THE DIRAC OPERATOR ON HYPERSURFACES*,**

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Dedicated to the memory of Professor Jan Rzewuski

(*Received March 15, 1995*)

Odd-dimensional Riemannian spaces that are non-orientable, but have a pin structure, require the consideration of the twisted adjoint representation of the corresponding pin group. It is shown here how the Dirac operator should be modified, also on even-dimensional spaces, to make it equivariant with respect to the action of that group when the twisted adjoint representation is used in the definition of the pin structure. An explicit description of a pin structure on a hypersurface, defined by its immersion in a Euclidean space, is used to derive a *Schrödinger transform* of the Dirac operator in that case. This is then applied to obtain — in a simple manner — the spectrum and eigenfunctions of the Dirac operator on spheres and real projective spaces.

PACS numbers: 02.40.+m, 03.65.Ge

1. Introduction

Most of the research on the Dirac operator on Riemannian spaces is restricted to the case of orientable manifolds. It is of some interest to treat also

* This paper is based, in part, on a lecture given at the *Workshop on Spinors, Twistors, and Conformal Invariants* at the Erwin Schrödinger Institute in Vienna (19–25 September 1994). During the *Workshop*, there was a short ceremony honoring the memory of Professor Jan Rzewuski.

** Research supported in part by the Polish Committee on Scientific Research (KBN) under Grant No. 2 P302 112 7.

the non-orientable case that requires the introduction of pin structures. In physics, even in the orientable case, one considers spinor fields transforming under space and time reflections, which are covered by elements of a suitable pin group. The generalization to the non-orientable case involves interesting subtleties. First of all, for a real vector space with a quadratic form of signature (k, l), the Clifford construction yields two groups, $\mathrm{Pin}_{k,l}$ and $\mathrm{Pin}_{l,k}$, which need not be isomorphic; see [1] and Sec. 3.1 for a precise statement. This fact is of interest also to physics [2]. There are non-orientable spaces with a metric tensor field of signature (k, l) admitting either a $\mathrm{Pin}_{k,l}$-structure or a $\mathrm{Pin}_{l,k}$-structure. If a space admits a $\mathrm{Spin}_{k,l}$-structure, then it is orientable and admits both these structures. Real projective spaces and quadrics provide the simplest examples of such situations [3–5]. If the dimension $k + l$ is even, then one can use either the adjoint or the twisted adjoint representation of $\mathrm{Pin}_{k,l}$. If one uses the twisted adjoint representation, as one has to do when $k + l$ is odd, then the classical Dirac operator (see, *e.g.*, [6–8]) needs to be modified to make it equivariant with respect to the action of the pin group [5, 9]. In this paper, the relation between the adjoint and the twisted adjoint representation of the pin group is considered in some detail (Section 3). In Section 4, the definition of spin and pin structures is illustrated on the example of spheres and real projective spaces. The form of the modified Dirac operator is recalled in Section 5. A canonical pin structure on a hypersurface immersed in a Euclidean space is described in Section 6 and shown to have a trivial associated bundle of 'Dirac' or 'Pauli' spinors . A convenient formula for the 'Schrödinger transform' of the modified Dirac operator on such hypersurfaces is derived in Section 7. As an illustration, the spectrum and the eigenfunctions of the Dirac operator on real projective spaces are found on the basis of the corresponding results for spheres (Section 8).

2. Notation and preliminaries

This paper is a continuation of [5] and [9]; it uses the notation and terminology introduced there. To make the paper self-contained, some of the notation is summarized below.

2.1. Clifford algebras and pin groups

Throughout this paper, by an algebra I mean an associative algebra with a unit element. A homomorphism of algebras is understood to map one unit into another. A representation of an algebra is a homomorphism of the algebra into the algebra End S of all endomorphisms of a vector space S. If V is a finite-dimensional vector space, then V^* denotes its dual and the value $f(v)$ of the 1-form $f \in V^*$ on $v \in V$ is often denoted by

$\langle v, f\rangle$. If $h : V \to W$ is a linear map (homomorphism of vector spaces), then its *transpose* ${}^t h : W^* \to V^*$ is defined by $\langle v, {}^t h(f)\rangle = \langle h(v), f\rangle$ for every $v \in V$ and $f \in W^*$. Let V be a real, m-dimensional vector space with an isomorphism $h : V \to V^*$ which is symmetric, $h = {}^t h$, and such that the quadratic form $V \to \mathbb{R}$, given by $v \mapsto \langle v, h(v)\rangle$ is of signature (k, l), $k + l = m$. One says that the pair (V, h) is a *quadratic space* of dimension m and signature (k, l). The corresponding *Clifford algebra* (see, *e.g.*, [1, 8, 10])

$$\mathrm{Cl}(h) = \mathrm{Cl}^0(h) \oplus \mathrm{Cl}^1(h),$$

contains $\mathbb{R} \oplus V$ and is $\mathbb{Z}_2$-graded by the *main automorphism* α characterized by $\alpha(1) = 1$ and $\alpha(v) = -v$ for every $v \in V$. Every $a \in \mathrm{Cl}(h)$ is decomposed into its even and odd components, a_0 and a_1, respectively, such that $a_\varepsilon \in \mathrm{Cl}^\varepsilon(h)$ and $a = a_0 + a_1 = \alpha(a_0 - a_1)$. For every $v \in V$, its Clifford square is $v^2 = \langle v, h(v)\rangle$. Assume V to be oriented and let $(e_1, \ldots, e_m)$ be an orthonormal frame in V of the preferred orientation. The square of the volume element $\mathrm{vol}(h) = e_1 \ldots e_m$ is

$$\mathrm{vol}(h)^2 = i(h)^2, \quad \text{where} \quad i(h) \in \{1, \mathrm{i}\}.$$

For every $v \in V$ one has $\mathrm{vol}(h)v = (-1)^{m+1} v\,\mathrm{vol}(h)$. Therefore, if m is even, then α is an inner automorphism, $\alpha(a) = \mathrm{vol}(h)\, a\, \mathrm{vol}(h)^{-1}$.

It follows from the universality of Clifford algebras that the Clifford map

$$V \to \mathrm{Cl}(h), \quad v \mapsto \mathrm{vol}(h)v,$$

extends to the homomorphism of algebras,

$$j : \mathrm{Cl}((-1)^{m+1} i(h)^2 h) \to \mathrm{Cl}(h), \tag{1}$$

such that $j(1) = 1$ and $j(v) = \mathrm{vol}(h)v$ for $v \in V$. For m even, this homomorphism is bijective and respects the $\mathbb{Z}_2$-grading of the algebras. If m is even and $\mathrm{vol}(h)^2 = 1$, then $j : \mathrm{Cl}(-h) \to \mathrm{Cl}(h)$ is an isomorphism of algebras. If m is even and $\mathrm{vol}(h)^2 = -1$, then the algebras $\mathrm{Cl}(h)$ and $\mathrm{Cl}(-h)$ are not isomorphic and j is an inner automorphism of $\mathrm{Cl}(h)$ given by $j(a) = \frac{1}{2}(1 + \mathrm{vol}(h))a(1 - \mathrm{vol}(h))$. If m is odd, then the homomorphism (1) is onto the even subalgebra $\mathrm{Cl}^0(h)$. In this case, the volume element corresponding to $\mathrm{vol}(h)^2 h$ has a positive square. Therefore, if m is odd and h is such that $\mathrm{vol}(h)^2 = 1$, then $j(\mathrm{vol}(h)) = 1$ and there is the exact sequence of homomorphisms of algebras,

$$0 \to \mathrm{Cl}^-(h) \to \mathrm{Cl}(h) \xrightarrow{j} \mathrm{Cl}^0(h) \to 0,$$

where $\mathrm{Cl}^-(h) = \{a \in \mathrm{Cl}(h) : \mathrm{vol}(h)a = -a\}$ is the subalgebra of anti--selfdual elements of $\mathrm{Cl}(h)$. There is no analogous sequence for m odd and

h such that $\mathrm{vol}(h)^2 = -1$. For m odd, the algebras $\mathrm{Cl}(h)$ and $\mathrm{Cl}(-h)$ are never isomorphic. The algebras $\mathrm{Cl}^0(h)$ and $\mathrm{Cl}^0(-h)$ are isomorphic irrespective of m and h. An element $u \in V$ is said to be a *unit vector* if either $u^2 = 1$ or $u^2 = -1$. The group $\mathrm{Pin}(h)$ is defined as the subset of $\mathrm{Cl}(h)$ consisting of products of all finite sequences of unit vectors; the group multiplication is induced by the Clifford product.[1] The spin group is $\mathrm{Spin}(h) = \mathrm{Pin}(h) \cap \mathrm{Cl}^0(h)$. The Lie algebra $\mathrm{spin}(h)$ of $\mathrm{Spin}(h)$ can be identified with the subspace of $\mathrm{Cl}^0(h)$ spanned by all elements of the form $uv - vu$, where $u, v \in V$. The Lie bracket in $\mathrm{spin}(h)$ coincides with the commutator induced by the Clifford product.

If $V = \mathbb{R}^{k+l}$ and one wants to specify the signature (k, l) of h, then one writes $\mathrm{vol}_{k,l}$, $\mathrm{Cl}_{k,l}$, $\mathrm{Pin}_{k,l}$ and $\mathrm{Spin}_{k,l}$ instead of $\mathrm{vol}(h)$, $\mathrm{Cl}(h)$, $\mathrm{Pin}(h)$ and $\mathrm{Spin}(h)$, respectively; a similar notation is used for the orthogonal groups $\mathrm{O}(h)$ and $\mathrm{SO}(h)$. Since the groups $\mathrm{Spin}(h)$ and $\mathrm{Spin}(-h)$ are isomorphic, one writes Spin_m instead of $\mathrm{Spin}_{m,0} = \mathrm{Spin}_{0,m}$. Since $\mathrm{vol}^2_{2n,0} = \mathrm{vol}^2_{0,2n}$ one can also write vol_{2n} instead of $\mathrm{vol}_{2n,0}$ or $\mathrm{vol}_{0,2n}$.

