DISSIPATIVE QUANTUM HYDRODYNAMICS*

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Dedicated to the memory of Professor Jan Rzewuski

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The concise discussion of various aspects of dissipative quantum hydrodynamics is given. Particular attention is paid to the Gisin formulation of dissipative quantum mechanics.

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The concept of *quantum hydrodynamics* is used to describe various formulations of very different physical systems. The common features of all of them is that the equations of motion, describing time evolution of the relevant degrees of freedom of the system, have the form similar to the hydrodynamic formulation of the Schrödinger equation proposed back in 1926 by Madelung [1]. Thus we talk about quantum hydrodynamics in context of simple quantum mechanics, as proposed by its founding father, and we talk about it discussing dynamics of superfluid helium isotopes [2, 3, 4]. Somehow unexpected is that the dynamics of purely classical one dimensional magnetic systems can also be cast into the quantum hydrodynamic form [5].

Consider simple quantum mechanical system described by means of the Schrödinger equation:

$$i\hbar\partial_t\psi(r,t) = -\frac{\hbar^2}{2m}\nabla^2\psi(r,t) + V(r,t)\psi(r,t)\,, \tag{1}$$

where m is the particle mass, and $V(r,t)$ is an external potential of the forces influencing the particle motion. Following Madelung substitute $\psi(r,t) \to$

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$\sqrt{\rho(r,t)}\exp(i\phi(r,t))$ and then separate Eq. (1) into its real and imaginary part. Resulting equations read:

$$\begin{aligned} \partial_t \rho(r,t) &= -\nabla\cdot\rho(r,t)\boldsymbol{u}(r,t)\,, \\ \partial_t \boldsymbol{u}(r,t) + \boldsymbol{u}(r,t)\cdot\nabla\boldsymbol{u}(r,t) &= -\nabla\mu_Q(\rho,\nabla\rho) - \tfrac{1}{m}\nabla V(r,t)\,, \end{aligned} \tag{2}$$

where $\boldsymbol{u}(r,t) = \hbar/m\nabla\phi(r,t)$ is the (potential) velocity field of the "quantum" fluid and μ_Q is the quantum chemical potential. It is this last term which distinguish the Madelung fluid dynamic equations from the usual Euler equations of hydrodynamics. Indeed $\mu_Q = -(\hbar^2/2m)\rho^{-1}(\nabla^2\rho - 1/2\rho(\nabla\rho)^2)$ is the only quantity in Eq. (1) containing Planck constant, and it measures the "internal quantum pressure" responsible for wave packet spreading. Alternatively the gradient of the quantum chemical potential can be written as derivative of the quantum stress tensor $\sigma_{ij}^Q = \hbar^2/4m\partial_i\partial_j\ln(\rho)$. Note also that the dependence of the quantum chemical potential μ_Q on density, and its gradients, is essentially different than in generalization of conventional hydrodynamics often used in the theory of phase transitions, where the inhomogeneity of the order parameter (density in the van der Waals like theories) are of importance [6, 7, 8]. The other important difference between the Madelung and Euler hydrodynamics is that the value of circulation

$$\Gamma = \oint \boldsymbol{u}\cdot d\boldsymbol{r} = n\frac{\hbar}{m}\,, \tag{3}$$

is quantized. The benefits and/or shortcomings of the Madelung formulation of quantum mechanics are discussed by Białynicki–Birula *et al.*, *viz.* Ref. [1].

The technically almost trivial, but physically deep and far-reaching, is the generalization of the Madelung formulation proposed by Gross in connection with the mean field description of interacting many-boson systems [3]. The easiest way of deriving this quantum hydrodynamic is to consider second-quantization equation of motion for the boson field operators $\hat\psi(r,t)$ stemming from the second-quantization Hamiltonian

$$\begin{aligned} \hat H =& \frac{\hbar^2}{2m}\int dr\nabla\hat\psi^\dagger(r,t)\cdot\nabla\hat\psi(r,t) \\ &+ \tfrac{1}{2}\int\int drdr'\hat\psi^\dagger(r,t)\hat\psi^\dagger(r',t)U(r-r')\hat\psi(r,t)\hat\psi(r',t)\,, \end{aligned} \tag{4}$$

where $U(r-r')$ is the interparticle interaction potential. Evaluating then the coherent states $|\alpha\rangle$ mean value of that equation[1], and splitting resulting

[1] Coherent states are defined as right eigenstates of the field operator $\hat\psi$, that is $\hat\psi(r,t)|\alpha\rangle = \alpha(r,t)|\alpha\rangle$.

equation into real and imaginary parts, we obtain equation analogous to the Madelung equations (2) with density ρ given as the absolute value of α and velocity as the gradient of its phase [4]. The main difference between the Madelung equation and the Gross equation (being bosonic field version of the Ginzburg–Landau equation, known from superconductivity and phase transition physics) is in the replacement of the external potential V by the self-consistent potential $\mathcal{V}(\rho) = V(r,t) + \int dr U(r-r')\rho(r',t)$.

