bimod** of R-bimodules with bifunctors of tensor products $\otimes$ and $\otimes^{opp}$ is an example of a monoidal category.

A monoidal category is said to be strict if a natural associativity from $\mathrm{Nat}[\otimes \circ (\mathrm{id} \times \otimes), \otimes \circ (\otimes \times \mathrm{id})]$ is choosen to be trivial,



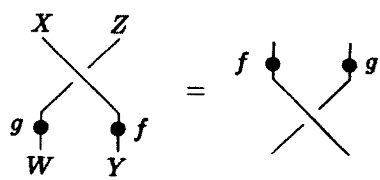
$= \quad a \simeq \mathrm{id} \in \mathrm{lin}[(X \otimes Y) \otimes Z, X \otimes (Y \otimes Z)].$

We consider strict monoidal categories only. Yetter introduced (labelled) tangles [Freyd and Yetter 1992] and in this section an interpretation of tangle drawings is recalled for the sake of completeness. Labelled diagrams, as above with no crossing, form a strict monoidal category. Labelled tangles, as below allowing crossing, form a braided strict monoidal category. A natural transformation $B \in \mathrm{Nat}(\otimes, \otimes^{opp})$ consists of a family of morphisms $B \equiv \{B_{\cdot,\cdot}\}$,



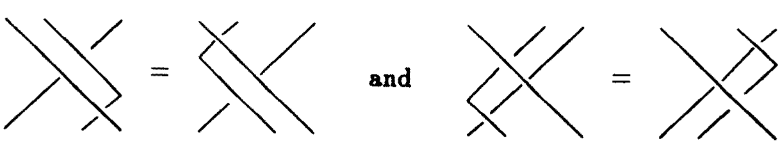
$= \quad B_{X,Y} \in \mathrm{lin}(X \otimes Y, Y \otimes X),$

such that for all morphisms $f \in \mathrm{hom}(X, Y)$ and $g \in \mathrm{hom}(Z, W)$ the naturality condition holds



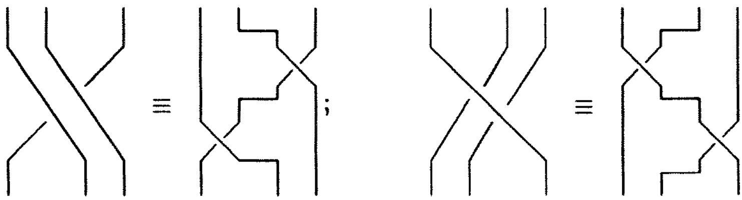
(3.1)

Then morphisms f and g are said to be 'over' and 'under' B-morphisms respectively. In particular $B_{V,W}$ is a B-morphism which means that a pair of tetragons holds (naturality of B),



(3.2)

Definition 3.1 (Prebraiding) A natural transformation $B \in \mathrm{Nat}(\otimes, \otimes^{opp})$ is said to be a prebraiding if a pair of the Mac Lane hexagons holds (*i.e.* if bifunctor $\otimes$ is a B-morphism),



(3.3)

A monoidal category is prebraided if it is equipped with a prebraiding.

The hexagons (3.3) express the Mac Lane coherence of a prebraiding B with the associativity $a \in \mathrm{Nat}[\otimes \circ (\mathrm{id} \times \otimes), \otimes \circ (\otimes \times \mathrm{id})]$. In a strict monoidal category, $a \simeq \mathrm{id}$, the Mac Lane hexagons are reduced to trigons, a strict hexagon is equivalent to a trigon. By the Mac Lane hexagons and by of naturality of a prebraiding B (tetragons (3.2)) the prebraid hexagon holds

(3.4)

A braided monoidal category is a monoidal category with a pair of braidings

$$B \in \mathrm{Nat}(\otimes, \otimes^{opp}) \quad \text{and} \quad B^{-1} \in \mathrm{Nat}(\otimes^{opp}, \otimes)$$

such that $B \circ B^{-1} = \mathrm{id}_{\otimes^{opp}}$, $\quad B^{-1} \circ B = \mathrm{id}_{\otimes}$, $\quad (B_{U,W})^{-1} = (B^{-1})_{W,U}$,

A braiding B is said to be involutive if $B^{-1} = B$. This condition is not natural. A braided monoidal category with an involutive braiding was introduced by Mac Lane in 1971 under the name symmetric monoidal category.

The switch. An involutive braiding $s \equiv \{s_{\cdot,\cdot}\} \in \mathrm{Nat}(\otimes, \otimes^{opp})$, which sends $v \otimes w \in V \otimes W$ to $w \otimes v$, is said to be the switch,

$$s_{V,W} v \otimes w \equiv w \otimes v. \tag{3.5}$$

Braid groups. A presentation for the braid group B_n on a n strings can be given by generating set $\{b_i,\ i = 1, \ldots, n-1\}$ and relations

$$\begin{aligned} b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1} \qquad \text{for } i = 1, \ldots, n-2\,, \\ b_i b_j &= b_j b_i \qquad \text{for } |i-j| \geq 2\,. \end{aligned} \tag{3.6}$$

Definition 3.2 (Braid operator) A R-bimodule endomorphism $\mathcal{B} \in \mathrm{End}_R(M^{\otimes 2})$ is said to be a braid operator if $\mathcal{B}$ extends to group map,

$$\rho \in \mathrm{group}(B_n, \mathrm{End}(M^{\otimes n})),$$
$$B_n \ni b_i \overset{\rho}{\longmapsto} \mathrm{id}_{M^{\otimes(i-1)}} \otimes \mathcal{B} \otimes \mathrm{id}_{M^{\otimes(n-i-1)}} \in \mathrm{End}(M^{\otimes n}),\ \forall n. \tag{3.7}$$

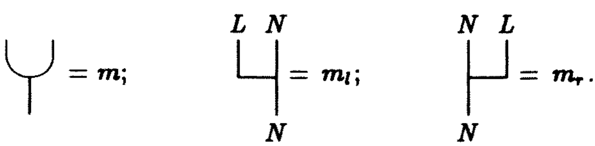
It follows that $\mathcal{B} \in \mathrm{End}(M^{\otimes 2})$ is a braid operator iff $\mathcal{B}$ is an invertible solution of the braid equation which is a particular case of (3.4) in $\mathrm{End}(M^{\otimes 3})$. The braid equation means that a braiding B is a B-morphism.

