LYAPUNOV EXPONENTS ANALYSIS OF AUTONOMOUS AND NONAUTONOMOUS SETS OF ORDINARY DIFFERENTIAL EQUATIONS*

K. Grygiel and P. Szlachetka

Nonlinear Optics Division, Institute of Physics
Adam Mickiewicz University
Grunwaldzka 6, 60-780 Poznań, Poland

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We compare the Lyapunov exponents for nonautonomous and autonomous versions of the same dynamical system governed by a set of ordinary differential equations (ODE), for a large class of physical systems admitting of the extraction of explicitly time-dependent terms in ODE. We have found some advantages of the Lyapunov analysis in the nonautonomous version. The main advantage is that we are able to solve the problem of Lyapunov exponents even though the time-dependent external force is nondifferentiable. Optical Kerr effect in a cavity with an external time-dependent field is considered numerically as an example.

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1. Introduction

Interest in the general theory of stability [1–2] and in particular that of Lyapunov stability [3–4] has a long and illustrious history. The spectrum of Lyapunov exponents has proved to be a most useful diagnostic providing a qualitative characterisation of the dynamical behavior of systems. There are several methods of calculating Lyapunov exponents for discrete maps as well as for continuous flow [5–9]. A comparison of the different methods for computing Lyapunov exponents has been performed in Ref. [7]. The methods have been tested with respect to their accuracy and efficiency.

In this paper we carry out a Lyapunov analysis of non-Hamiltonian systems with explicitly time-dependent parameters. Physically this means, for example, dissipative systems with time-dependent external forces. The

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dynamics of such systems are governed by a nonautonomous set of ordinary differential equations (NODE). Contrary to NODE, an autonomous set of ordinary differential equations (AODE) describes dynamical systems without explicitly time-dependent parameters. However, it is always possible formally to autonomize NODE into AODE. Therefore, a given nonautonomous dynamical system can be studied by using the above two cases of ODE. The main aim of this paper is to compare the Lyapunov exponents analysis for NODE and AODE within the same dynamical model.

In Section 2 we consider the maximal Lyapunov exponent method as well as Wolf's et al procedure [9] for the calculation of the whole spectrum of exponents for a continuous flow. We find some advantages in the case of NODE analysis leading to certain theoretical predictions. The latter are tested numerically in Section 3 for the dynamical system of an anharmonic oscillator with external modulated field in a cavity (optical Kerr effect). Finally, Section 4 contains some concluding remarks.

2. Lyapunov Exponents for NODE and AODE

We consider an arbitrary dynamical system. If we have a time-dependent external force, or time-dependent damping, *etc.*, we can write a set of ordinary differential equations (ODE) in non-autonomous version (NODE)

$$\dot{x}_i = \mathcal{F}_i^N(x_1, \ldots, x_n, t), \qquad i = 1, \ldots, n, \tag{1}$$

or the respective autonomized version (AODE)

$$\begin{aligned} \dot{x}_i &= \mathcal{F}_i^A(x_1, \ldots, x_n, x_{n+1}), \qquad i = 1, \ldots, n, \\ \dot{x}_{n+1} &= 1. \end{aligned} \tag{2}$$

Here x_i is a coordinate in the i-th direction of some point on the trajectory $\{x_i(t)\}$ in n-dimensional phase space. Obviously, the dimension of the phase space in the case of AODE is increased by one due to the formal elimination of the independent variable t by putting $t \to x_{n+1}$. The two cases (1) and (2) are physically equivalent and lead to analogous results. The evolution governed by (1) in n-dimensional space is the same as the evolution governed by (2) in the n-subspace of the $(n+1)$-dimensional complete space. The coordinate x_{n+1} is a "dummy" variable and is sometimes referred to as the non-chaotic variable. Although the difference in nonautonomous and autonomous notation of the same dynamical system is but formal, the Lyapunov analysis in these two cases exhibits some advantages of the nonautonomous approach which, according to our knowledge, have as yet not been disclosed.