2.2. Notation concerning smooth manifolds and bundles

All manifolds, maps and bundles are assumed to be smooth; manifolds are paracompact and bundles are locally trivial. If $\pi : E \to M$ and $\sigma : F \to N$ are two bundles , then the pair (f, f') of maps $f : M \to N$ and $f' : E \to F$ is a *morphism of bundles* if $\sigma \circ f' = f \circ \pi$. A bundle is trivial if it is isomorphic to a Cartesian product of its base by the typical fiber. A map $s : M \to E$ is a *section* of π if $\pi \circ s = \mathrm{id}_M$. For every manifold M, there is the *tangent bundle* $TM \to M$. If $f : M \to N$ is a map of manifolds, then $Tf : TM \to TN$ is the derived map of their tangent bundles and (f, Tf) is a morphism of bundles. For $x \in M$, there is the linear map $T_x f : T_x M \to T_{f(x)}N$ of the fiber $T_x M$ of the bundle $TM \to M$ into the corresponding fiber of the other bundle. Given a bundle $\sigma : F \to N$ and a map $f : M \to N$, one defines the bundle $\pi : E \to M$ *induced* by f from σ as follows: $E = \{(x, q) \in M \times F : \sigma(q) = f(x)\}$ and $\pi(x, q) = x$. There is then also a canonical map, $f' : E \to F$, given by $f'(x, q) = q$ and the pair (f, f') is a morphism of bundles. A *Riemannian space* is a connected manifold M with a metric tensor field g which need not be definite; if it is, then one refers to M as a *proper* Riemannian space. For every $x \in M$, the metric tensor defines a symmetric isomorphism $g_x : T_x M \to T_x{}^* M$. If M is a Riemannian space, then there is a quadratic space (V, h) such that,

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[1] This definition of the pin group follows [5, 8, 11] and can be traced to Cartan, see Sections 12, 97 and 127 in [12]. An equivalent definition, using the notion of spinor norm, and based on the (semi-)simplicity of the Clifford algebras, is in [13, 14].

for every $x \in M$, there is a linear isometry $p : V \to T_x M$, *i.e.* a linear isomorphism such that ${}^t p \circ g_x \circ p = h$. One says that (V, h) is *local model* of the Riemannian space and that p is an *orthonormal frame at* x. If (e_μ) is an orthonormal frame in V, then p can be identified with the collection of vectors (p_μ), where $p_\mu = p(e_\mu)$, $\mu = 1, \ldots, m = \dim V = \dim M$.

If ω is a differential form on a manifold, then $\mathrm{d}\omega$ is its exterior derivative. Wedge denotes the exterior product of forms. If X is a vector field on M and ω is a $(p+1)$-form, then $X \lrcorner\, \omega$ is the p-form such that $(X \lrcorner\, \omega)(X_1, \ldots, X_p) = \omega(X, X_1, \ldots, X_p)$ for every collection $(X_1, \ldots, X_p)$ of vector fields on M. In particular, if $f : M \to \mathbb{R}$ and X is a vector field, then $X \lrcorner\, \mathrm{d}f = \langle X, \mathrm{d}f \rangle = X(f)$ is the derivative of the function f in the direction of the vector field X.

By a group is meant here a Lie group; a subgroup is a closed Lie subgroup. An exact sequence of group homomorphisms $1 \to K \xrightarrow{k} G \xrightarrow{l} H \to 1$ is said to define G as an *extension of* H *by* K. Two extensions, $K \xrightarrow{k} G \xrightarrow{l} H$ and $K \xrightarrow{k'} G' \xrightarrow{l'} H$, of the group H by the group K, are *equivalent* if there is an isomorphism of groups $f : G \to G'$ such that $f \circ k = k'$ and $l' \circ f = l$. Given a representation $\gamma : G \to \mathrm{GL}(S)$ of the group G in a vector space S and a homomorphism $\iota : G \to G'$ of groups, one says that a representation $\gamma' : G' \to \mathrm{GL}(S)$ *extends* γ (relative to ι) if $\gamma' \circ \iota = \gamma$.

A *principal bundle* with structure group G ('principal G-bundle') and projection π of its total space P to the base manifold M is sometimes represented, symbolically, by the sequence $G \to P \xrightarrow{\pi} M$. The group G is assumed to act on P to the right: there is a map $\delta : P \times G \to P$ such that, if $\delta(a)(p) = \delta(p, a)$, then $\pi \circ \delta(a) = \pi$, $\delta(a) \circ \delta(b) = \delta(ba)$ and $\delta(1_G) = \mathrm{id}_P$, where $p \in P$, $a, b \in G$ and 1_G is the unit of G. One writes pa instead of $\delta(p, a)$. A principal bundle admitting a section f is trivial, *i.e.* isomorphic (in the category of principal bundles) to the product bundle $M \times G \to M$; a *trivializing map* (isomorphism of principal bundles) is given by $(x, a) \mapsto f(x)a$, where $x \in M$ and $a \in G$. Let there be given a left action of the group G on the manifold S, *i.e.* a map $\gamma : G \times S \to S$ such that, if $\gamma(a)(\varphi) = \gamma(a, \varphi)$, then $\gamma(a) \circ \gamma(b) = \gamma(ab)$ and $\gamma(1_G) = \mathrm{id}_S$ for every $a, b \in G$ and $\varphi \in S$. One then defines the bundle $\pi_E : E \to M$, *associated* with P by γ. Its typical fiber is S and its total space E, often denoted by $P \times_\gamma S$, is the set of all equivalence classes of the form $[(p, \varphi)]$, where $(p, \varphi) \in P \times S$ and $[(p', \varphi')] = [(p, \varphi)]$ if, and only if, there exists $a \in G$ such that $p' = pa$ and $\varphi = \gamma(a)\varphi'$. The projection is given by $\pi_E([(p, \varphi)]) = \pi(p)$. If S is a vector space, then the associated bundle is a *vector bundle.* A homomorphism $\iota : G \to G'$ of groups defines a left action of G on G', viz. $(a, b) \mapsto \iota(a)b$, where $a \in G$ and $b \in G'$; the corresponding bundle $P \times_\iota G' \to M$, associated with $P \to M$, is a principal G'-bundle.

3. Representations of the pin groups

3.1. The vector representations

For every invertible $u \in V$, the map $v \mapsto -uvu^{-1}$ is a reflection in the hyperplane orthogonal to the vector u; this observation leads to the definition of the *twisted adjoint* vector representation ρ of the group $\mathrm{Pin}(h)$ in V: for every $a \in \mathrm{Pin}(h)$ the map $\rho(a) : V \to V$, given by

$$\rho(a)\,v = \alpha(a)\,v\,a^{-1} \tag{2}$$

is orthogonal,

$$^t\rho(a) \circ h \circ \rho(a) = h\,, \tag{3}$$

and there is the exact sequence of group homomorphisms

$$1 \to \{1, -1\} \to \mathrm{Pin}(h) \xrightarrow{\rho} \mathrm{O}(h) \to 1\,.$$

Replacing in (2) the vector v by the μth vector e_μ of an orthonormal frame in V, one obtains

$$e_\nu \rho^\nu{}_\mu(a) = \alpha(a) e_\mu a^{-1}\,. \tag{4}$$

In this equation, and elsewhere in this paper, there is tacitly assumed a summation (the *Einstein convention*) over the range of tensor indices appearing in contragredient pairs.

The *adjoint* vector representation Ad is defined by

$$\mathrm{Ad}(a)\,v = a\,v\,a^{-1}$$

and leads to the exact sequences of group homomorphisms

$$1 \to \left\{ \begin{matrix} \{1,-1\} \\ \{1,-1,\mathrm{vol}(h),-\mathrm{vol}(h)\} \end{matrix} \right\} \to \mathrm{Pin(h)} \xrightarrow{\mathrm{Ad}} \left\{ \begin{matrix} \mathrm{O}(h) \\ \mathrm{SO}(h) \end{matrix} \right\} \to 1 \; \left\{ \begin{matrix} \text{for } m \text{ even}, \\ \text{for } m \text{ odd}. \end{matrix} \right.$$

The homomorphisms ρ and Ad coincide when restricted to $\mathrm{Spin}(h)$. For every quadratic space (V, h), irrespective of the parity of m, there is the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(h) \xrightarrow{\rho} \mathrm{SO}(h) \to 1\,, \tag{5}$$

where $\mathbb{Z}_2 = \{1, -1\}$.

For every *even*-dimensional quadratic space (V, h), one can consider four central extensions of $\mathrm{O}(h)$ by $\mathbb{Z}_2$, associated with the groups $\mathrm{Pin}(\pm h)$, namely

$$\rho \text{ and } \mathrm{Ad} : \mathrm{Pin}(h) \to \mathrm{O}(h), \text{ and } \rho \text{ and } \mathrm{Ad} : \mathrm{Pin}(-h) \to \mathrm{O}(h)\,,$$

but, in each case, only two among the four are inequivalent. Indeed, if m is even, then

$$\rho = \mathrm{Ad} \circ j \tag{6}$$

as may be seen by checking that both sides of (6) coincide on the generating subset V. More precisely:

(i) if $\mathrm{vol}(h)^2 = 1$, then the extensions

$$\mathbb{Z}_2 \to \mathrm{Pin}(\pm h) \xrightarrow{\mathrm{Ad}} \mathrm{O}(h)$$

are equivalent to the corresponding extensions

$$\mathbb{Z}_2 \to \mathrm{Pin}(\mp h) \xrightarrow{\rho} \mathrm{O}(h)\,;$$

(ii) if $\mathrm{vol}(h)^2 = -1$, then the extensions

$$\mathbb{Z}_2 \to \mathrm{Pin}(\pm h) \xrightarrow{\mathrm{Ad}} \mathrm{O}(h)$$

are equivalent to the corresponding extensions

$$\mathbb{Z}_2 \to \mathrm{Pin}(\pm h) \xrightarrow{\rho} \mathrm{O}(h)\,.$$

To summarize, we have

Proposition 1. *For every real quadratic space* (V, h), *there are two inequivalent central extensions of* $\mathrm{O}(h)$ *by* $\mathbb{Z}_2$, *given by*

$$\mathbb{Z}_2 \to \mathrm{Pin}(h) \xrightarrow{\rho} \mathrm{O}(h) \ \text{ and } \ \mathbb{Z}_2 \to \mathrm{Pin}(-h) \xrightarrow{\rho} \mathrm{O}(h)\,, \tag{7}$$

where ρ *is as in* (2). *By restriction to* $\mathrm{Spin}(h)$ *each of these extensions reduces to the one given by* (5).

Note that for $k = l$ (neutral signature) the groups $\mathrm{Pin}(h)$ and $\mathrm{Pin}(-h)$ are isomorphic, but the extensions (7) are not. There are also extensions of $\mathrm{O}(h)$ by $\mathbb{Z}_2$ that do not come from the Clifford construction [15]. The (untwisted) adjoint representation seems to be the first to have attracted attention. It has been much used by physicists in the theory of the Dirac equation of the electron; see, *e.g.*, [6,16]. The twisted representation is implicit in É. Cartan's approach to spinors, see §58 and §97 in [12]. Explicitly, it has been defined by Atiyah *et al.* in [17]. It follows from the preceding remarks that, for even-dimensional spaces, one can use either of the two representations, but in the case of odd dimensions, only ρ provides a cover of the full orthogonal group. For this reason and for uniformity, from now on, only ρ is used in the definition of pin structures.

3.2. The spinor representations

In this paper, by a *spinor representation* of a group Pin(h) or Spin(h) is understood a representation obtained by restriction, to the group, of a representation of the algebra Cl(h) in a finite-dimensional *complex* vector space S, the space of *spinors.* If $\gamma : \mathrm{Cl}(h) \to \mathrm{End}S$ is any representation of the algebra, then the group representation, obtained by restriction to Pin(h), is denoted by the same letter γ; similar abuses of notation and terminology are made throughout the paper. Given an orthonormal frame (e_μ) in V, one defines the 'Dirac matrices' (automorphisms of S) by $\gamma_\mu = \gamma(e_\mu)$. The following Proposition summarizes well known facts about complex representations of real Clifford algebras [5, 8, 10, 17].