4. Braided derivation

Let L and N be R-bimodules and let $\{R, L, N\}$ be generating objects of a free monoidal category with a ring R as two-sided identity with respect to the tensor multiplication bifunctor $\otimes_R$. Let B be a braiding in this category such that $B_{R,*} = B_{*,R} = s$. This braiding is generated by $s, B_{L,L}, B_{L,N}, B_{N,L}$ and $B_{N,N}$.

Let $m \in \mathrm{lin}_R(L^{\otimes 2}, L)$ be a multiplication on L, $A \equiv \{L, m\}$ be R-algebra and $\{N, m_l, m_r\}$ be (A, A)-module with

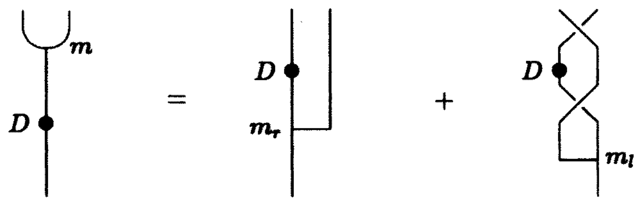
$$m_l \in \mathrm{lin}_R(L \otimes N, N), \quad m_r \in \mathrm{lin}_R(N \otimes L, N),$$



Definition 4.1 (Braided derivation). A B-dependent $\Bbbk$-derivation of R-algebra $A \equiv \{L, m\}$ to (A, A)-module $\{N, m_l, m_r\}$ is $\Bbbk$-linear map $D \in \mathrm{lin}_\Bbbk(L, N)$,

$\in \mathrm{lin}_\Bbbk(L, N)\,,$

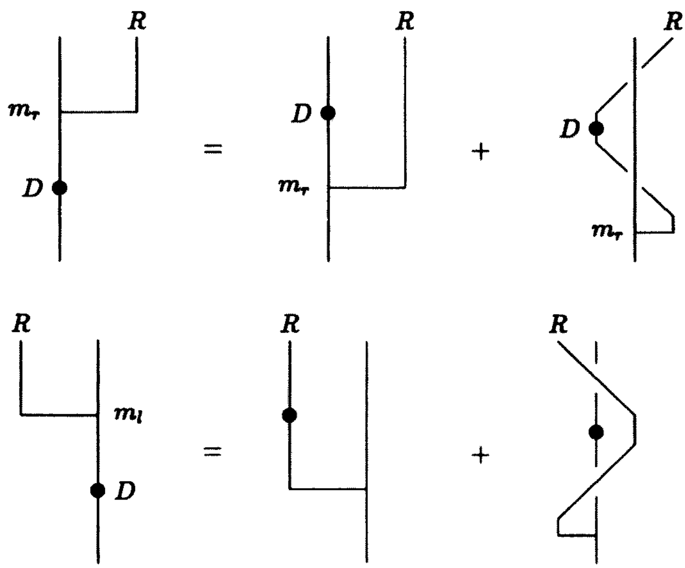
that satisfies the Leibniz rule,



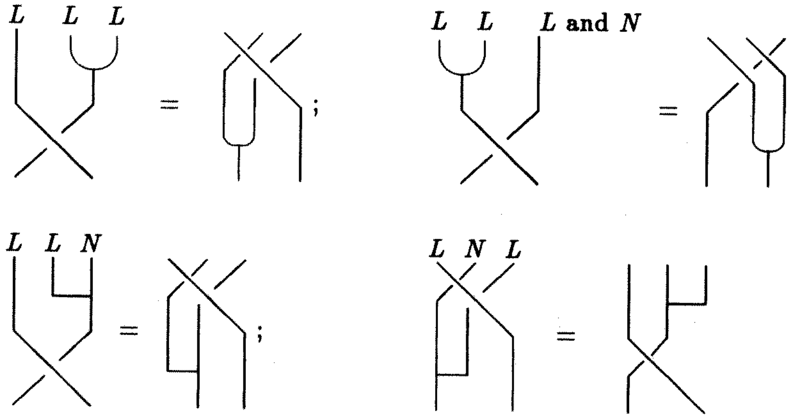
(4.1)

A right $\Bbbk$-module of B-dependent $\Bbbk$-derivations of R-algebra A to (A, A)-module N is denoted by $B\mathrm{der}_{\Bbbk}(A, N)$.

A B-dependent $\Bbbk$-derivations of R-bimodules are defined in a similar way by Leibniz rules



Lemma 4.2 *Let the following B-morphisms conditions hold,*



(4.2)

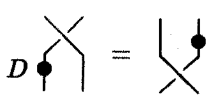
Then R-derivation (4.1) $D \in B\,\mathrm{der}_R(A,N)$ of an associative R-algebra $A \equiv \{L, m\}$ to A-bimodule $\{N, m_l, m_r\}$ is consistent with the associativity of an algebra A.

Omitted proof can be done using diagrammatic notation.

Define

$$\mathrm{der}_{\Bbbk}(A,N) \equiv s\mathrm{der}_{\Bbbk}(A,N)\,. \tag{4.3}$$

A $\Bbbk$-subspace $[\mathrm{der}_{\Bbbk}(A,N)] \bigcap [B\mathrm{der}_{\Bbbk}(A,N)]$ consists of $\Bbbk$-derivations which are under B-morphisms,

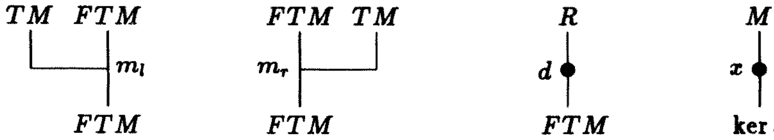


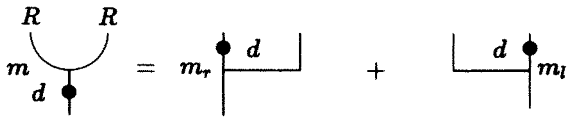
$$\Longrightarrow \quad D \in [\mathrm{der}_{\Bbbk}(A,N)] \bigcap [B\mathrm{der}_{\Bbbk}(A,N)]. \tag{4.4}$$

5. Derivations of a tensor algebra

More notations:

M	is an R-bimodule;
$\varepsilon \in \mathrm{alg}(TM, R)$	is an augmentation epimorphism ($\equiv$ counit);
$\ker \varepsilon \lhd TM$	is the augmentation ideal, $M \in \ker \varepsilon$;
$d \in \mathrm{der}_{\Bbbk}(R, FTM)$	is a $\Bbbk$-derivation;