To calculate the spectrum of Lyapunov exponents we have to linearize the set of equations (1) or (2) in so-called tangent space [5–9]:
(NODE)

$$\dot{\delta x_i} = \sum_{j=1}^{n} \frac{\partial \mathcal{F}_i^N}{\partial x_j} \delta x_j\,, \qquad i = 1, \ldots, n\,. \tag{3}$$

(AODE)

$$\dot{\delta x_i} = \sum_{j=1}^{n+1} \frac{\partial \mathcal{F}_i^A}{\partial x_j} \delta x_j\,, \qquad i = 1, \ldots, n+1\,. \tag{4}$$

In other words, we linearize the system (1) or (2) along the trajectory $\{x_i(t)\}$. The variable δx_i is the principal axis of the n-dimensional (or $(n+1)$-dimensional for AODE) ellipsoid attached to the trajectory in the point $x_i(t)$ and describes the expanding or contracting nature of the system in phase space.

We restrict ourselves to the case when we can extract from $\mathcal{F}_i^N$ those terms which are only time-dependent (they may be referred to as a "force"). Thus, we can re-write the sets (1) and (2) as follows:
(NODE)

$$\dot{x}_i = F_i^N(x_1, \ldots, x_n) \,+\, f_i(t)\,, \tag{5}$$

(AODE)

$$\begin{aligned} \dot{x}_i &= F_i^A(x_1, \ldots, x_n) \,+\, f_i(x_{n+1})\,, \\ \dot{x}_{n+1} &= 1 \quad \text{and} \quad x_{n+1}(0) = 0\,. \end{aligned} \tag{6}$$

From (5) and (6) one easily obtains the set of linearized ODE for δx_i in tangent space. Remembering that now we have

$$F_i^N = F_i^A = F_i\,, \tag{7}$$

we get from Eqs (3) and (4) the following set of linearized equations:
(NODE)

$$\dot{\delta x_i} = \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j} \delta x_j\,, \tag{8}$$

(AODE)

$$\begin{aligned} \dot{\delta x_i} &= \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j} \delta x_j \,+\, \frac{\partial f_i}{\partial x_{n+1}} \delta x_{n+1}\,, \\ \dot{\delta x}_{n+1} &= 0\,. \end{aligned} \tag{9}$$

The term $\frac{\partial f_i}{\partial x_{n+1}}\,\delta x_{n+1}$ with the additional equation $\dot{\delta x}_{n+1}=0$ occurs only in the autonomous case, Eq. (9). Here appears the first disadvantage of the autonomous approach. The function f_i must possess its first derivative at any point. This requirement is not satisfied in the case when the force f_i is, for example, in the form of sharp pulses. The existence of the first derivative, however, is not necessary in the nonautonomous case, Eq. (8). The linearized equations (8) involved no term similar to $\frac{\partial f_i}{\partial x_{n+1}}\,\delta x_{n+1}$ due to the differentiation by x_i $(i=1...n)$ performed above, and because t is an independent variable. The set (8) for NODE is the same as the linearized equations in the absence of the force f_i.

Since δx_{n+1} is constant in time, we can write (9) in the following form:
(AODE)

$$\dot{\delta x}_i=\sum_{j=1}^{n}\frac{\partial F_i}{\partial x_j}\,\delta x_j+\frac{\partial f_i}{\partial x_{n+1}}\,\delta x_{n+1}(0)\,. \tag{10}$$

Generally, the maximal Lyapunov exponent (MLE) for a continuous flow is defined in the form:

$$\lambda_{\mathbf{max}}=\lim_{t\to\infty}\sup\tfrac{1}{t}\,\ln\|\delta x\|\,, \tag{11}$$

where $\|\ldots\|$ is the norm (any norm) of the vector $\delta x(t)=\{\delta x_i(t)\}$. It seems that the Euclidean norm is the most popular one.

The MLE characterizes a system from the stability point of view. Chaos involves the existence of a positive Lyapunov exponent.

A more detailed analysis is given by the whole spectrum of Lyapunov exponents [5–9]. On writing (8) and (9) in matrix form we obtain the Jacobi matrix of linearized equations in tangent space:
(NODE)

$$J^N_{n\times n}=\begin{pmatrix}\frac{\partial F_1}{\partial x_1} & \ldots & \frac{\partial F_1}{\partial x_j} & \ldots & \frac{\partial F_1}{\partial x_n}\\ \ldots & & \ldots & & \ldots\\ \frac{\partial F_i}{\partial x_1} & \ldots & \frac{\partial F_i}{\partial x_j} & \ldots & \frac{\partial F_i}{\partial x_n}\\ \ldots & & \ldots & & \ldots\\ \frac{\partial F_n}{\partial x_1} & \ldots & \frac{\partial F_n}{\partial x_j} & \ldots & \frac{\partial F_n}{\partial x_n}\end{pmatrix}, \tag{12}$$