Proposition 2. *Let (V, h) be a quadratic space of dimension m and let ν denote a positive integer.*

(i) If m is even, *$m = 2\nu$, then the algebra* Cl(h) *is central simple and, as such, has only one, up to equivalence, faithful and irreducible* Dirac representation *γ in a vector space S, which turns out to be of complex dimension 2^ν. The restriction of γ to $\mathrm{Cl}^0(h)$ decomposes into the direct sum $\gamma_+ \oplus \gamma_-$ of two complex-inequivalent* Weyl representations. *In a notation adapted to the decomposition $S = S_+ \oplus S_-$ of the space of Dirac spinors into the direct sum of the spaces S_+ and S_- of Weyl spinors, the Dirac matrices are of the form*

$$\gamma_\mu = \begin{pmatrix} 0 & \gamma_\mu^- \\ \gamma_\mu^+ & 0 \end{pmatrix}.$$

(ii) If m is odd, *$m = 2\nu - 1$, then the algebra $\mathrm{Cl}^0(h)$ is central simple and has a faithful and irreducible* Pauli representation *in a space of complex dimension $2^{\nu-1}$. This representation extends to two representations, σ and $\sigma \circ \alpha$, of the full algebra* Cl(h) *in the same space, by putting $\sigma(\mathrm{vol}(h)) = i(h)\,\mathrm{id}$. These representation, also referred to as Pauli representations of* Cl(h), *are complex-inequivalent and irreducible, but faithful only when $i(h) = \mathrm{i}$. A faithful, but reducible,* Cartan representation *γ of* Cl(h) *is defined as $\gamma = \sigma \oplus (\sigma \circ \alpha)$. Therefore, if $\sigma_\mu = \sigma(e_\mu)$, then*

$$\gamma_\mu = \begin{pmatrix} \sigma_\mu & 0 \\ 0 & -\sigma_\mu \end{pmatrix}.$$

The commutant of the Cartan representation γ is generated by $\gamma(\mathrm{vol}(h))$. The elements of the carrier spaces of the representations γ and σ are now called Cartan and Pauli spinors, respectively.

The names of Dirac, Weyl and Pauli are used by physicists mainly in connection with spinors associated with vector spaces of low dimension. In mathematics, the Weyl representations $\gamma_\pm$ are usually denoted by $\Delta^\pm$ and sometimes referred to as half-spinor representations [8, 13]. The Cartan representation seldom appears because it is decomposable. In this paper, I identify the representations by using one of the above names; thus the letters γ and γ' can denote any one of the spinor representations, depending on the context.

If $\gamma : \mathrm{Cl}(h) \to \mathrm{End}S$ is as in Prop. 2, then the *helicity* automorphism of the representation γ in S is $\gamma(\mathrm{vol}(h)) = \gamma_1 \dots \gamma_m$ so that Weyl (resp., Pauli) spinors are its eigenvectors for m even (resp., odd). The foregoing remarks can be supplemented by

Proposition 3. *Let (V, h) be a quadratic space of dimension $m = 2\nu$ (resp., $2\nu - 1$). There is a faithful representation γ of the Clifford algebra $\mathrm{Cl}(h)$ in a complex vector space S of dimension 2^ν such that the representations γ and $\gamma \circ \alpha$ are complex-equivalent. The representation is unique, up to complex equivalence, and irreducible (resp., decomposable into two irreducibles). By restriction to the even subalgebra $\mathrm{Cl}^0(h)$, the representation γ decomposes into the direct sum of two irreducible representations, each defined in a complex space of dimension $2^{\nu-1}$. The isomorphism γ_{m+1} intertwining the representations γ and $\gamma \circ \alpha$ can be taken to act on the Dirac (resp., Cartan) spinor (φ, ψ) so that $\gamma_{m+1}(\varphi, \psi) = (\mathrm{i}\varphi, -\mathrm{i}\psi)$ (resp., $\gamma_{m+1}(\varphi, \psi) = (-\psi, \varphi)$). Irrespective of the parity of m, one has*

$$\gamma_{m+1}^2 = -\mathrm{id}_S \quad \text{and} \quad \gamma_{m+1}\gamma_\mu + \gamma_\mu\gamma_{m+1} = 0\,, \tag{8}$$

for $\mu = 1, \dots, m$.

The intertwining isomorphism is not unique; see [5] for a precise statement on the 'Dirac intertwiner' $\mathrm{i}\gamma_{m+1}$. By applying a spinor representation γ to both sides of (4), one obtains

$$\gamma_\nu\, \rho^\nu{}_\mu(a) = \gamma \circ \alpha(a)\, \gamma_\mu\, \gamma(a)\,. \tag{9}$$

3.3. Extension of a spinor representation from dimension m to $m + 1$

For every pair (k, l) of non-negative integers, there is the *isomorphism* of algebras,

$$\iota : \mathrm{Cl}_{k,l} \to \mathrm{Cl}^0_{k,l+1} \quad \text{given by} \quad \iota(a_0 + a_1) = a_0 + a_1 e_{k+l+1}\,.$$

Proposition 4. *If γ' is a representation of $\mathrm{Cl}_{k,l+1}$ in a complex vector space (of spinors), then $\gamma = \gamma' \circ \iota$ is a representation of $\mathrm{Cl}_{k,l}$ in the same space. In particular:*

(i) If $k+l$ is even and γ' is a Pauli representation, then γ is the Dirac representation.

(ii) If $k+l$ is odd and γ' is the Dirac representation, then γ is the Cartan representation. Moreover, if $\gamma'_{\pm}$ are the Weyl components of γ', i.e. $\gamma'|\mathrm{Cl}^0_{k,l+1} = \gamma'_+ \oplus \gamma'_-$, then $\gamma_\pm = \gamma'_\pm \circ \iota$ are the Pauli components of γ, i.e. $\gamma = \gamma_+ \oplus \gamma_-$.

Proof. Since ι is injective, if γ' is faithful, then so is γ. In case *(i)*, the Pauli representation γ' is faithful unless $\mathrm{vol}^2_{k,l+1} = 1$. If $\mathrm{vol}^2_{k,l+1} = 1$, then the kernel of γ' is either the subalgebra $\mathrm{Cl}^+_{k,l+1}$ of selfdual elements or the subalgebra $\mathrm{Cl}^-_{k,l+1}$ of anti-selfdual elements of $\mathrm{Cl}_{k,l+1}$, see Sec. 2.1. Since $\mathrm{Cl}^{\pm}_{k,l+1} \cap \mathrm{Cl}^0_{k,l+1} = \{0\}$, the representation $\gamma = \gamma' \circ \iota$ is faithful in every case when $m = k+l$ is even. In case *(ii)*, the Dirac representation γ' of $\mathrm{Cl}_{k,l+1}$ in S is faithful. Therefore, the representation γ is also faithful. Let $\gamma_i = \gamma'(e_i)$, $i = 1, \ldots, m+1 = k+l+1$, be the Dirac matrices. Then $\gamma \circ \alpha(a) = \gamma_{m+1}\gamma(a)\gamma^{-1}_{m+1}$ for every $a \in \mathrm{Cl}_{k,l}$ and, by Prop. 3, γ is the Cartan representation. Since $\gamma(e_i) = \gamma'(e_i e_{m+1}) = \gamma_i \gamma_{m+1}$ for $i = 1, \ldots, m$, the helicity automorphisms of γ' and $\gamma = \gamma' \circ \iota$ are equal. Therefore, the decompositions of S into spaces of Weyl and Pauli spinors coincide. □

By iteration of the above, one can obtain, for $k+l$ *odd,* two Pauli representations of $\mathrm{Cl}_{k,l+2}$ extending the Cartan representation of $\mathrm{Cl}_{k,l}$. Similarly, for $k+l$ *even,* there are two Weyl representations of $\mathrm{Cl}^0_{k,l+2}$ extending the Dirac representation of $\mathrm{Cl}_{k,l}$. One cannot, however, go beyond that without changing the dimension of the space of spinors underlying the representations.

By restriction, the isomorphism of algebras ι gives rise to the *monomorphism* of groups

$$\iota : \mathrm{Pin}_{k,l} \to \mathrm{Spin}_{k,l+1} \,. \tag{10}$$

The corresponding monomorphism of the (pseudo-)orthogonal groups,

$$\kappa : \mathrm{O}_{k,l} \to \mathrm{SO}_{k,l+1} \;\text{ such that }\; \kappa \circ \rho = \rho \circ \iota \,, \tag{11}$$

satisfies $\kappa(A)\, e_\mu = A\, e_\mu$, $\mu = 1, \ldots, k+l$, and $\kappa(A)\, e_{k+l+1} = (\det A)\, e_{k+l+1}$ for every $A \in \mathrm{O}_{k,l}$.

By restricting the representations γ' and γ, referred to in Prop. 4, to the groups $\mathrm{Pin}_{k,l+1}$ and $\mathrm{Pin}_{k,l}$ one obtains representations of pin and spin groups; there are statements about extensions of spinor representations of these groups analogous to those appearing in the Proposition.

4. Pin structures and bundles of spinors

4.1. Definitions

Let (V, h) be a local model of an m-dimensional Riemannian manifold M and let $\pi : P \to M$ be the bundle of all orthonormal frames of M. A $\mathrm{Pin}(h)$-*structure* on M is a principal $\mathrm{Pin}(h)$-bundle $\varpi : Q \to M$, together with a morphism $\chi : Q \to P$ of principal bundles over M associated with the epimorphism $\rho : \mathrm{Pin}(h) \to \mathrm{O}(h)$. The morphism condition means that $\varpi = \pi \circ \chi$ and that, for every $q \in Q$ and $a \in \mathrm{Pin}(h)$, one has $\chi(qa) = \chi(q)\rho(a)$. The expression $\mathrm{Pin}_{k,l}$-*structure* is used when one wants the signature of h to appear explicitly. For brevity, we shall describe a $\mathrm{Pin}(h)$-structure by the sequence

$$\mathrm{Pin}(h) \to Q \xrightarrow{\chi} P \xrightarrow{\pi} M . \tag{13}$$

Another pin structure over the same manifold M, $\mathrm{Pin}(h) \to Q' \xrightarrow{\chi'} P \xrightarrow{\pi} M$ is said to be *equivalent* to the structure (13) if there is a diffeomorphism $f : Q \to Q'$ such that $\chi' \circ f = \chi$ and $f(qa) = f(q)a$ for every $q \in Q$ and $a \in \mathrm{Pin}(h)$.

If M is orientable and admits a $\mathrm{Pin}(h)$-structure, then it has a spin structure. In an abbreviated style, similar to that of (13), it may be described by the sequence of maps

$$\mathrm{Spin}(h) \to SQ \to SP \to M , \tag{14}$$

where SP is now an $\mathrm{SO}(h)$-bundle. One often abbreviates the expression 'M has a spin structure' to 'M is spin'. Equivalence of spin structures is defined similarly to that of pin structures.