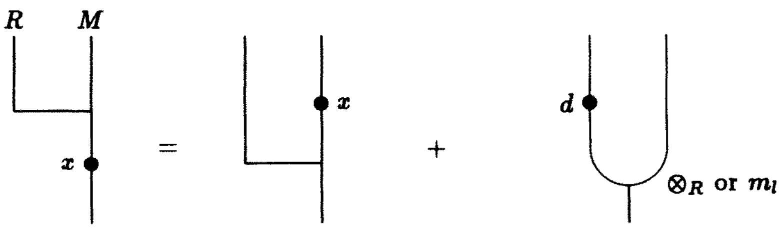


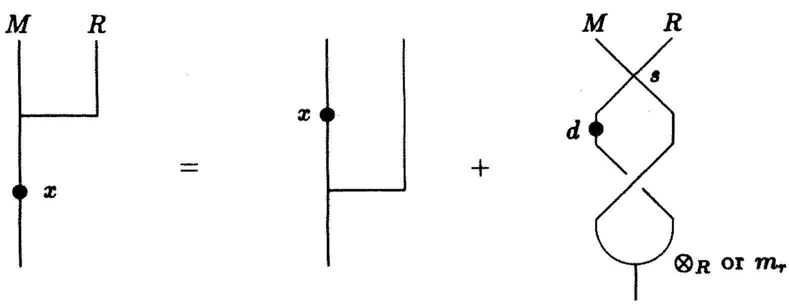


B	is a braiding in a category generated by $\{R, M\}$,
$B_{k,l}$	$\equiv B_{M^{\otimes k}, M^{\otimes l}}$;
$\mathrm{Br}_R(M^{\otimes 2})$	$\subset \mathrm{End}_R(M^{\otimes 2})$ is a subset of braid operators;
$\mathcal{B}$	$\equiv B\vert M^{\otimes 2} \equiv B_{1,1} \in \mathrm{Br}_R(M^{\otimes 2})$ is a braid operator;
$a \in \mathrm{End}_R M$,	$Ta \in \mathrm{alg}_R TM, \quad a_n \equiv (Ta)\vert M^{\otimes n}$;
π^r	$\in \mathrm{lin}_R(FTM, M^{\otimes r})$, denote R-module epimorphism:

$$\mathrm{id}_{FTM} = \sum \pi^r, \quad \pi^r|M^{\otimes s} = \begin{cases} \mathrm{id} & \text{if } r = s\,, \\ 0 & \text{otherwise}\ . \end{cases}$$

Definition 5.1 (Derivation of R-bimodule) A B-dependent $\Bbbk$-derivation of R-bimodule M to R-bimodule FTM is d-dependent $\Bbbk$-linear map $x \in \mathrm{lin}_{\Bbbk}(M, FTM)$ that satisfies the Leibniz rules ($d \equiv 0$ if $|x| = -1$),



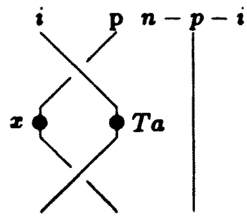


A set of k-derivations of R-bimodule M to R-bimodule FTM is denoted by $B\mathrm{der}_{\Bbbk}(M, FTM)$; $\mathrm{der}_{\Bbbk}(R, FTM) \times B\mathrm{der}_{\Bbbk}(M, FTM)$ is a k-space.

Definition 5.2 (Linear injection). Given a braiding B and $a \in \mathrm{End}_R M$ (see notations) we define a linear injections

- $B\mathrm{der}_{\Bbbk}(M^{\otimes p\geq 1}, FTM) \ni x \mapsto D_x(\mathcal{B}) \in \mathrm{lin}_{\Bbbk}(\oplus_{i\geq p} M^{\otimes i}, FTM)$,

$$D_n(\mathcal{B}) \equiv D_x(\mathcal{B})|M^{\otimes n\geq p\geq 1} = \sum_i \{ \quad \}. \qquad (5.1)$$



- $\mathrm{lin}_R(M^{\otimes p\geq 1}, f^{(p-1)FTM}) \ni x \mapsto D_x \in \mathrm{End}_R FTM$,
  by $D_x|f^{(p-1)}TM \equiv 0$ and by (5.1).
- $\mathrm{der}_{\Bbbk}(R, FTM) \times B\mathrm{der}_{\Bbbk}(M, F\ker\varepsilon) \ni (d, x) \mapsto D_{d,x} \in \mathrm{End}_{\Bbbk} FTM$,
  by $D_{d,x}|R \equiv d$ and by (5.1) for $p = 1$.

Let $D_n(\mathcal{B}) \equiv D_x(\mathcal{B})|M^{\otimes n}$. For $x \in B\mathrm{der}_{\Bbbk}(M, FTM)$:

$$D_2(\mathcal{B}) \equiv x \otimes \mathrm{id}_M + \sum_{p\geq -1} (B^{-1})_{1+p,1} \circ [(\pi^{1+p} \circ x) \otimes a] \circ B_{1,1}\,.$$

Let FTM be TM-bimodule with

$$m = m_r = \otimes \qquad \text{and} \qquad m_l = \otimes \circ (Ta \times \mathrm{id}_{TM})\,. \qquad (5.2)$$

Inserting (5.2) into B-morphisms conditions (4.2) we get that $a \in \mathrm{End}M$ is a B-morphism.