(AODE)

$$J^A_{(n+1)\times(n+1)}=\begin{pmatrix}\begin{pmatrix}J^N_{n\times n}\end{pmatrix} & \begin{pmatrix}\ldots\\ \frac{\partial f_i}{\partial x_{n+1}}\\ \ldots\end{pmatrix}\\ 0\;\ldots\;\ldots & 0\end{pmatrix}. \tag{13}$$

The Jacobi matrices are used to calculate the spectrum of Lyapunov exponents λ_i in the following way:

$$\lambda_i=\lim_{t\to\infty}\tfrac{1}{t}\,ln\;|\,\chi_i(t)\,|\,, \tag{14}$$

where the $\chi_i(t)$ are the eigenvalues of the matrix

$$\Gamma(t) \equiv \mathrm{Exp}\left[\int_0^t dt'\, J^{N(A)}(t')\right] . \tag{15}$$

In consequence, our eigenvalue problem leads to the solution of the equation

$$\det\left[\Gamma^{N(A)} - \chi I\right] = 0 .$$

There are two essential differences between NODE and AODE. For the autonomous system (9) the Jacobi matrix (13) possesses all the $n+1$ eigenvalues and at least one of them is always equal to zero. Therefore, the spectrum of Lyapunov exponents is given by $(\lambda_1, \lambda_2, ..., \lambda_n, 0)$. For the nonautonomous system (8) the Jacobi matrix is given by (12) and in consequence the spectrum has the form $(\lambda_1, \lambda_2, ..., \lambda_n)$.

Moreover, we wish to stress that for nonautonomous (in contradistinction to autonomous) systems we cannot construct a solution of (8) in the form [10]:

$$\delta x_i \equiv \dot{x}_i . \tag{16}$$

This is due to the presence of a force $f_i(t)$ in (5). The solution (16), however, is necessary to the proof that for each trajectory which does not terminate at a fixed point at least one Lyapunov exponent vanishes [10]. This causes that for AODE $\lambda_{\mathbf{max}} \geq 0$. However, for a nonautonomous system, where the identity (16) is not valid, the maximal Lyapunov exponent can be negative, even if its trajectory does not terminate at a fixed point. Negativity of the MLE by no means constitutes a drawback but is rather natural consequence of our nonautonomous approach. On the other hand, the NODE case of nonlinear systems (5) provides an important advantage: namely, for the NODE case, one is able to calculate the spectrum of Lyapunov exponents even if the force f_i is no longer differentiable. This is so becouse the Jacobi matrix "loses" the f_i term due to its independence of the dynamical variables.

In the next Section, we show numerically the possibilities of the existence of a negative maximal Lyapunov exponent in the analysis of a nonautonomous ODE for a physical situation with trajectory in the form of a limit cycle.

3. Numerical example

In this Section, we shall consider in more detail a continuous flow described by the following set of equations:
(NODE)

$$\begin{aligned}\frac{d\xi_1}{dt} &= -\frac{\gamma}{2}\xi_1 + \xi_1^2\xi_2 + \xi_2^3 + A_0[1 + \sin(wt)]\,,\\ \frac{d\xi_2}{dt} &= -\frac{\gamma}{2}\xi_2 - \xi_1^3 - \xi_1\xi_2^2\,.\end{aligned} \tag{17}$$

The AODE version has the form:

$$\begin{aligned}\frac{d\xi_1}{dt} &= -\frac{\gamma}{2}\xi_1 + \xi_1^2\xi_2 + \xi_2^3 + A_0(1 + \sin(w\xi_3)\,,\\ \frac{d\xi_2}{dt} &= -\frac{\gamma}{2}\xi_2 - \xi_1^3 - \xi_1\xi_2^2\,,\\ \frac{d\xi_3}{dt} &= 1 \quad \text{and} \quad \xi_3(0) = 0\,.\end{aligned} \tag{18}$$

This is the well known case of a nonlinear oscillator in an external field and describes Kerr oscillations in nonlinear medium in a cavity [11–13]. The external electromagnetic field has a time-dependent (modulated with $\sin(\omega t)$) amplitude, and a frequency equal to that of the basic field in the cavity. The field intensity in the cavity is given by $|\xi|^2$, where $\xi = \xi_1 + i\xi_2$.