Another formulation of the quantum hydrodynamics can be obtain by considering the Schrödinger and Gross equation as "classical" field equations of motion for fields described by the Hamiltonians:

$$\mathcal{H}\{\alpha,\alpha^*\} = \int dr \left(\frac{\hbar^2}{2m}|\nabla\alpha|^2 + V(r)|\alpha(r)|^2 \right) \tag{5}$$

or

$$\mathcal{H}\{\alpha,\alpha^*\} = \int dr \left(\frac{\hbar^2}{2m}|\nabla\alpha|^2 + \tfrac{1}{2}U(r-r')|\alpha(r)|^2|\alpha(r')|^2 \right) \tag{6}$$

for Schrödinger and Gross equations, respectively.
The equations of motion can be obtained from the above Hamiltonian considering field $\alpha(r)$, and its complex conjugate $\alpha^*(r)$, as classical field with the Lie–Poisson bracket

$$\{\alpha(r), \alpha^*(r')\} = \tfrac{1}{i\hbar}\delta(r-r')\,. \tag{7}$$

It is relatively straightforward to rewrite the above formal setup, of either Madelung or Gross hydrodynamics, in terms of density and velocity fields [4].

$$\mathcal{H}(\rho, \boldsymbol{u}) = \int dr \tfrac{1}{2}\rho \boldsymbol{u}^2 + \mathcal{V}(\rho) + \frac{\hbar^2}{2m}\int dr\, (\nabla\sqrt{\rho})^2\,, \tag{8}$$

where $\mathcal{V}(\rho)$ is the interaction part of the Hamiltonian. The Lie–Poisson brackets for these fields are formally equivalent to original Landau proposal [2], and have been thoroughly discussed in [9]. Quite remarkable is the fact that one encounters similar construction in purely classical models of one dimensional magnetism theory [5]. Indeed, the one dimensional Heisenberg ferromagnet, described by the spin Hamiltonian:

$$\mathcal{H} = -J\sum_i \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}\,, \tag{9}$$

where summation is over the lattice sites, can be converted, in the continuum limit, into nonlinear Schrödinger equation, which in turn can be cast into fluid form with the Hamiltonian written below, where the density ρ equals to the magnetic chain energy density (per unit chain length) and the velocity field τ equals to the energy density current divided by ρ.

$$\mathcal{H}(\rho,\tau) = \tfrac{1}{2}\rho\tau^2 - \tfrac{1}{8}\rho^2 + \tfrac{1}{2}(\partial_R\sqrt{\rho})^2\,, \tag{10}$$

where R is the continuous position along the chain. Note that in addition to the "quantum pressure" term the Hamiltonian density contains quadratic in density contribution which leads to a negative pressure term in fluid dynamic interpretation. Existence of that term is responsible for many unusual properties of the Heisenberg chain dynamic, for example its complete integrability, existence of solitons *etc.* [5].

There are many applications of the above outlined quantum hydrodynamics. Its main shortcomings, when applied to the many body systems rather than considered as "fundamental" theory, is that it describes symplectic — dissipation free — dynamics. While it is not obvious why we should add a dissipative term to the Schrödinger equation (1), it is clear that terms like that should show up in the description of even superfluid liquid.

In spite of many efforts the theory of quantum dissipative systems is far from being completely understood [10, 11]. Recently Enz [12] has given comprehensive review of the possible quantization procedures for dissipative systems. Particularly interesting proposal for description of the quantum dissipative systems was given by Gisin [13]. In order to describe the evolution of the quantum state $|\psi\rangle$ of a *dissipative* system Gisin introduced a phenomenological equation

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle + i\lambda\left(\frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} - \hat{H}\right)|\psi\rangle\,, \tag{11}$$

in which $\hat{H}$ is the system Hamiltonian and $\lambda \geq 0$ is the dimensionless damping constant. The structure of the bracketed term on the rhs of the Gisin equation ensures that the norm of the state vector is preserved during the system evolution. That property distinguishes Gisin equation from several other dissipative quantum mechanical "generalizations" of the Schrödinger equation [14] and permits for retaining most of the conventional interpretation of the quantum mechanics. Other interesting property of the equation (11) is that the time evolution of the original Hamiltonian eigenstates is conservative (no damping). When the initial wave packed, consisting of several of those eigenstates, evolves in time then it will eventually reach the final state which will be the lowest eigenstate present in the initial wave packet. That implies that when one considers the evolution of a coherent state [15, 16, 17] then the system is *always* proceeding towards the ground state.