Let M be a Riemannian space with a $\mathrm{Pin}(h)$-structure (13) and let γ be a spinor representation of the group $\mathrm{Pin}(h)$ in S, as described in Prop. 2. The complex vector bundle $\pi_E : E \to M$, with typical fiber S, associated with Q by γ, is the *bundle of spinors of type* γ. If the dimension m of M is even (resp., odd), then E is called a bundle of Dirac (resp., Cartan) spinors. For m odd, $m = 2\nu - 1$, one can also take the representation $\sigma : \mathrm{Pin}(h) \to \mathrm{GL}(2^{\nu-1}, \mathbb{C})$ to define the bundle of Pauli spinors over M. Similarly, if m is even and M has a spin structure, then there are two bundles of Weyl spinors over M.

Let M be a Riemannian manifold with a pin structure (13). A *spinor field* (some authors say: a 'pinor' field) of type γ on M is a section of π_E. The (vector) space of such sections is known to be in a natural and bijective correspondence with the set of all maps $\psi : Q \to S$ equivariant with respect to the action of $\mathrm{Pin}(h)$, $\psi \circ \delta(a) = \gamma(a^{-1})\psi$, for every $a \in \mathrm{Pin}(h)$. It is convenient to refer to ψ itself as a spinor field of type γ on M. Depending on whether E is a bundle of Dirac, Weyl, Cartan or Pauli spinors, one refers to its sections as Dirac, Weyl, Cartan or Pauli spinor fields, respectively.

The existence of a pin (or spin) structure on a Riemannian manifold M imposes topological conditions on M. They are expressed in terms of the Stiefel-Whitney classes $w_i \in H^i(M, \mathbb{Z}_2)$ associated with the tangent bundle of the manifold; see, *e.g.* [1] and the applications of the Karoubi theorem given in [3–5].

4.2. Remarks on the triviality of associated bundles

Proposition 5. *The vector bundle $E \to M$, associated with the principal G-bundle $Q \to M$ by a representation γ of G in S, is trivial if, and only if, there exists a group G', a homomorphism $\iota : G \to G'$, and an extension $\gamma' : G' \to \mathrm{GL}(S)$ of γ, such that the associated principal G'-bundle $Q \times_\iota G' \to M$ is trivial.*

Proof. Indeed, if E is trivial as a vector bundle, then there is a trivializing map $E \to M \times S$, $[(q, \varphi)] \mapsto (\pi(q), g(q)\varphi)$, such that $g : Q \to \mathrm{GL}(S)$ and $g(qa) = g(q) \circ \gamma(a)$ for every $q \in Q$, $\varphi \in S$ and $a \in G$. Taking $G' = \mathrm{GL}(S)$ and $\iota = \gamma$ one sees that $\gamma' = \mathrm{id}$ extends γ. The principal bundle $Q \times_\iota G' \to M$ is trivial because it has a global section corresponding to the equivariant map $e : Q \to G'$, where $e(q) = \iota(q)^{-1}$ for every $q \in Q$. Conversely, given an extension γ' of γ and a homomorphism $\iota : G \to G'$ such that $Q \times_\iota G' \to M$ is trivial, there is a map $e : Q \to G'$ such that $e(qa) = \iota(a^{-1})e(q)$ for every $q \in Q$ and $a \in G$. If $g : Q \to \mathrm{GL}(S)$ is given by $g(q) = \gamma'(e(q)^{-1})$, then the map $[(q, \varphi)] \mapsto (\pi(q), g(q)\varphi)$, which is well-defined because of $\gamma' \circ \iota = \gamma$, trivializes the vector bundle $E \to M$. □

If G is a subgroup of G', then there is the principal G-bundle $\pi : G' \to G'/G$. The action of G on G' given by the left translations defines the associated principal G'-bundle $G' \times_\gamma G' \to G'/G$ that is trivial: a trivializing map is given by $[(a, b)] \mapsto (\pi(a), ab)$, where $a, b \in G'$.

Corollary. If there is a representation γ' of G' in S extending the representation $\gamma : G \to \mathrm{GL}(S)$, then the bundle $G' \times_\gamma S \to G'/G$, associated with $\pi : G' \to G'/G$ by γ, is trivial.

Indeed, a trivializing isomorphism is given by $[(a, \varphi)] \mapsto (\pi(a), \gamma'(a)\varphi)$, where $a \in G'$ and $\varphi \in S$.

4.3. Examples

A. The spheres. For every $m > 1$, the unit sphere $\mathbb{S}_m \subset \mathbb{R}^{m+1}$ has a unique spin structure described by

$$\mathrm{Spin}_m \to \mathrm{Spin}_{m+1} \to \mathrm{SO}_{m+1} \to \mathbb{S}_m\,. \tag{15}$$

Its bundle of Dirac (m even) or Pauli (m odd) spinors is trivial [18] by virtue of Prop. 5 and its *Corollary*. The projection $\varpi : \mathrm{Spin}_{m+1} \to \mathbb{S}_m$ is given by $\varpi(a) = a\, e_{m+1}\, a^{-1}$, where $(e_1, \ldots, e_{m+1})$ is the canonical frame in $\mathbb{R}^{m+1}$. Consider a Dirac or a Pauli representation of Spin_m in $\mathrm{GL}(S)$ and let γ' be one of its extensions to Spin_{m+1}. For every $\Psi : \mathbb{S}_m \to S$ the map $\psi : \mathrm{Spin}_{m+1} \to S$ given by $\psi(a) = \gamma'(a^{-1})\, \Psi(\varpi(a))$ is a spinor field on the sphere; every such field can be so obtained. This observation is implicit in the work of Schrödinger [19] on the Dirac equation on low-dimensional spheres; see also [20].

Remark 1. Consider the group Spin_{m+1} as a subgroup of $\mathrm{Pin}_{0,m+1}$ or $\mathrm{Pin}_{m+1,0}$. The map $\varsigma : \mathrm{Spin}_{m+1} \to \mathrm{Spin}_{m+1}$ given by $\varsigma(a) = e_{m+1}\, a\, e_{m+1}^{-1}$ is an involutive automorphism of Spin_{m+1}, preserving the subgroup Spin_m. If $\tau : \mathbb{S}_m \to \mathbb{S}_m$ is the symmetry $x \mapsto e_{m+1}\, x\, e_{m+1}^{-1}$, then $\varpi \circ \varsigma = \tau \circ \varpi$. Since $\varsigma(ab) = \varsigma(a)b$ for every $b \in \mathrm{Spin}_m$, the pair (τ, ς) is an automorphism of the spin structure of the sphere and if ψ is a spinor field, then so is $\psi_\varsigma = \psi \circ \varsigma$. If m is even and b is an odd element of $\mathrm{Pin}_{0,m}$ or $\mathrm{Pin}_{m,0}$, then $\varpi(a\, b\, \mathrm{vol}_{m+1}) = -\varpi(a)$; if m is odd, then $\varpi(a\, \mathrm{vol}_{m+1}) = -\varpi(a)$.

Remark 2. The case of m odd, $m = 2\nu - 1$, deserves an additional comment. There are two Weyl representations γ_+ and γ_- of $\mathrm{Spin}_{2\nu}$ in S_+ and S_-, respectively. By restriction to $\mathrm{Spin}_{2\nu-1}$, they give two equivalent Pauli representations: there is an isomorphism $\theta : S_+ \to S_-$ such that $\theta \circ \gamma_+(b) = \gamma_-(b) \circ \theta$ for every $b \in \mathrm{Spin}_{2\nu-1}$. For every $a \in \mathrm{Spin}_{2\nu}$, the linear map $\gamma_-(a) \circ \theta \circ \gamma_+(a^{-1})$ is an isomorphism of S_+ on S_-; as a function of a it is constant on the fibers of ϖ; therefore, it defines a map $\Theta : \mathbb{S}_{2\nu-1} \to \mathrm{Iso}\,(S_+, S_-)$ such that $\Theta(\varpi(a)) \circ \gamma_+(a) = \gamma_-(a) \circ \theta$ for every $a \in \mathrm{Spin}_{2\nu}$. Since $\varpi(a\, \mathrm{vol}_{2\nu}) = -\varpi(a)$, one has $\Theta(-x) = -\Theta(x)$ for every $x \in \mathbb{S}_{2\nu-1}$. The representations $\gamma_\pm$ give rise to the associated bundles $\mathrm{Spin}_{2\nu} \times_{\gamma_\pm} S_\pm$ of Pauli spinors ever $\mathbb{S}_{2\nu-1}$. According to the *Corollary*, the trivializing isomorphisms $\mathrm{Spin}_{2\nu} \times_{\gamma_\pm} S_\pm \to \mathbb{S}_{2\nu-1} \times S_\pm$ are given by $[(a, \varphi_\pm)]_\pm \mapsto (\varpi(a), \gamma_\pm(a)\varphi_\pm)$, where $a \in \mathrm{Spin}_{2\nu}$, $\varphi_\pm \in S_\pm$ and $[(a, \varphi_\pm)]_\pm = [(a', \varphi'_\pm)]_\pm$ if, and only if, there is $b \in \mathrm{Spin}_{2\nu-1}$ such that $a' = ab$ and $\varphi_\pm = \gamma_\pm(b)\varphi'_\pm$. The two bundles of Pauli spinors are isomorphic: an isomorphism $\mathrm{Spin}_{2\nu} \times_{\gamma_+} S_+ \to \mathrm{Spin}_{2\nu} \times_{\gamma_-} S_-$ is given by $[(a, \varphi_+)]_+ \mapsto [(a, \theta(\varphi_+)]_-$ and the corresponding isomorphism $\mathbb{S}_{2\nu-1} \times S_+ \to \mathbb{S}_{2\nu-1} \times S_-$ by $(\varpi(a), \varphi_+) \mapsto (\varpi(a), \Theta(\varpi(a)\varphi_+)$.

B. Real projective spaces. Recall that the real, m-dimensional projective space $\mathbb{P}_m$ is orientable if, and only if, m is odd. There is the canonical map $\mathbb{S}_m \to \mathbb{P}_m$, $x \mapsto [x] = \{x, -x\}$. The symmetry τ of $\mathbb{S}_m$, defined in *Remark 1*, descends to a symmetry τ' of $\mathbb{P}_m$, $\tau'([x]) = [\tau(x)]$ for $x \in \mathbb{S}_m$. If k is a positive integer, then

$$\mathrm{vol}_{4k}^2 = 1\,, \quad \text{but} \quad \mathrm{vol}_{4k+2}^2 = -1\,,$$

and $\quad \mathrm{vol}^2_{4k+1,0} = \mathrm{vol}^2_{0,4k+3} = 1\,, \quad$ but $\quad \mathrm{vol}^2_{4k+3,0} = \mathrm{vol}^2_{0,4k+1} = -1\,.$

To treat simultaneously the spaces $\mathbb{P}_{4k}$ and $\mathbb{P}_{4k+2}$, define

$$\mathrm{Pin}^\star_{2k} = \mathrm{Pin}_{2k,0} \quad \text{and} \quad \mathrm{vol}^\star_{2k+1} = \mathrm{vol}_{2k+1,0} \quad \text{for } k \text{ even}\,,$$

and
$$\mathrm{Pin}^\star_{2k} = \mathrm{Pin}_{0,2k} \quad \text{and} \quad \mathrm{vol}^\star_{2k+1} = \mathrm{vol}_{0,2k+1} \quad \text{for } k \text{ odd}\,.$$

For *m even* there is the monomorphism of groups $l : \mathrm{O}_m \to \mathrm{SO}_{m+1}$ given by $l(A)e_\mu = (\det A)Ae_\mu$ for $\mu = 1, \ldots, m$ and $l(A)e_{m+1} = (\det A)e_{m+1}$. By an argument similar to the one used in [5] to determine the spin structures on real projective quadrics it follows that:

(i) The space $\mathbb{P}_{4k+1}$ has no spin structure.