Theorem 5.3. *Let for a braiding B, $a \in \mathrm{End}_R M$ be B-morphism. Then*

$$\forall x \in B\mathrm{der}_{\Bbbk}(M, FTM) \implies D_x(\mathcal{B}) \in B\mathrm{der}_{\Bbbk}(TM, FTM)\,.$$

Proof. We must show that Leibniz rule (4.1) holds for TM-module FTM (5.2). Let $x \in \mathrm{lin}_{\Bbbk}(M^{\otimes p\geq 1}, FTM)$. Then

$$\sum_{i=0}^{n-p} = \sum_{i=0}^{k-p} + \sum_{i=k}^{n-p} + \sum_{i=k+1-p}^{k-1}\,.$$

Let $n = k + l$. Consider four strings in diagrammatic notation

$$n = \{k,\ l\} = \{k,\ i-k,\ p,\ n-p-i\}\,.$$

We need to use that a is a B-morphism and the coherence conditions for a braiding (3.4). This gives

$$S_{k,l} \equiv D_{k+l} - D_k \otimes \mathrm{id}_l - B^{-1} \circ (D_l \otimes a_k) \circ B_{k,l} = \sum_{i=k+1-p}^{k-1} [B^{-1} \circ (x \otimes a_i) \circ B_{i,p}] \otimes \mathrm{id}^{\otimes(n-p-i)}. \quad (5.3)$$

The last sum in (5.3) violate the Leibniz rule (4.1). This sum is nonzero iff $x \in \mathrm{lin}_{\Bbbk}(M^{\otimes p \geq 2}, FTM)$. □

6. Graded derivations

We will show that braided derivations include as a particular case graded derivations. Let $\lambda \in \Bbbk \setminus \{0\}$ and $x \in B\mathrm{der}_{\Bbbk}(M, FTM)$. Then

$$D_{k+l}(\lambda \cdot \mathcal{B}) = D_k(\lambda \cdot \mathcal{B}) \otimes \mathrm{id}_l + \sum_{r \geq -1} \lambda^{-rk} \cdot (B^{-1})_{l+r,k} \circ [D_l(\lambda \cdot \mathcal{B}) \otimes a_k] \circ B_{k,l}.$$

Let $f, g \in \mathrm{End}_R M$, $Tf, Tg \in \mathrm{alg}\, TM$. Consider a model of a braid operator

$$B_{1,1} \equiv \mathcal{B} = (f \otimes g) \circ s = s \circ (g \otimes f), \quad f \circ g = g \circ f,$$
$$B_{k,l} = [(Tf^k | M^{\otimes l}) \otimes (Tg^l | M^{\otimes k})] \circ s_{k,l},$$
$$B^{-1} \circ [D_l \otimes \mathrm{id}_k] \circ B_{k,l} = \mathrm{id}_k \otimes (Tf^{-k} \circ D \circ Tf^k),$$
$$D_{(Tf) \circ x, f \circ a}(\mathcal{B}) \circ Tf = Tf \circ D_{x \circ f, a \circ f}(\mathcal{B});$$
$$D_x(\mathcal{B}) \text{ is a } B\text{-morphism iff } Tf \circ x = x \circ f.$$

Let $x \circ f = \lambda \cdot Tf \circ x$ and $a = \mathrm{id}_M$. Then

$$D_x \circ Tf = \lambda \cdot Tf \circ D_x\,,$$
$$B^{-1} \circ [D_l \otimes \mathrm{id}_k] \circ B_{k,l} = \lambda^k \cdot \mathrm{id}_k \otimes D_l\,.$$

7. Special derivations

Let $x \in \mathrm{lin}_R(M, FTM)$ and $a \in \mathrm{End}_R M$. A derivation (5.1) for $p = 1$ is given by the data $\{d, x, a, \mathcal{B}\}$. In this section we put for simplicity $d = 0$ and $\mathcal{B} = s$. We wish to study derivations with the property

$$D_{x,a} \in \mathrm{der}_R(TM, FTM) \implies (D_{x,a})^2 \in \mathrm{der}_R(TM, FTM)$$

for TM-bimodules FTM with $m_l = \otimes \circ (Ta \times \mathrm{id}_{TM})$ and $m_l = \otimes \circ (Ta^2 \times \mathrm{id}_{TM})$ respectively. A sufficient condition is

$$\boxed{(Ta) \circ x = -x \circ a.} \tag{7.1}$$

Let $\mu, \nu, \lambda = \pm 1$ and define

$$\begin{aligned} &[\cdot, \cdot]_\lambda : \mathrm{lin}(M, FTM) \otimes \mathrm{lin}(M, FTM) \longrightarrow \mathrm{lin}(M, FTM), \\ &[x, y]_\lambda \equiv D_{x,a} \circ y - \lambda \cdot D_{y,a} \circ x ; \\ &L_\lambda(a) \equiv \{x \in \mathrm{lin}_R(M, FTM), (Ta) \circ x = \lambda \cdot x \circ a\} . \end{aligned} \tag{7.2}$$

For TM-bimodule FTM (5.3) if $x \in L_\lambda(a)$ then a derivation $D_{x,a} \in \mathrm{der}(TM, FTM)$ is said to be special. We have

$$\begin{gathered} [L_\mu, L_\nu] \subset L_{\mu\nu} , \\ \text{for } x \in L_\lambda , \quad (Ta) \circ D_{x,a} = \lambda \cdot D_{x,a} \circ (Ta) , \end{gathered}$$

and the following (anti)-commutative diagram holds,

$$\begin{array}{ccccc} M & \xrightarrow{\;x\;} & TM & \xrightarrow{\;D_{x,a}\;} & TM \\ \Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle Ta} & & \Big\downarrow{\scriptstyle Ta} \\ M & \xrightarrow{\;\pm x\;} & TM & \xrightarrow{\;\pm D_{x,a}\;} & TM \end{array}$$

For involutive braid $S = \lambda \cdot s$ we have an isomorphism of algebras: the λ-commutator of special derivations is a special derivation, for $x, y \in L_\lambda$,

$$[D_{x,a}, D_{y,a}] \equiv D_{x,a} \circ D_{y,a} - \lambda \cdot D_{y,a} \circ D_{x,a} \tag{7.3}$$

$$= D_{[x,y],a^2} . \tag{7.4}$$

Example 7.1. Let S be R-bimodule, $a \in \mathrm{End}_R S$, $L_{-1}(a) \subset \mathrm{End}_R S$ and

$$\gamma \in \mathrm{lin}_R(M, L_{-1}(a)) , \text{ i.e. for } v \in M , \quad \{\gamma_v, a\} = 0 .$$

If $x, y \in \mathrm{End}_R M$ then $\{x, y\} \equiv [x, y]_{\lambda=-1} = x \circ y + y \circ x$. For zero grade x and $a \neq \mathrm{id}_M$, $D_{x,a}$ does not factors to exterior and symmetric factor algebras.