Now, the damping, nonlinear term in Eq.(11) can be obtained following the rules of the metriplectic formulation of classical dynamics [9, 12] by supplementing the antisymmetric Lie–Poisson bracket (7) by a symmetric part

$$\{\{\alpha(r), \alpha^*(r')\}\} = \frac{\lambda}{\hbar}\left(\delta(r-r') - \frac{\alpha(r)\alpha^*(r')}{||\alpha||^2}\right) . \tag{12}$$

The resulting metriplectic bracket $[\alpha, \beta] = \{\alpha, \beta\} + \{\{\alpha, \beta\}\}$ is the sum of expressions (7) and (12). The Gisin equation (11) is now easily obtained on evaluation of the metriplectic bracket of the $\alpha(r,t)$ with the Hamiltonian (5).

As stated before the standing of the Gisin generalization of the Schrödinger wave mechanics *as a fundamental theory* is not free from serious difficulties and is open for criticism. As seen from more applied view point it appears as a coarse grained description of many body system in which many irrelevant degrees of freedom were "averaged" over to provide dissipation accounted for by the damping constant λ [11]. This later interpretation of the Gisin equations works well in the magnetism theory. Indeed, the Gisin description is equivalent to the Gilbert–Landau–Lifshitz one for damped magnetic system. We shall illustrate that analyzing dynamics of an anisotropic quantum spin chain [16, 18, 19].

Consider collection of N spins $\boldsymbol{S}$ located on a one-dimensional chain and described by the Heisenberg Hamiltonian

$$\hat{H} = -\sum_{n=1}^{N}\left[J\hat{\boldsymbol{S}}_n\hat{\boldsymbol{S}}_{n+1} + B\hat{S}_n^Z + C(\hat{S}_n^Z)^2\right] , \tag{13}$$

where $J(>0)$ is a ferromagnetic exchange constant, $B(>0)$ is an external magnetic field, $(C>0)$ is the value of the local magnetic anisotropy. Summation in Eq. (13) runs over all chain sites.

We begin by assuming that the quantum state $|\psi\rangle$ of the whole chain is a direct product of *spin coherent states* $|\mu_n\rangle$ for each spin in the chain

$$|\psi\rangle = \bigotimes_{n=1}^{N} |\mu_n\rangle . \tag{14}$$

The coherent state of each individual spin is defined, using the conventional polar angles representation of spins, as follows [20]:

$$|\mu_n\rangle = |\theta_n, \phi_n\rangle = (1+|\mu_n|^2)^{-S}\exp(\mu_n\hat{S}^-)|0_n\rangle , \tag{15}$$

where $\mu_n = \tan(\theta_n/2)\exp(i\phi_n)$, $\hat{S}_n^{\pm} = \hat{S}_n^X \pm i\hat{S}_n^Y$ and $|0_n\rangle$ is the ground state of a single n-th spin, *i.e.* $\hat{S}_n^Z|0_n\rangle = S|0_n\rangle$. From equation (11) it follows [18] that the time evolution of the mean value for any operator $\hat{A}$ obeys (the generalized Ehrenfest equation of motion)

$$\frac{d}{dt}\langle\psi|\hat{A}|\psi\rangle = \left\langle\psi\left|\partial_t\hat{A}\right|\psi\right\rangle + \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{A}]|\psi\rangle - \frac{\lambda}{\hbar}\Big(\langle\psi|[\hat{H},\hat{A}]_+|\psi\rangle - 2\langle\psi|\hat{H}|\psi\rangle\langle\psi|\hat{A}|\psi\rangle\Big)\,, \qquad (16)$$

where $[\bullet,\bullet]$ and $[\bullet,\bullet]_+$ denote commutator and anticommutator, respectively, and we have assumed that $\langle\psi|\psi\rangle = 1$.

Using expression (16) for operators $\hat{S}_n^+$ and the Hamiltonian (13), taking into account the parametrization (15) and going over to the continuum limit we obtain two nonlinear partial *differential* equations [18] for the evolution of parameters $\theta(R,t)$ and $\phi(R,t)$ (here R is the spatial variable along the chain). The surprising result is that this system of equations is equivalent to system of classical equations arising from Gilbert–Landau–Lifshitz theory [18], when one writes these equations with the help of standard spherical parametrization (θ,ϕ) of spin vectors. The only difference is that one needs to replace in quantum equation the factors $C(S-\frac{1}{2})$ by CS.