(ii) The space $\mathbb{P}_{2k}$ has two inequivalent $\mathrm{Pin}^\star_{2k}$-structures $(+)$ and $(-)$,

$$\mathrm{Pin}^\star_{2k} \xrightarrow{i_\pm} \mathrm{Spin}_{2k+1} \to \mathrm{SO}_{2k+1} \to \mathbb{P}_{2k}\,, \tag{16}$$

corresponding to the two monomorphisms of groups i_+ and i_- given by

$$i_\pm(a) = \begin{cases} a & \text{for } a \in \mathrm{Spin}_{2k}\,, \\ \pm a\,\mathrm{vol}^\star_{2k+1} & \text{for } a = -\alpha(a) \in \mathrm{Pin}^\star_{2k}\,, \end{cases}$$

so that $\rho \circ i_\pm = l \circ \rho$. The bundle of Dirac spinors associated with each of the pin structures on $\mathbb{P}_{2k}$ is trivial: this follows from the *Corollary* and the observation that the Dirac representation of $\mathrm{Pin}^\star_{2k}$ extends to the Pauli representation of Spin_{2k+1}. The projection $\varpi' : \mathrm{Spin}_{2k+1} \to \mathbb{P}_{2k}$ is given by $\varpi'(a) = [\varpi(a)]$. The pair (τ', ς) is now an isomorphism of one pin structure on $\mathbb{P}_{2k}$ onto the other, as may be seen from the easy-to-check equality $\varsigma(a\, i_+(b)) = \varsigma(a)\, i_-(b)$ for every $a \in \mathrm{Spin}_{2k+1}$ and $b \in \mathrm{Pin}^\star_{2k}$.

(iii) The space $\mathbb{P}_{4k-1}$ has two inequivalent spin structures,

$$\mathrm{Spin}_{4k-1} \to \mathrm{Spin}_{4k}/\mathbf{Z}_2^\pm \to \mathrm{SO}_{4k}/\mathbb{Z}_2 \to \mathbb{P}_{4k-1}\,, \tag{17}$$

where $\mathbf{Z}_2^\pm = \{1, \pm\mathrm{vol}_{4k}\}$ and $\mathbb{Z}_2$ is the center of SO_{4k}. The bundle of Pauli spinors associated with each of these structures is trivial [18]. To see this in detail, let $a \mapsto [a]_\pm = \{a, \pm a\,\mathrm{vol}_{4k}\}$ be the canonical homomorphisms of Spin_{4k} onto $\mathrm{Spin}_{4k}/\mathbf{Z}_2^\pm$. The Pauli representation of Spin_{4k-1} extends to representations $\gamma'_\pm$ of $\mathrm{Spin}_{4k}/\mathbf{Z}_2^\pm$, descending from the Weyl representation $\gamma_\pm$ of Spin_{4k} in $S_\pm$ such that $\gamma_\pm(\mathrm{vol}_{4k}) = \pm\mathrm{id}_{S_\pm}$, namely $\gamma'_\pm([a]_\pm) = \gamma_\pm(a)$.

The automorphism ς descends to an isomorphism of groups, $\varsigma' : \mathrm{Spin}_{4k}/\mathbf{Z}_2^+ \to \mathrm{Spin}_{4k}/\mathbf{Z}_2^-$, such that $\varsigma'([a]_+) = [\varsigma(a)]_-$. The pair (τ', ς') is now an isomorphism of one spin structure on $\mathbb{P}_{4k-1}$ onto the other, but not an equivalence of spin structures. The inequivalence of the two structures described in *(ii)* and *(iii)* is proved in [3].

5. The Dirac operator

5.1. Covariant differentiation of spinor fields

Let again (13) be a pin structure on an m-dimensional Riemannian space M. The Levi–Civita connection form on P lifts to a spin(h)-valued spin connection 1-form ω on Q. For every $q \in Q$, there is the orthonormal frame $\chi(q) = (\chi_\mu(q)) \in P$, where $\chi_\mu(q) \in T_{\varpi(q)}M$ for $\mu = 1, \ldots, m$. The spin connection defines on Q the collection (∇_μ) of m *basic* horizontal vector fields such that, for $\mu = 1, \ldots, m$ and every $q \in Q$,

$$\nabla_\mu \lrcorner\, \omega = 0 \quad \text{and} \quad T_q\varpi(\nabla_\mu(q)) = \chi_\mu(q)\,.$$

For every $a \in \mathrm{Pin}(h)$ they transform according to

$$\nabla_\mu(qa) = T_q\delta(a)\,\nabla_\nu(q)\,\rho^\nu{}_\mu(a)\,. \tag{18}$$

Let (e_μ) be a frame in V and let γ be a spinor representation of $\mathrm{Pin}(h)$ in S. Defining $\gamma^\mu = h^{\mu\nu}\gamma_\nu$, where $(h^{\mu\nu})$ is the inverse of the matrix $(\langle e_\nu, h(e_\mu)\rangle)$, and using Eqs (3) and (4), one obtains

$$\gamma \circ \alpha(a)\,\gamma^\mu = \rho^\mu{}_\nu(a^{-1})\,\gamma^\nu\,\gamma(a)\,. \tag{19}$$

Let $\psi : Q \to S$ be a spinor field of type γ. Its *covariant derivative* is a map $\nabla\psi : Q \to \mathrm{Hom}(V, S)$ such that, for every $v = v^\mu e_\mu \in V$, one has $\langle v, \nabla\psi\rangle = v^\mu\nabla_\mu\psi$, where

$$\nabla_\mu\psi = \nabla_\mu \lrcorner\, \mathrm{d}\,\psi\,.$$

5.2. The classical and the modified Dirac operators

In the notation of the preceding paragraph, the *classical Dirac operator* D^{cl} is given by

$$D^{\mathrm{cl}}\psi = \gamma^\mu\nabla_\mu\psi\,. \tag{20}$$

According to (18) and (19), the classical Dirac operator maps a spinor field of type γ into a spinor field of type $\gamma \circ \alpha$.

Let γ_{m+1} be the isomorphism intertwining the representations γ and $\gamma \circ \alpha$, as described in Prop. 2. The *modified Dirac operator*,

$$D = \gamma_{m+1} D^{\mathrm{cl}}, \tag{21}$$

preserves the type of the spinor field and the corresponding eigenvalue equation $D\psi = \lambda\psi$ is meaningful on non-orientable pin manifolds [5, 9]. To summarize, one has

Proposition 6. *The classical Dirac operator (20) maps a spinor field of type γ into a spinor field of type $\gamma \circ \alpha$; the modified Dirac operator (21) preserves the type of spinor fields.*

If the dimension m of M is *even*, then one can use the vector representation Ad in the definition of the pin structure on M. The classical Dirac operator preserves then the type of spinor fields and there is no need for its modification. Using the decomposition of the space of Dirac spinors into the sum of spaces of Weyl spinors and the form of the Dirac matrices given in part *(i)* of Prop. 2, one can represent the classical and the modified Dirac operators as

$$D^{\mathrm{cl}} = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & \mathrm{i}D^- \\ -\mathrm{i}D^+ & 0 \end{pmatrix},$$

where $D^{\pm}$ are the *Weyl operators*: they act on Weyl spinor fields and change their helicity. For an even-dimensional *spin* manifold, if ψ is a Dirac spinor field, then so is $\gamma_{m+1}\psi$. Therefore, the operators D and $-D$ are equivalent, $-D = \gamma_{m+1} D \gamma_{m+1}^{-1}$, and the spectra of both D and D^{cl} are symmetric.

If M is an *odd*-dimensional *spin* manifold, then the interesting object is the *Pauli operator* $D_0 = \sigma^\mu \nabla_\mu$ acting on Pauli spinor fields. According to Prop. 2 and 3, one can write

$$D^{\mathrm{cl}} = \begin{pmatrix} D_0 & 0 \\ 0 & -D_0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix}.$$

In this case, however, the operators D_0 and $-D_0$ are not equivalent and the spectrum of D_0 need not be symmetric. Each of the operators D_0 and $-D_0$ is 'equally good'. In other words, the spectrum of an odd-dimensional spin manifold is defined only up to 'mirror symmetry', $\lambda \mapsto -\lambda$.

Irrespective of the parity of m, if M is a spin manifold and ψ is a Dirac or Cartan spinor field, then $\gamma_{m+1}\psi$ is a spinor field of the same type. From (8) one obtains $(1+\gamma_{m+1})^{-1} = \frac{1}{2}(1-\gamma_{m+1})$ and $D = (1+\gamma_{m+1})\,D^{\mathrm{cl}}(1+\gamma_{m+1})^{-1}$ so that if $D^{\mathrm{cl}}\psi = \lambda\psi$ then $D\psi' = \lambda\psi'$, where $\psi' = (1+\gamma_{m+1})\psi$.

6. Pin structures on hypersurfaces

6.1. Existence

Let M be a hypersurface in a proper Riemannian spin $(m+1)$-manifold N, defined by an isometric immersion $f : M \to N$. The hypersurface need not be orientable. The normal bundle $T^\perp M$ is a line bundle and the Whitney sum $TM \oplus T^\perp M$ is isomorphic to the pullback of TN to M by Tf. Since N is spin, its first and second Stiefel–Whitney classes vanish and the Whitney theorem gives

$$w_1(TM) + w_1(T^\perp M) = 0 \quad \text{and} \quad w_2(TM) + w_1(TM)w_1(T^\perp M) = 0\,.$$

Therefore, according to the Karoubi theorem [1], the hypersurface M has a $\mathrm{Pin}_{0,m}$-structure. For example, since $\mathbb{P}_n$ is a spin manifold for $n \equiv 3$ mod 4, and the real projective quadric $Q_{k,l} = (\mathbb{S}_k \times \mathbb{S}_l)/\mathbb{Z}_2$ is orientable for $k+l$ even, the natural immersion $Q_{k,l} \to \mathbb{P}_{k+l+1}$ gives, for $k + l \equiv 2 \bmod 4$, a spin structure on the quadric with a proper Riemannian metric, see [4].