For $\lambda = -1$ and $v, w \in M$ we put

$$x = \gamma_v\,,\ \ y = \gamma_w\,,\ \ \Gamma_v \equiv D_{\gamma_v, a}\,,$$
$$\Gamma | S^{\otimes 2} = \gamma \otimes \mathrm{id}_S + a \otimes \gamma :\ M \longrightarrow \mathrm{End}(S^{\otimes 2})\,.$$

Then we recover the case related to works by Kähler [1960] and by Królikowski [1993] ($a = \gamma_5$)

$$\{\Gamma_v, \Gamma_w\} = D_{\{\gamma_v, \gamma_w\}, a^2}\,.$$

Example 7.2. Let $x \in \mathrm{lin}(M, M^{\otimes 2})$. One can solve the special condition (7.1)

$$(Ta) \circ x = -x \circ a$$

for minimal polynomials $a^n = (-1)^n \cdot \mathrm{id}_M$. Let $a^2 = \mathrm{id}_M$, $i, j, k \in \mathrm{spec}\, a \equiv \{-1, +1\}$ and

$$x^{ij}_k \equiv [(\mathrm{id}_M + ia) \otimes (\mathrm{id}_M + ja)] \circ x \circ (\mathrm{id}_M + ka)\,.$$

Then the special condition (7.1) is reduced to the system

$$(1 + ijk) \cdot x^{ij}_k = 0\,, \qquad \text{no sums.}$$

This is the condition for $\mathbb{Z}/2\mathbb{Z}$-graded cogebra x,

$$\mathrm{im}\{x|(\mathrm{id}_M + a)\} \subset \mathrm{im}\{(\mathrm{id}_M - a) \otimes (\mathrm{id}_M + a) \oplus (\mathrm{id}_M + a) \otimes (\mathrm{id}_M - a)\}\,,$$
$$\mathrm{im}\{x|(\mathrm{id}_M - a)\} \subset \mathrm{im}\{(\mathrm{id}_M + a) \otimes (\mathrm{id}_M + a) \oplus (\mathrm{id}_M - a) \otimes (\mathrm{id}_M - a)\}\,.$$

Note that a-dependent coassociator is $(D_x)^2 | M \equiv (x \otimes \mathrm{id}_M + a \otimes x) \circ x$. This allows to consider a-coassociative (and a-associative) (graded associative) $\mathbb{Z}/2\mathbb{Z}$-graded cogebras and algebras with Hochschild-like cohomology. Such algebras are non associative in the usual sense if $a \neq -1$.

8. Braided free or cofree Hopf algebras

Hopf algebras in a braided monoidal category ($\equiv$ a braided Hopf algebras) has been introduced by Majid [1991-1993]. In this section we construct two important examples: (pre)braided free Hopf algebra and (pre)braided cofree Hopf algebra as the deformation of bifree Hopf algebra.

Let $TM \equiv \{M^{\otimes}, \otimes\}$ be a tensor algebra free on R-bimodule M. Let a prebraiding K be generated by a prebraid operator $K_{1,1} \equiv \mathcal{K} \in \mathrm{End}_R(M^{\otimes 2})$ and be such that

$$K_{0,0} \equiv K | R \otimes R \equiv s\,,$$
$$K_{1,0} \equiv K | M \otimes R \equiv s\,,$$
$$K_{0,1} \equiv K | R \otimes M \equiv s\,.$$

Denote briefly by KM an associative unital R-algebra with R-bimodule $FKM \equiv M^{\otimes} \otimes M^{\otimes}$, with a $\mathcal{K}$-dependent multiplication

$$[(\otimes) \otimes (\otimes)] \circ (\mathrm{id} \otimes K \otimes \mathrm{id})$$

and with unit $1 \otimes 1$. For each algebra map $C(\mathcal{K}) \in \mathrm{alg}(TM, KM)$, $C1 \equiv 1 \otimes 1$, a pair $\{TM, C(\mathcal{K})\}$ is a free bialgebra. We noted in section 2 (see (2.1)), that every R-linear map $C_1 \in \mathrm{lin}_R(M, FKM)$ determine unique algebra map $C(\mathcal{K}) \in \mathrm{alg}(TM, KM)$ such that $C(\mathcal{K})|M \equiv C_1$. We put

$$C_1 \equiv 1 \otimes \mathrm{id}_M + \mathrm{id}_M \otimes 1.$$

Then due to braid equation (3.4) this comultiplication $C(\mathcal{K})$ is coassociative. An algebra map $\varepsilon \in \mathrm{alg}_R(TM, R)$ such that $M \in \ker \varepsilon$, is a counit for a comultiplication $C(\mathcal{K})$. Therefore $\{TM, C(\mathcal{K}), \varepsilon\}$ is a free biunital (*i.e.* unital and counital) and biassociative (*i.e.* an associative and coassociative) bialgebra. Due to braid equation (3.4) a comultiplication $C(\mathcal{K})$ is a K-morphism. Moreover the unit and the counit are K-morphisms and therefore a free bialgebra $\{TM, C(\mathcal{K}), \varepsilon\}$ is K-braided. This free R-bialgebra possess K-braided antipod $S(\mathcal{K})$ and we have a free K-braided Hopf algebra

$$fHM(\mathcal{K}) \equiv \{TM, C(\mathcal{K}), \varepsilon, S(\mathcal{K})\}.$$

In this example we get

$$S(\mathcal{K})|R = \mathrm{id}, \quad S(\mathcal{K})|M = -\mathrm{id}, \quad S(\mathcal{K})|M^{\otimes 2} = \mathcal{K}.$$

Theorem 8.1 (Majid 1993). *Let a biunital and biassociative Hopf algebra be K-braided, i.e. all involved maps be K-morphisms. Then K-braided antipod $S(\mathcal{K})$ must be an algebra map,*

$$S(\mathcal{K}) \in \mathrm{alg}(\otimes, \otimes \circ K),$$
$$S(\mathcal{K}) \circ \otimes = (\otimes \circ K) \circ [S(\mathcal{K}) \otimes S(\mathcal{K})].$$

Note that $C(\mathcal{K})|M^{\otimes n} = \sum_{i=0}^{n} C^{i,n-i}(\mathcal{K})$ and the operators $C^{p,q}(\mathcal{K}) \in \mathrm{lin}_R[M^{\otimes(p+q)}, M^{\otimes p} \otimes M^{\otimes q}]$ for $p = 1$ or $q = 1$ are the same as *braided integers* introduced by Majid in a paper on *free braided differential calculus* [1993],

$$\begin{bmatrix} 1+p \\ p \end{bmatrix}; \mathcal{K} = C^{p,1}(\mathcal{K}), \qquad [1+p\,; \mathcal{K}] = C^{1,p}(\mathcal{K}).$$