The equivalence of the Gilbert–Landau–Lifshitz description of damped Heisenberg model with the Gisin formulation of dissipative quantum mechanics follows from the use of the coherent states representation. Although this procedure has been widely used in the field of magnetism following its successful application in quantum optics, recent results shed some doubts on its limits of validity in description of magnetic systems [21].

It interesting to see how the Gisin equation looks like in the Madelung representation. Using again the hydrodynamic representation of the wave function $\psi(r,t) \to \sqrt{\rho(r,t)}\exp(i\phi(r,t))$ we can obtain from Eq. (11) the hydrodynamic equations of quite unusual form. First the continuity equation (*cf.* Eq. (2a) reads now:

$$\partial_t\rho(r,t) + \nabla\cdot\rho(r,t)\boldsymbol{u}(r,t) = \frac{2\lambda}{\hbar}\left[\rho\langle H\rangle - \left\{\tfrac{1}{2}\rho\boldsymbol{u}^2 + \frac{\hbar^2}{2m}(\nabla\sqrt{\rho})^2 + \rho V\right\}\right]. \qquad (17)$$

where $\langle H\rangle = \langle\psi|\hat{H}|\psi\rangle/\langle\psi|\psi\rangle$. Although this is rather odd looking equation it still describes conservation of probability, for the integral of the right hand side of it vanishes identically for *each* wave function ψ on virtue of Eq. (8). For $\lambda \to 0$ we recover the Madelung continuity equation. The classical limit ($\hbar \to 0$) is a bit more tricky. It is easier to discuss it in "many body" interpretation of the Gisin equation where the damping coefficient λ is a quantity which one should have derived from the complete theory, and, therefore, it is given as a an averaged over those system degrees of freedom which are assumed to be less relevant than these described by the wave function ψ. In this case the value of λ will be "generically" proportional to $\hbar^2$. The rhs of Eq. (17) will then vanish in classical limit.

The Madelung fluid Euler equation is now, of course, dissipative, and again one finds here difference as compared with, what one has in mind talking about dissipative fluid dynamics, namely the Navier–Stokes equation. Indeed we have

$$\partial_t \boldsymbol{u}(r,t)+\boldsymbol{u}(r,t)\cdot\nabla \boldsymbol{u}(r,t) = -\frac{1}{m}\nabla V - \nabla\mu_Q + \frac{\hbar\lambda}{2m}\nabla^2\boldsymbol{u} + \frac{\hbar\lambda}{2m}\nabla\left[\boldsymbol{u}\cdot\nabla\ln(\rho)\right] . \quad (18)$$

The rhs of this equation contains two terms familiar from the Madelung formulation which survive the limit $\lambda \to 0$. Two last terms describe quantum dissipation. The first term $(\hbar\lambda/2m)\nabla^2\boldsymbol{u}$ looks like the usual viscous damping term known from the Navier–Stokes theory. One is tempted therefore to call the coefficient $\eta_Q = \hbar\lambda/2m$ the quantum kinematic viscosity. The physical interpretation of that term is quite similar to that in classical theory as it describes the transfer of momentum from faster moving parcel of the "fluid" into those moving less rapidly. The second term $\propto \nabla[\boldsymbol{u}\cdot\nabla\ln(\rho)]$ is this one which has no classical analogy. It describes damping of momentum which gets stronger the higher are the gradients of the density field.

Eqs (17), (18) form the set of dissipative quantum hydrodynamics equations which are essentially different from the Madelung theory and also from these one would obtain from the theory proposed by Kostin [14]. The difference stem from broken Galilean and gauge invariance of the Kostin formulation, in which the dissipation is constructed such as to reproduce the linear velocity damping on the level of the Ehrenfest equations. This is not the case in our model for the rhs of Eq. (18) is just the full gradient, thus the momentum is conserved in our model. Kostin formulation shares some similarity with Galilean noninvariant mesoscopic level hydrodynamics proposed for description of two dimensional fluid adsorbates in order to produce finite values of two dimensional transport coefficients [22].

As shown above one can reformulate recent theory of quantum dissipative systems following the hydrodynamic, or fluid like, picture proposed almost 70 years ago by Madelung. As with many of these formal procedures they usefulness for solving, or even elucidate, fundamental problems can be questioned. For variety of applications they may be more useful, as shown by example from magnetism theory [18] (*cf.* also [11]). There are several new areas of the many body physics where the Gisin formulation, and its hydrodynamic interpretation given here, might be useful. Some of them were mentioned in [12], other include the few boson systems in an atomic trap. Objects like that are recently of great interest in view of their potential role in explaining the physics of the Bose–Einstein condensation in weakly interacting systems.

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