According to general theory [8], two immersions of M into N, which are homotopic one to another, give rise to isomorphic pin structures on M. The circle $\mathbb{S}_1$ is known to have two inequivalent spin structures: the trivial one, $\mathrm{Spin}_1 = \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathrm{U}_1 \to \mathrm{U}_1 = \mathrm{SO}_2 \to \mathbb{S}_1$ and the non-trivial structure, corresponding to the 'squaring' map, $\mathrm{Spin}_2 = \mathrm{U}_1 \xrightarrow{sq} \mathrm{U}_1 = \mathrm{SO}_2$. Up to homotopy, all immersions of $\mathbb{S}_1$ into $\mathbb{R}^2$ are classified by integers: with $n \in \mathbb{Z}$, $n \neq 0$, there is associated the class represented by the immersion $\mathrm{U}_1 \to \mathbb{C} = \mathbb{R}^2$ given by $z \mapsto z^n$. One can easily verify that the spin structures on the circle corresponding to immersions with n even (resp., odd) are trivial (resp., non-trivial).

6.2. Construction

Let

$$\mathrm{Spin}_{m+1} \to Q' \xrightarrow{\chi'} P' \xrightarrow{\pi'} N \tag{22}$$

be a spin structure on N and let $\pi : P \to M$ be the O_m-bundle of all orthonormal frames on the hypersurface M immersed by f isometrically in N. Define the map $f' : P \to P'$ so that if $p = (p_\mu) \in P$ and $x = \pi(p)$, then the frame $f'(p) = (f'_i(p))$ at $f(x) \in N$ $(i = 1, \ldots, m+1)$ is given by

$$f'_\mu(p) = (T_x f)(p_\mu) \quad \text{for} \quad \mu = 1\,, \ldots\,, m$$

and $f'_{m+1}(p)$ is a unit vector at $f(x)$, orthogonal to $T_x f(T_x M)$ and oriented in such a way that $f'(p) \in P'$. It is clear that f' is an injection. Let

$\iota : \mathrm{Pin}_{0,m} \to \mathrm{Spin}_{m+1}$ be as in (10) for $k = 0$ and $l = m$ and let $\kappa : \mathrm{O}_m \to \mathrm{SO}_{m+1}$ be the corresponding monomorphism of the orthogonal groups, $\kappa \circ \rho = \rho \circ \iota$. For every $A \in \mathrm{O}_m$ and $p \in P$, one has $f'(pA) = f'(p)\kappa(A)$. Therefore, the pair (f, f') is a morphism of principal bundles. The $\mathrm{pin}_{0,m}$-structure on M is now given as a bundle $\chi : Q \to P$ induced from the bundle $\chi' : Q' \to P'$ by the map f'. Explicitly,

$$Q = \{(p, q') \in P \times Q' : f'(p) = \chi'(q')\}, \quad \chi(p, q') = p .$$

The action of $\mathrm{Pin}_{0,m}$ on Q is given by

$$(p, q')a = (p\rho(a), q'\iota(a)) \tag{23}$$

so that $\chi((p, q')a) = \chi(p, q')\rho(a)$ for every $a \in \mathrm{Pin}_{0,m}$.

The pin structure on a hypersurface M, constructed in this manner, is said to be *induced* by the immersion $f : M \to N$.

Let $\gamma' : \mathrm{Spin}_{m+1} \to \mathrm{GL}\,(S)$ be a spinor representation and let $\psi' : Q' \to S$ be a spinor field on N of type γ'. It follows from (23) that its restriction ψ to Q, $\psi(p, q') = \psi'(q')$, is a spinor field on M of type $\gamma = \gamma' \circ \iota$.

6.3. Spinors on hypersurfaces in Euclidean spaces

As an important special case, consider a hypersurface M immersed in the Euclidean space $\mathbb{R}^{m+1}$ with its standard flat proper-Riemannian metric.

Proposition 7. *Let f be an isometric immersion of a hypersurface M in the Euclidean space $\mathbb{R}^{m+1}$. The pin-structure on M, induced by the immersion,*

$$\mathrm{Pin}_{0,m} \to Q \xrightarrow{\chi} P \xrightarrow{\pi} M ,$$

is such that the bundle of Dirac (m even) or Pauli (m odd) spinors on M is trivial.

Since the spin structure (22) of the ambient space reads now

$$\mathrm{Spin}_{m+1} \to \mathbb{R}^{m+1} \times \mathrm{Spin}_{m+1} \to \mathbb{R}^{m+1} \times \mathrm{SO}_{m+1} \to \mathbb{R}^{m+1} ,$$

the map $f' : P \to \mathbb{R}^{m+1} \times \mathrm{SO}_{m+1}$, defined in the preceding paragraph, can be written as $f' = (f \circ \pi, F)$, where

$$F : P \to \mathrm{SO}_{m+1} \tag{24}$$

satisfies $F(pA) = F(p)\kappa(A)$ for every $p \in P$ and $A \in \mathrm{O}_m$. Let $k : P \to M \times \mathrm{SO}_{m+1}$ be the map $k = (\pi, F)$. The definition of Q can be simplified to read

$$Q = \{(x, a) \in M \times \mathrm{Spin}_{m+1} : (x, \rho(a)) \in k(P)\} .$$

Since k is injective, the projection $\chi : Q \to P$ is well-defined. If $p = \chi(x, a)$, then the definition of Q implies

$$F(p) = \rho(a) . \tag{25}$$

The action of $\mathrm{Pin}_{0,m}$ on Q is now given by $(x, a)b = (x, a\iota(b))$, where $(x, a) \in Q$ and $b \in \mathrm{Pin}_{0,m}$. The bundle $Q \times_\iota \mathrm{Spin}_{m+1} \to M$, obtained by extending the structure group $\mathrm{Pin}_{0,m}$ of Q to Spin_{m+1}, is isomorphic with the trivial bundle $M \times \mathrm{Spin}_{m+1} \to M$: an isomorphism is given by $[((x, a), a')] \mapsto (x, aa')$, where $(x, a) \in Q$ and $a' \in \mathrm{Spin}_{m+1}$. The spinor representation $\gamma' : \mathrm{Spin}_{m+1} \to \mathrm{GL}(S)$ extends $\gamma = \gamma' \circ \iota$. For m even (resp., odd), one takes γ' to be a Pauli (resp., Weyl) representation, so that γ is the Dirac (resp., Pauli) representation. Applying Prop. 4 to the present case, with $G = \mathrm{Pin}_{0,m}$ and $G' = \mathrm{Spin}_{m+1}$, one obtains that the bundle of spinors $Q \times_\gamma S \to M$ is trivial. Let $\psi : Q \to S$ be a spinor field of type γ on M, *i.e.* $\psi(x, a\iota(b)) = \gamma(b^{-1})\psi(x, a)$ for every $(x, a) \in Q$ and $b \in \mathrm{Pin}_{0,m}$. The map $Q \to S$, given by $(x, a) \mapsto \gamma'(a)\psi(x, a)$, is constant on the fibers of $Q \to M$. There thus exists a map

$$\Psi : M \to S \tag{26}$$

such that

$$\Psi(x) = \gamma'(a)\psi(x, a) \tag{27}$$

for every $(x, a) \in Q$. Conversely, for every map (26), the *Schrödinger transformation* (27) defines a spinor field ψ of type $\gamma = \gamma' \circ \iota$ on M. An equivalent way of defining the map Ψ associated with the spinor field ψ of type γ is to consider the latter's extension ψ' to the trivial bundle $M \times \mathrm{Spin}_{m+1} \to M$ such that $\psi'(x, a) = \psi(x, a)$ for every $(x, a) \in Q$ and $\psi'(x, ab) = \gamma'(b^{-1})\psi'(x, a)$ for every $x \in M$ and $a, b \in \mathrm{Spin}_{m+1}$. If $s : M \to M \times \mathrm{Spin}_{m+1}$ is the standard section $s(x) = (x, 1)$, then

$$\Psi = \psi' \circ s .$$

7. A formula for the Dirac operator on orientable hypersurfaces

7.1. The general case

Assume, for simplicity, that the hypersurface M, immersed isometrically in $\mathbb{R}^{m+1}$, is connected, orientable and has been oriented by distinguishing a connected component SP of its bundle P of all orthonormal frames. The pin structure on M, induced by the immersion f, can be now restricted to the group Spin_m by taking $SQ = \chi^{-1}(SP)$, see (13) and (14).

The map (24) restricted to SP defines the *Gauss map* $n : M \to \mathbb{S}_m \subset \mathbb{R}^{m+1}$ of unit normals to M. It is given by $n(\pi(p)) = F(p)e_{m+1}$ and, by virtue of (25), for every $(x, a) \in Q$, one has

$$n(x) = a\, e_{m+1}\, a^{-1}\,, \tag{28}$$

where the product on the right is given by multiplication in $\mathrm{Pin}_{0,m+1}$. Let γ be a spinor representation of $\mathrm{Pin}_{0,m+1}$ in S; by restriction, it gives rise to representations of its subgroups,

$$\mathrm{Spin}_m \to \mathrm{Spin}_{m+1} \to \mathrm{Pin}_{0,m+1} \xrightarrow{\gamma} \mathrm{GL}(S)\,.$$

Since the first two arrows are standard injections, there is now no need to introduce a separate notation for the restrictions and to distinguish γ and γ' as in Prop. 4. In particular, for every $i = 1, \ldots, m+1$, one has the Dirac matrix $\gamma_i = \gamma(e_i)$ and

$$\text{if} \quad \sigma_{ij} = \gamma_i\gamma_j + \delta_{ij}, \quad \text{then} \quad \sigma_{ij} + \sigma_{ji} = 0 \tag{29}$$

for $i, j = 1, \ldots m+1$.

The following identities are useful:

$$\gamma_i\sigma_{jk} = \gamma_{[i}\gamma_j\gamma_{k]} + \delta_{ik}\gamma_j - \delta_{ij}\gamma_k\,, \tag{30}$$

$$\sigma_{ij}\sigma_{kl} = \gamma_{[i}\gamma_j\gamma_k\gamma_{l]} + \delta_{ik}\sigma_{jl} - \delta_{jk}\sigma_{il} + \delta_{jl}\sigma_{ik} - \delta_{il}\sigma_{jk} + \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\,, \tag{31}$$

where it is understood that there is antisymmetrization over the indices included in square brackets. By applying γ to both sides of (28), one obtains, for every $(x, a) \in Q$,

$$\boldsymbol{n}(x) = \gamma(a)\gamma_{m+1}\gamma(a^{-1}), \quad \text{where} \quad \boldsymbol{n} = \gamma \circ n = \sum_i n_i\gamma_i\,. \tag{32}$$

According to Prop. 6, the modified Dirac operator D maps a spinor field ψ of type γ into a spinor field of the same type. There thus exists a linear differential operator $\mathcal{D}$, the *Schrödinger transform* of D, acting on maps from M to S, such that

$$\gamma(a)(D\psi)(x, a) = (\mathcal{D}\Psi)(x)\,, \tag{33}$$

where $(x, a) \in Q$ and Ψ is given by (27). Symbolically,

$$\mathcal{D} = \gamma(a)D\gamma(a^{-1})\,.$$

Since D anticommutes with γ_{m+1}, one obtains from (32)

$$\mathcal{D}n + n\mathcal{D} = 0 . \tag{34}$$

Let ∂_i denote the (constant) vector field on $\mathbb{R}^{m+1}$ given by differentiation with respect to the ith Cartesian coordinate x_i, *i.e.* $\partial_i \varphi = e_i \lrcorner \, \mathrm{d}\varphi$ for every function φ on $\mathbb{R}^{m+1}$. The field of unit normals, $n = \sum_i n_i \, \partial_i$, defines a collection of $\frac{1}{2}m(m+1)$ vector fields $n_i \partial_j - n_j \partial_i \quad (1 \le i < j \le m+1)$ tangent to the hypersurface.