Cofree tensor (coassociative) R-cogebra CM cogenerated on R-bimodule M is algebraic dual of a tensor algebra $TM^* \equiv \{M^{*\otimes}, \otimes\}$p of a dual R-bimodule M^*. Therefore $CM = \{M^{\otimes}, \triangle \equiv \otimes^*\}$. A computation shows that $\triangle \equiv \otimes^* = C(0)$ and that $\triangle$ is a shuffle (free) comultiplication. The construction of cofree cogebras was considered by Fox [1993]. Therefore coassociative counital cogebra $\{M^{\otimes}, C(\mathcal{K}), \varepsilon\}$ is $\mathcal{K}$-deformation of cofree (shuffle) cogebra $\{M^{\otimes}, \triangle, \varepsilon\}$ and a free Hopf algebra

$$fHM(\mathcal{K}) \equiv \{TM, C(\mathcal{K}), \varepsilon, S(\mathcal{K})\},$$

is $\mathcal{K}$-deformation of bifree (*i.e.* free and cofree) Hopf algebra $fHM(0) \equiv \{TM, \triangle, \varepsilon, S(0)\}$.

Note that exists an associative $\mathcal{K}$-deformation of a tensor product $\otimes \to Q(\mathcal{K})$, $Q(0) \equiv \otimes$, such that

$$Q(\mathcal{K}) \equiv [C(\mathcal{K}^T)]^T,$$
$$E(\mathcal{K}) \equiv \otimes \circ C(\mathcal{K}) = Q(\mathcal{K}) \circ \triangle \in \mathrm{End}_R(M^{\otimes}),$$
$$E(0)|M^{\otimes n} = (n+1) \cdot \mathrm{id}.$$

Denote briefly by MK an coassociative counital R-cogebra with R-bimodule $FMK \equiv M^{\otimes} \otimes M^{\otimes}$, with a $\mathcal{K}$-dependent comultiplication

$$(\mathrm{id} \otimes K \otimes \mathrm{id}) \circ [\triangle \otimes \triangle]$$

and with counit $\varepsilon \otimes \varepsilon$. Then one can show that $Q(\mathcal{K}) \in \mathrm{cog}(MK, CM)$. Therefore we get K-braided cofree Hopf algebra

$$cHM(\mathcal{K}) \equiv \{CM, Q(\mathcal{K}), u, S(\mathcal{K})\}$$

as nonequivalent $\mathcal{K}$-deformation of bifree Hopf algebra.

Theorem 8.2. *Exists unique homomorphism $\mathcal{W}(\mathcal{K})$ of $\mathcal{K}$-deformed free Hopf algebra into $\mathcal{K}$-deformed cofree Hopf algebra,*

$$\mathcal{W}(\mathcal{K}) \in \mathrm{hopf}\{fHM(\mathcal{K}), cHM(\mathcal{K})\},$$

such that $\mathcal{W}(\mathcal{K})|R \oplus M \equiv \mathrm{id}$. Omitted proof use the prebraid equation (3.4). The operator $\mathcal{W}(\mathcal{K})$ commutes with antipod $S(\mathcal{K})$ and moreover we have $\mathcal{W}(0) = \mathrm{id}$. Therefore $\mathcal{W}(\mathcal{K})$ is $\mathcal{K}$-deformation of identity. One can show also that $\mathcal{W}(\mathcal{K})$ coincide with symmetrizer/alternator introduced by Woronowicz [1989, pp. 153-155].

Corollary 8.3. The subspace $\ker \mathcal{W}(\mathcal{K})$ is a two-sided biideal in a free tensor Hopf R-algebra $fHM(\mathcal{K})$ and $\mathrm{im}\mathcal{W}(\mathcal{K})$ is a sub-Hopf algebra of cofree Hopf R-algebra $cHM(\mathcal{K})$.

Definition 8.4. A factor Hopf algebra $M^{\wedge}_{\mathcal{K}} \equiv fHM(\mathcal{K})/\ker\mathcal{W}(\mathcal{K})$ is said to be the exterior Hopf algebra of R-bimodule M.

9. Factor braiding

Let A be an algebra and $I \lhd A$ be a two-sided ideal. A braid $B \in \mathrm{End}(A^{\otimes 2})$ factors to the braid on the factor algebra A/I iff

$$B(I \otimes A + A \otimes I) \subset I \otimes A + A \otimes I . \tag{9.1}$$

Corollary 9.1. Let $\pi \in \mathrm{alg}(A, A/I)$ be algebra epimorphism and let a braid B on A factors to a braid $B^\wedge$ on A/I, $B^\wedge \circ (\pi \otimes \pi) = (\pi \otimes \pi) \circ B$. Then a factor multiplication $\wedge \equiv m/I$, $\wedge \circ (\pi \otimes \pi) = \pi \circ m$, is a $B^\wedge$-morphism,

$$(\wedge \otimes \mathrm{id}_{A/I}) \circ B^\wedge_{A/I,(A/I)\otimes(A/I)} = B^\wedge \circ (\mathrm{id}_{A/I} \otimes \wedge) .$$

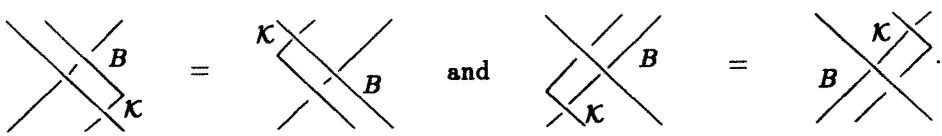
If $I \equiv \ker \mathcal{W}$ then a sufficient condition that a braid operator B factors (9.1) is the existence of a map $\widetilde{B}$ such that

$$(\mathcal{W} \otimes \mathcal{W}) \circ B = \widetilde{B} \circ (\mathcal{W} \otimes \mathcal{W}) . \tag{9.2}$$

Lemma 9.2. *Let $\mathcal{K}$ and $\mathcal{B} \in \mathrm{End}(M^{\otimes 2})$ be braid operators. The following assertions are equivalent*

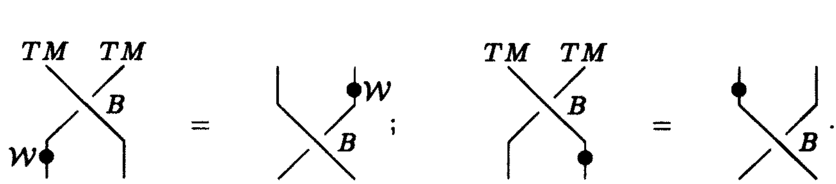
(i) A braid $\mathcal{K}$ is a B-morphism for $B_{1,1} \equiv \mathcal{B}$.
(ii) The Hopf algebra map $\mathcal{W}(\mathcal{K})$ is a B-morphism.