To determine explicitly the Schrödinger transform $\mathcal{D}$ of the modified Dirac operator, consider the extension ψ' of a spinor field of type γ on M defined at the end of Sec. 6.3. The connection form on SQ extends to a spin_{m+1}-valued connection form (ω'_{ij}) on $M \times \mathrm{Spin}_{m+1}$. With the conventions of Eq. (29), the covariant exterior derivative of ψ' is

$$\mathrm{hor}\, \mathrm{d}\psi' = \mathrm{d}\psi' - \tfrac{1}{4} \sum_{i,j} \sigma_{ij} \, \omega'_{ij} \, \psi' .$$

The connection form pulled back to M by the standard section s is

$$\omega_{ij} = s^* \omega'_{ij} = n_i \, \mathrm{d}n_j - n_j \, \mathrm{d}n_i .$$

Let $\varepsilon = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_{m+1}$ denote the canonical volume form on $\mathbb{R}^{m+1}$ and let $\varepsilon_i = e_i \lrcorner \, \varepsilon$ so that $\mathrm{d}\psi \wedge \varepsilon_i = \varepsilon \, \partial_i \psi$ and $\omega_{kl} \wedge \varepsilon_i = (n_k \partial_i n_l - n_l \partial_i n_k) \, \varepsilon$ on M. Noting that the m-form $n \lrcorner \, \varepsilon$ is the volume form of the hypersurface, using an expression of the Dirac operator with the help of differential forms [9, 24] and pulling it back to M by the section s, one obtains

$$(n \lrcorner \, \varepsilon) \mathcal{D} \Psi = \sum_{i,j} \sigma_{ij} \, n_i \, n \lrcorner \left(\left(\mathrm{d}\Psi - \tfrac{1}{4} \sum_{k,l} \sigma_{kl} \, \omega_{kl} \, \Psi \right) \wedge \varepsilon_j \right) .$$

With the help of Eqs (30) and (31) one finds

$$\mathcal{D} = \sum_{i,j} \sigma_{ij} \, n_i \, \partial_j + \tfrac{1}{2} \mathrm{div}\, n , \tag{35}$$

where the 'intrinsic divergence' div is given by

$$\mathrm{div}\, n = \sum_{i,j} (\delta_{ij} - n_i \, n_j) \, \partial_i \, n_j .$$

The differential operator $\sum_{i,j} \sigma_{ij} n_i \, \partial_j$ has been studied by Delanghe and Sommen [21] who refer it to an unpublished thesis by Lounesto; see also [22, 23] and the bibliography given there. The divergence term in (35) is essential for $\mathcal{D}$ to correspond to the intrinsic (modified) Dirac operator on M. In the special case when M is the hyperplane given by $x_{m+1} = 0$, one has $n_i = \delta_{i,m+1}$ and $\mathcal{D}$ reduces to $\gamma_{m+1} \sum_{\mu=1}^{m} \gamma_\mu \partial_\mu$.

7.2. The case of odd-dimensional hypersurfaces

If the hypersurface M is odd-dimensional, $m = 2\nu - 1$, and orientable, then it is enough to consider its bundle of Pauli spinors, which is of fiber dimension $2^{\nu-1}$. Let γ_i, $i = 1, \ldots, 2\nu$, be the Dirac matrices associated with the ambient space $\mathbb{R}^{2\nu}$ and put $\gamma_{2\nu+1} = \gamma_1 \cdots \gamma_{2\nu}$. The matrix $\gamma_{2\nu+1}$ is unchanged by the Schrödinger transformation; its eigenvectors are Pauli spinors and the space of Cartan spinors S, associated with M, decomposes into the sum $S_+ \oplus S_-$ of spaces of Pauli spinors. Since $\gamma_{2\nu+1}$ commutes with the operator $\mathcal{D}$ (and also with D), it suffices to consider eigenfunctions of $\mathcal{D}$ with values in one or the other space of Pauli spinors; see *Remark 2* in Sec. 4.3; it applies, *mutatis mutandis*, to all odd-dimensional hypersurfaces in $\mathbb{R}^{m+1}$. Taking the Dirac matrices γ_i in the form described in part *(i)* of Prop. 2 one can write

$$\boldsymbol{n} = \begin{pmatrix} 0 & \boldsymbol{n}^- \\ \boldsymbol{n}^+ & 0 \end{pmatrix},$$

where, for every $x \in \mathbb{R}^{m+1}$, $\boldsymbol{n}^\pm(x) : S_\pm \to S_\mp$. The matrices σ_{ij}, corresponding to even elements of the Clifford algebra, preserve the helicity,

$$\sigma_{ij} = \begin{pmatrix} \sigma_{ij}^+ & 0 \\ 0 & \sigma_{ij}^- \end{pmatrix}.$$

The same is true of the Dirac operator,

$$\mathcal{D} = \begin{pmatrix} \mathcal{D}^+ & 0 \\ 0 & \mathcal{D}^- \end{pmatrix},$$

and (34) gives

$$\mathcal{D}^\pm \boldsymbol{n}^\mp + \boldsymbol{n}^\mp \mathcal{D}^\mp = 0\,.$$

If $\Phi : M \to S_+$ is an eigenfunction of $\mathcal{D}^+$ with eigenvalue λ, then $\boldsymbol{n}^+\Phi : M \to S_-$ is an eigenfunction of $\mathcal{D}^-$ with eigenvalue $-\lambda$.

7.3. The case of a foliation of $\mathbb{R}^{m+1}$ by hypersurfaces

Consider an open subset U of $\mathbb{R}^{m+1}$ foliated by a family of hypersurfaces. The Gauss map defines now a field n on U. Let $\partial/\partial r = \sum_i n_i \partial_i$ be the derivative along n. Define the classical Dirac operator in $\mathbb{R}^{m+1}$ as

$$\boldsymbol{\partial} = \sum_i \gamma_i\, \partial_i\,.$$

A simple computation, based on (29) gives

$$\boldsymbol{n}\,\boldsymbol{\partial} = \mathcal{D} - \left(\frac{\partial}{\partial r} + \tfrac{1}{2}\operatorname{div} n\right) . \tag{36}$$

Similar formulae, expressing the split of the Dirac operator D on a manifold with boundary into parts tangential and transversal to the boundary, are used in the index theory of D [7, 25].

For m odd, in the notation of Sec. 7.2, one has

$$\boldsymbol{\partial} = \begin{pmatrix} 0 & \boldsymbol{\partial}^- \\ \boldsymbol{\partial}^+ & 0 \end{pmatrix} .$$

Eq. (36) gives now

$$\boldsymbol{n}^{\mp}\boldsymbol{\partial}^{\pm} = \mathcal{D}^{\pm} - \left(\frac{\partial}{\partial r} + \tfrac{1}{2}\operatorname{div} n\right) .$$

8. Applications

8.1. *The spectrum of the sphere* $\mathbb{S}_m$

Let the integer m be > 1. The set $U = \{x = (x_i) \in \mathbb{R}^{m+1} : x \neq 0\}$ is foliated by the spheres $r = \text{const.} > 0$, where $r = (x_1^2 + \ldots + x_{m+1}^2)^{1/2}$. Since now $n_i = x_i/r$, one has $\operatorname{div} n = m/r$. The differential operators $r(n_i\partial_j - n_j\partial_i)$ generalize the operator $\vec{L} = \vec{r} \times \vec{p}$ of 'orbital angular momentum' and the operator $r\sum_{i,j}\sigma_{ij}n_i\partial_j$ corresponds to the 'spin-orbit coupling' term $\vec{\sigma}\vec{L}$ of quantum mechanics in $\mathbb{R}^3$. If $r = 1$, then, by virtue of (31),

$$\Big(\sum_{i,j}\sigma_{ij}x_i\partial_j\Big)^2 + (m-1)\sum_{i,j}\sigma_{ij}x_i\partial_j + \Delta = 0 ,$$

where Δ is the Laplace operator on $\mathbb{S}_m$. Together with (35), this gives

$$(\mathcal{D} - \tfrac{1}{2}m)(\mathcal{D} + \tfrac{1}{2}m - 1) + \Delta = 0 .$$

If $\Psi : \mathbb{S}_m \to S$ is an eigenfunction of $\mathcal{D}$, then it is also an eigenfunction of Δ. Every eigenvalue of Δ on $\mathbb{S}_m$ is known to be of the form $-l(l+m-1)$ for some $l = 0, 1, \ldots$. Therefore, if $\mathcal{D}\Psi = \lambda\Psi$, then $(\lambda - \frac{1}{2}m)(\lambda + \frac{1}{2}m - 1) = l(l+m-1)$, *i.e.* either $\lambda = l + \frac{1}{2}m$ or $\lambda = -l - \frac{1}{2}m + 1$. If $\mathcal{D}\Psi = (-\frac{1}{2}m + 1)\Psi$, then $\Delta\Psi = 0$; therefore, Ψ is a constant and (35) gives $\mathcal{D}\Psi = \frac{1}{2}m\Psi$. Since

$m \neq 1$, the equality $(-\frac{1}{2}m+1)\Psi = \frac{1}{2}m\Psi$ implies $\Psi = 0$ and so the number $-\frac{1}{2}m+1$ is not an eigenvalue: the spectrum of $\mathcal{D}$ on $\mathbb{S}_m$ is contained in the set $\{\pm(l+\frac{1}{2}m) : l = 0, 1, \ldots\}$.

(i) m even. To show that every element of this set is an eigenvalue and to compute its multiplicity, assume first that m is *even*, $m = 2\nu$ and let S be the 2^ν-dimensional space of spinors. Consider the space $H_{m,l}(S)$ of S-valued harmonic polynomials on $\mathbb{R}^{m+1}$, homogeneous of degree l. Since $H_{m,l}(S) = H_{m,l}(\mathbb{C}) \otimes S$, one has

$$\dim H_{m,l}(S) = (s_{m,l} + s_{m,l+1}) \dim S\,, \tag{37}$$

where

$$s_{m,l} = \binom{m+l-1}{l}\,.$$

Let $\boldsymbol{x}$ be the linear map of multiplication of a function $\Phi : \mathbb{R}^{m+1} \to S$ by $\sum_i \gamma_i x_i$, *i.e.* $(\boldsymbol{x}\,\Phi)(x) = \sum_i \gamma_i x_i\, \Phi(x)$. The validity of the following identity is easy to check:

$$\boldsymbol{\partial}\, \boldsymbol{x}^2 - \boldsymbol{x}^2\, \boldsymbol{\partial} = 2\boldsymbol{x}\,. \tag{38}$$

If Φ is a harmonic polynomial of degree l, then the functions $\boldsymbol{\partial}(\boldsymbol{x}\,\Phi)$ and $\boldsymbol{x}\,(\boldsymbol{\partial}\,\Phi)$ are also such polynomials.

Lemma 1. *For every $\Phi \in H_{m,l}(S)$ one has*

$$(\boldsymbol{\partial}\,\boldsymbol{x} + \boldsymbol{x}\,\boldsymbol{\partial})\,\Phi = -(2l+m+1)\,\Phi\,, \tag{39}$$

$$\boldsymbol{\partial}(\boldsymbol{\partial}\,\boldsymbol{x} + 2)\,\Phi = 0\,. \tag{40}$$

Proof. Since Φ is homogeneous of degree l, the Euler identity reads $\sum_i x_i \partial_i \Phi = l\Phi$. Using Eq. (29), one obtains

$$\begin{aligned}\sum_{i,j} \gamma_i \gamma_j (\partial_i x_j + x_i \partial_j)\Phi &= \sum_{i,j} (\sigma_{ij} - \delta_{ij})(\delta_{ij} + x_j \partial_i + x_i \partial_j)\Phi \\ &= -(2l+m+1)\Phi\,.\end{aligned}$$

Since Φ is harmonic, $\boldsymbol{\partial}^2 \Phi = 0$, and

$$\boldsymbol{\partial}^2(\boldsymbol{x}\,\Phi) = -\sum_{i,j} \partial_i^2\,(x_j \gamma_j \Phi) = -2\sum_i \gamma_i\, \partial_i\, \Phi = -2\boldsymbol{\partial}\Phi. \qquad \square$$

Lemma 2. *The sequence*

$$\ldots \xrightarrow{\boldsymbol{\partial}} H_{m,l+1}(S) \xrightarrow{\boldsymbol{\partial}} H_{m,l}(S) \xrightarrow{\boldsymbol{\partial}} H_{m,l-1}(S) \xrightarrow{\boldsymbol{\partial}} \ldots \xrightarrow{\boldsymbol{\partial}} S \xrightarrow{\boldsymbol{\partial}} 0 \tag{41}$$

is exact and there is a decomposition

$$H_{m,l} = H'_{m,l}(S) \oplus H''_{m,l}(S)\,, \tag{42}$$

where

$$H'_{m,l}(S) = \{\Phi \in H_{m,l}(S) : \partial\Phi = 0\}$$

is the kernel of ∂ *and*

$$H''_{m,l}(S) = \{\boldsymbol{x}\Phi : \Phi \in H'_{m,l-1}(S)\}\,.$$

Proof. To show that the sequence is exact, one notices that $\partial H_{m,l+1}(S) \subset H'_{m,l}(S)$; if $\Phi \in H'_{m,l}(S)$, then, by (39), $\Phi = -(2l+m+1)^{-1}\partial(\boldsymbol{x}\Phi)$, *i.e.* $H'_{m,l}(S) \subset \partial H_{m,l+1}(S)$. By virtue of Eq. (40), the vector space $H''_{m,l}(S)$ is a subspace of $H_{m,l}(S)$ and the map $\boldsymbol{x} : H'_{m,l-1}(S) \to H''_{m,l}(S)$ is an isomorphism of vector spaces. The sum (42) is direct because if $\Phi \in H'_{m,l}(S) \cap H''_{m,l}(S)$, then Eqs (38) and (39) give $2\Phi = -(2l+m+1)\Phi$, thus $\Phi = 0$. To show that $H_{m,l} \subset H'_{m,l}(S) \oplus H''_{m,l}(S)$ one writes, as a consequence of (39),

$$(\partial\boldsymbol{x} + 2)\Phi + \boldsymbol{x}\,\partial\Phi = -(2l+m-1)\Phi\,.$$

According to (40), $(\partial\boldsymbol{x} + 2)\Phi \in H'_{m,l}(S)$. Since Φ is harmonic, $\boldsymbol{x}\,\partial\Phi$ is in $H''_{m,l}(S)$. Moreover, $m > 1$ and $l \geq 0$ imply $2l + m - 1 > 0$. □

By virtue of (36), if $\Phi' \in H'_{m,l}(S)$, then the restriction Ψ' of Φ' to the unit sphere is an eigenfunction of $\mathcal{D}$,

$$\mathcal{D}\Psi' = (l + \tfrac{1}{2}m)\Psi' \quad \text{and} \quad \Psi'(-x) = (-1)^l\Psi'(x)\,.$$

for $l = 0, 1, \ldots$.

Similarly, if $\Phi'' \in H''_{m,l+1}(S)$ then, by virtue of (39), the restriction Ψ'' of Φ'' to the unit sphere satisfies

$$\mathcal{D}\Psi'' = -(l + \tfrac{1}{2}m)\Psi'' \quad \text{and} \quad \Psi''(-x) = (-1)^{l+1}\Psi''(x)$$

for $l = 0, 1, \ldots$.

According to Lemma 2 the vector spaces $H'_{m,l}(S)$ and $H''_{m,l+1}(S)$ are isomorphic and

$$\dim H_{m,l+1}(S) = \dim H'_{m,l}(S) + \dim H'_{m,l+1}(S)\,. \tag{43}$$

Writing $\dim H'_{m,l}(S) = p_{m,l}\dim S$ and using (37) and (43), one obtains

$$p_{m,l} + p_{m,l+1} = s_{m,l} + s_{m,l+1}\,.$$

Since $p_{m,0} = s_{m,0} = 1$, one has $p_{m,l} = s_{m,l}$ for every $l = 0, 1, \ldots$.

(ii) m odd. If $m = 2\nu + 1$, then $S = S_+ \oplus S_-$ and the spaces of Pauli spinors $S_\pm$ are 2^ν-dimensional. Spinor fields on $\mathbb{S}_{2\nu-1}$ can be identified with maps from $\mathbb{S}_{2\nu-1}$ to one of the spaces of Pauli spinors, say $S = S_+$, but it is convenient to consider sequences such as

$$\ldots \to H_{m,l+1}(S_-) \xrightarrow{\partial^-} H_{m,l}(S_+) \xrightarrow{\partial^+} H_{m,l-1}(S_-) \to \ldots .$$

They are used to prove suitable modifications of Lemmas 1 and 2. For example, $H'_{m,l}(S_+)$ is now defined as the kernel of ∂^+ and $H''_{m,l}(S_+)$ as the image of $H'_{m,l-1}(S_-)$ by $\boldsymbol{x}^-$.

Irrespective of the parity of m, every eigenfunction of Δ on $\mathbb{S}_m$ is known to be the restriction of a harmonic polynomial in $\mathbb{R}^{m+1}$; therefore, every eigenfunction of $\mathcal{D}$ on $\mathbb{S}_m$ belongs to the restriction of either $H'_{m,l}(S)$ or $H''_{m,l}(S)$ for some $l = 0, 1, \ldots$. To summarize, one has

Proposition 8. *Let $m = 2\nu$ (resp., $m = 2\nu + 1$), where ν is a positive integer. The spectrum of the Dirac (resp., Pauli) operator on $\mathbb{S}_m$ is the set $\{\pm(l + \frac{1}{2}m) : l = 0, 1, \ldots\}$. Each of the eigenvalues $l + \frac{1}{2}m$ and $-l - \frac{1}{2}m$ occurs with the multiplicity $2^\nu \binom{m+l-1}{l}$. Let S be the 2^ν-dimensional space of Dirac (resp., Pauli) spinors and let $\Psi : \mathbb{S}_m \to S$ be an eigenfunction of the Schrödinger transform of the modified Dirac operator. If $\mathcal{D}\Psi = (l + \frac{1}{2}m)\Psi$, then Ψ is the restriction of $\partial(\boldsymbol{x}\Phi)$ to $\mathbb{S}_m$, where Φ is an S-valued harmonic polynomial on $\mathbb{R}^{m+1}$, homogeneous of degree l. If $\mathcal{D}\Psi = -(l + \frac{1}{2}m)\Psi$, then Ψ is the restriction of $\boldsymbol{x}(\partial\Phi)$, where Φ is a similar polynomial of degree $l + 1$.*

The same result on the spectrum and its multiplicity is quoted in [26] and derived in [27], by a method different from the one presented here; see also [28].

8.2. Application to real projective spaces

The simple description of the spectrum of the Dirac operator on spheres, given in the preceding paragraph, can be used to find the corresponding results for real projective spaces. Since a real projective space is locally isometric to its covering sphere, a spinor field on $\mathbb{P}_m$ is an eigenfunction of the Dirac operator D if, and only if, it descends from a corresponding eigenfunction on $\mathbb{S}_m$. Therefore, the spectrum of D on $\mathbb{P}_m$ is contained in that of D on $\mathbb{S}_m$.

By comparing (15) with (16), one sees that the total space $\mathrm{Spin}_{2\nu+1}$ defining the pin structures on $\mathbb{P}_{2\nu}$ is the same as the total space defining the spin structure on $\mathbb{S}_{2\nu}$. Let $\gamma : \mathrm{Spin}_{2\nu+1} \to \mathrm{GL}\,(S)$ be the Pauli representation and let $\varpi' : \mathrm{Spin}_{2\nu+1} \to \mathbb{P}_{2\nu}$ be the projection defined in part

B *(ii)* of Sec. 4.3. If $\Phi : \mathbb{P}_{2\nu} \to S$, then $\psi : \mathrm{Spin}_{2\nu+1} \to S$ defined by $\psi(a) = \gamma(a^{-1})\Phi(\varpi'(a))$ is a spinor field of type $\gamma \circ i_{\pm}$. Referring to the last sentence of *Remark 1*, one sees that every spinor field on $\mathbb{P}_{2\nu}$ can be so obtained. Therefore, every spinor field on $\mathbb{P}_{2\nu}$ comes from an *even* function $\Psi : \mathbb{S}_{2\nu} \to S$. A similar analysis applies to the case $m \equiv 3 \bmod 4$.

Proposition 9. *Let $m = 2\nu$, where ν is a positive integer (resp., $m = 2\nu+1$, where ν is a positive odd integer). The spectrum of the Dirac (resp., Pauli) operator on $\mathbb{P}_m$ is the set*

$$\{-\tfrac{1}{2} \pm (2l + \tfrac{1}{2}m + \tfrac{1}{2}) : l = 0, 1, \ldots\}.$$

The eigenvalue λ occurs in the spectrum with the multiplicity

$$2^{\nu}\binom{|\lambda| + \frac{1}{2}m - 1}{|\lambda| - \frac{1}{2}m}.$$

Note that, in this case, the spectrum is asymmetric.

Most of the work reported in this paper was done during my numerous visits to Trieste and Brussels. I thank Paolo Budinich, Paweł Nurowski, Michel Cahen, and Simone Gutt for their interest and helpful discussions.

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