Proof. A braid operator $\mathcal{K}$ is a B-morphism if



(9.3)

For $\mathcal{K} = \mathcal{B}$ (9.3) is just (3.2). The Hopf algebra map $\mathcal{W}(\mathcal{K})$ is a B-morphism if (see (3.1))



We must show an implication

$$\mathcal{K} \text{ is a } B\text{-morphism} \quad \Longrightarrow \quad \mathcal{W}(\mathcal{K}) \text{ is a } B\text{-morphism.}$$

Proof of this implication goes by induction on grades using braid hexagon (3.4). Most easily proof goes in diagrammatic notation. □

Every braid operator is a s-morphism, however the switch s (3.5) needs not to be a $\mathcal{B}$-morphism for general braid operator $\mathcal{B} \in \mathrm{Br}_R(M^{\otimes 2})$.

Lemma 9.3 *Let $\mathcal{K} \in \mathrm{Br}_R(M^{\otimes 2}) \subset \mathrm{End}(M^{\otimes 2})$ be a braid operator and let $\ker \mathcal{W}(\mathcal{K}) \lhd TM$ be biideal as in Corollary 8.3. A sufficient condition that a braiding B on $M^{\otimes}$, $B_{1,1} \equiv \mathcal{B} \in \mathrm{End}(M^{\otimes 2})$, factors to a braiding $B^{\wedge}(\mathcal{K}) \in \mathrm{End}(M^{\wedge}_{\mathcal{K}} \otimes M^{\wedge}_{\mathcal{K}})$ is that $\mathcal{K}$ is a B-morphism i.e. (9.3) holds.*

Proof. Proof is by induction. □

Therefore if a braid operator $\mathcal{K}$ is a B-morphism then in (9.2) $\widetilde{B} = B$ and exists a factor braiding $B^{\wedge}(\mathcal{K}) \in \mathrm{End}(M^{\wedge}_{\mathcal{K}} \otimes M^{\wedge}_{\mathcal{K}})$.

Let $B_{1,1} = (f \otimes g) \circ s$. Then a braid $\mathcal{K} \in \mathrm{Br}_R(M^{\otimes 2})$ is a B-morphismm iff

$$Tf \circ K = K \circ Tf \qquad \text{and} \qquad Tg \circ K = K \circ Tg\,.$$

Appendix A

The Rzewuski model of internal manifold

Professor Jan Rzewuski (1916-1994) invented a spinorial model of internal manifold of elementary particle based on direct product of two (or more) Lie groups. One of these group is a group E of external (space-time) symmetries and the second one is a group I of internal symmetries. Examples of external symmetries include conformal group $\mathrm{SU}_{2,2}$ and the covering group of the Poincaré group $P \subset \mathrm{SU}_{2,2}$. Examples of internal groups are SU_m, $\mathrm{SU}_3 \times \mathrm{SU}_2 \times \mathrm{U}_1$.

In this appendix we shall describe the Rzewuski model [Kocik and Rzewuski 1986-1995; Rzewuski 1989-1994 and references therein].

Let $\mathbb{C}$-space S be an irreducible left E-module (a spinor space) and $\mathbb{C}$-space V be an irreducible left I-module. Then dual $\mathbb{C}$-space $V^* \equiv \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is a right I-module. Therefore $S \otimes_{\mathbb{C}} V^* \simeq \mathrm{Hom}_{\mathbb{C}}(V, S)$ is E-I-bimodule (or equivalently a left $E \times I^{opp}$-module) and $V \otimes_{\mathbb{C}} S^* \simeq \mathrm{Hom}_{\mathbb{C}}(S, V)$ is I-E-bimodule. These bimodules are referred in Rzewuski's papers as 'matrix manifolds of multi-spinors' [*e.g.* Rzewuski 1989, 1993].

Rzewuski was considering also $\mathbb{Z}_2$-graded generalization of matrix manifolds [Rzewuski 1991], where $\mathbb{C}$-spaces S or/and V are $\mathbb{Z}_2$-graded and Lie groups E or I or both E and I are a supergroups correspondingly [see *e.g.*Berezin 1983].

Rzewuski (also jointly with Kocik [1986 and 1995]) gave classification of $(E \times I)$-homogeneous submanifolds ($E \times I$-orbits) in $(E \times I)$-bimodules. The aim of this classification was determination of $E \times I$-orbits foliated over E-orbit. An E-orbit M is said to be an external space, *e.g.* a complex Minkowski space. Let $W \subset S \otimes V^*$ be $E \times I$-homogeneous submanifold. A foliation given by projection $\pi : W \to M$ is said to be *consistent* if π intertwine the transitive action of $E \times I$ with the transitive action of an external group E,

$$\begin{array}{ccc} W & \xrightarrow{E\times I} & W \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{E} & M \end{array}$$

The following definition is a heart of the Rzewuski model.

Definition A.1 (Internal manifold of particle). Let $p \in M$. A typical fiber $\pi^{-1}p \subset W$ of consistent fibration $\pi : W \to M$ is said to be an *internal manifold of particle.*

This definition rises the problem: to determine all possible internal manifolds. Rzewuski proved the following amazing theorem (see also [Kocik and Rzewuski 1995]).

Theorem A.2 (Rzewuski 1993). *Let $E = SU_{2,2}$ or $E = P$. Let V be (left) SU_m-module, $m =$ dim V and let S be left E-module,* $\dim_{\mathbb{C}} S = 4$. *Then exists unique $(E \times SU_m)$-orbit W in a space of m-spinors* Hom(V, S) *with consistent fibration over compact Minkowski $\mathbb{C}$-space M.*

For the Poincaré group an internal manifold determined by Rzewuski has a form,

$$\frac{SO_{3,1} \times SU_m}{SO_2 \times SU_{m-2}}\,. \tag{A.1}$$

Determination of submanifolds of Hom(S, V) of fixed rank leads to generalization of the Penrose transform [Rzewuski 1985-1993], [Kocik and Rzewuski 1986-1995]. The case considered by Penrose is $\dim_{\mathbb{C}} S = 4$ and $\dim_{\mathbb{C}} V = 2$.

Rzewuski determined invariant metric tensors, measures and differential operators on homogeneous spaces (*e.g.* Laplace–Beltrami operator) together with their spectral analysis. The aim was study of invariant differential operators on internal manifold of particle in the Rzewuski model (A.1) and dynamics for invariant Lagrangians.

Appendix B

Identities for involutive braid

Let $\mathrm{Br}_R(M^{\otimes 2}) \subset \mathrm{End}_R(M^{\otimes 2})$ denote a set of pre-braid operators. Let M be R-bimodule, $B \in \mathrm{Br}_R(M^{\otimes 2})$ and let 'braided commutator' $C(B) \in$

$\mathrm{End}_R FTM$ be defined by recurrent relation

$$
\begin{aligned}
C(B)|(R\oplus M) &= \mathrm{id}\,,\\
C_r(B) &\equiv C(B)|M^{\otimes r\geq 2}\\
&= (\mathrm{id}_r - B_{1,r-1})\circ[\mathrm{id}_M\otimes C_{r-1}(B)] \in \mathrm{End}_R(M^{\otimes r})\,,\\
J_r(B) &\equiv C_r(B)\circ\left\{\sum_{0\leq i\leq r-1}(B_{1,r-1})^i\right\}\,,\\
C_3(B) &\equiv (\mathrm{id}_3 - B_{1,2})\circ(\mathrm{id}_3 - \mathrm{id}_M\otimes B_{1,1})\,,\\
J_3(B) &\equiv C_3(B)\circ[\mathrm{id}_3 + B_{1,2} + (B_{1,2})^2]\,.
\end{aligned}
\tag{B.1}
$$

Lemma B.1 (Jacobi identity). *Let $S = S^{-1} \in \mathrm{Br}_R(M^{\otimes 2})$ be an involutive braid. Then $J_2(S) = 0$ and $J_3(S) = 0$.*

Proof. An identity $J_3(S) = 0$ will be shown in two ways.

$$
\begin{aligned}
J_3(S) &= (\mathrm{id}_3 - S\otimes\mathrm{id}_M)\circ[\mathrm{id}_3 - (S_{1,2})^3]\\
&\quad +(S\otimes\mathrm{id}_M - \mathrm{id}_M\otimes S)\circ[\mathrm{id}_3 + S_{1,2} + (S_{1,2})^2] = 0 \quad \text{by} \quad S = S^{-1}\,.
\end{aligned}
$$

Let $\mathcal{W}(\mathcal{B})$ be the homomorphism of Hopf algebras as defined in Section 8. For involutive braid $S = S^{-1}$ we have

$$
\begin{aligned}
\mathcal{W}_3(\mathcal{S}) &= [\mathrm{id}_3 + S_{1,2}\circ(\mathrm{id}_M\otimes S)]\circ[\mathrm{id}_3 + S_{1,2} + (S_{1,2})^2]\\
&= [\mathrm{id}_M\otimes S + S_{1,2}]\circ[\mathrm{id}_3 + S_{1,2} + (S_{1,2})^2]\,.
\end{aligned}
\tag{B.2}
$$

Therefore $J_3(S) = \mathcal{W}_3(\mathcal{S}) - \mathcal{W}_3(\mathcal{S}) = 0$. □

Appendix C

Braid-dependent determinant

Let $x \in M^*$ and $c_x \equiv x\otimes\mathrm{id}_{TM}$ be contraction.

$$c\in \mathrm{lin}(M^*, F\mathrm{End}\,FTM)\,,\quad \tilde{c}\in\mathrm{alg}(TM^*, \mathrm{End}\,FTM)\,,\quad \tilde{c}|M^*\equiv c\,.$$

The homomorphism of Hopf algebras (see Section 8) $\mathcal{W}(\mathcal{K})$ determine the pairing of exterior Hopf algebras,

$$
\begin{array}{ccc}
TM^*\otimes TM & \ni \xi\otimes X \longmapsto \tilde{c}_\xi\mathcal{W}(\mathcal{K})X \in & R\\
\Big\downarrow{\scriptstyle \pi_\mathcal{K}\otimes\pi_\mathcal{K}} & & \Big\|\\
M_\mathcal{K}^{*\wedge}\otimes M_\mathcal{K}^\wedge & \xrightarrow{\det_\mathcal{K}} & R
\end{array}
$$

$$(\pi_{\mathcal{K}}\xi)(\pi_{\mathcal{K}}X) \equiv \det{}_{\mathcal{K}}(\pi_{\mathcal{K}}\xi \otimes \pi_{\mathcal{K}}X) \equiv \begin{cases} \tilde{c}_\xi \mathcal{W}(\mathcal{K})X & \text{if } |\xi| = |X|, \\ 0 & \text{otherwise,} \end{cases} \tag{C.1}$$

or

$$\pi_{\mathcal{K}}\xi \circ \pi_{\mathcal{K}} \equiv \tilde{c}_\xi \circ \mathcal{W}(\mathcal{K}) = \tilde{c}_{\mathcal{W}^T(\mathcal{K})\xi} \,.$$

Let s be the switch (3.5). Then

$$\det{}_{-s} = \text{determinant}\,,$$
$$\det{}_{+s} = \text{permanent}\,.$$

A honest bosonic (symmetric) algebra is paired by permanent. A honest fermionic (exterior) algebra is paired by determinant.

For $\xi \in TM^*$, $X \in TM$, $|\xi| = |X|$,

$$R \in \quad \det(\pi\xi \otimes \pi X) \equiv \tilde{c}_\xi \mathcal{W}(-s)X\,,$$
$$R \in \quad \mathrm{per}(\pi\xi \otimes \pi X) \equiv \tilde{c}_\xi \mathcal{W}(+s)X\,,$$

$$(\xi^1 \wedge \ldots \wedge \xi^n)(x_1 \wedge \ldots \wedge x_m) = \delta^n_m \cdot \left\{ \begin{matrix} \det \\ \mathrm{per} \end{matrix} \right\} \{\xi^i x_j\}\,.$$

A pairing (C.1) with $\mathcal{W}(\mathcal{K})$ generalize the determinant/permanent for an arbitrary braid operator $\mathcal{K}$.

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