To calculate the spectrum of Lyapunov exponents and the MLE we need the linearized equations of motion. The Eqs (8) and (9) lead immediately to the following sets of equations:
(NODE)

$$\begin{aligned}\frac{d\delta\xi_1}{dt} &= -\frac{\gamma}{2}\delta\xi_1 + \xi_1^2\delta\xi_2 + 2\xi_1\xi_2\delta\xi_1 + 3\xi_2^2\delta\xi_2\,,\\ \frac{d\delta\xi_2}{dt} &= -\frac{\gamma}{2}\delta\xi_2 - 3\xi_1^2\delta\xi_1 - 2\xi_1\xi_2\delta\xi_2 - \xi_2^2\delta\xi_1\,,\end{aligned} \tag{19}$$

(AODE)

$$\begin{aligned}\frac{d\delta\xi_1}{dt} &= -\frac{\gamma}{2}\delta\xi_1 + \xi_1^2\delta\xi_2 + 2\xi_1\xi_2\delta\xi_1 + 3\xi_2^2\delta\xi_2 + A_0\, w\, cos(w\,\xi_3)\,\delta\xi_3\,,\\ \frac{d\delta\xi_2}{dt} &= -\frac{\gamma}{2}\delta\xi_2 - 3\xi_1^2\delta\xi_1 - 2\xi_1\xi_2\delta\xi_2 - \xi_2^2\delta\xi_1\,,\\ \frac{d\delta\xi_3}{dt} &= 0\,.\end{aligned} \tag{20}$$

Wolf *et al.* method

We have calculated the spectrum of Lyapunov exponents of the above Kerr model for fixed parameters values $A_0 = 3$ (amplitude modulation) and $\gamma = 0.1$ (loss in the cavity). We have employed the algorithm outlined in [9] together with a standard fourth-order Runge–Kutta integration scheme. For the initial condition $\xi_1(0) = \xi_2(0) = 1.0$ we have numerically solved the Eqs (17) together with (19) for NODE and Eqs (18) together with (20) for AODE. As a "knob" parameter we used the frequency of modulation w, varying w in steps of $\Delta w = 0.01$ in the range $0 \le w \le 4.5$. The step length used in the integration was $\Delta t = 0.01$ throughout which, for our purposes, it gave sufficiently accurate results.

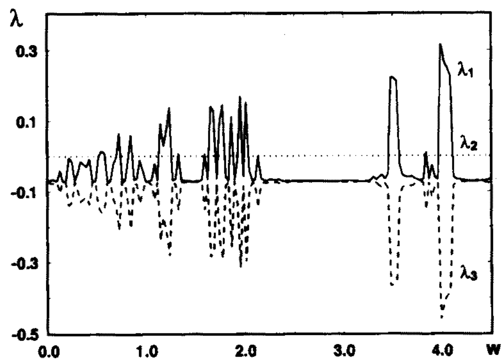

Fig. 1. The Lyapunov spectrum $\lambda_1, \lambda_2, \lambda_3$ for AODE (Eqs (18) and (20)).

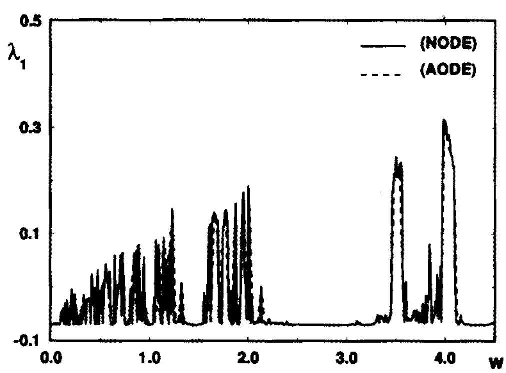

Fig. 2. Lyapunov exponents λ_1 for NODE and AODE ($\Delta w = 0.03$).

Our results are displayed in Fig. 1, where we plot the spectrum of Lyapunov exponents (λ_1, λ_2, λ_3) for AODE. The numerical analysis of the spectrum for NODE exhibits a lack of λ_3. The detailed comparison of the

NODE and AODE cases is displayed in Fig. 2. We note very good agreement between the nonautonomous and autonomous case of the ODE solutions. Some differences appear due to numerical errors, mainly in regions where the Lyapunov exponents change rapidly. We observe a similar behavior of the second Lyapunov exponents λ_2 for both cases.

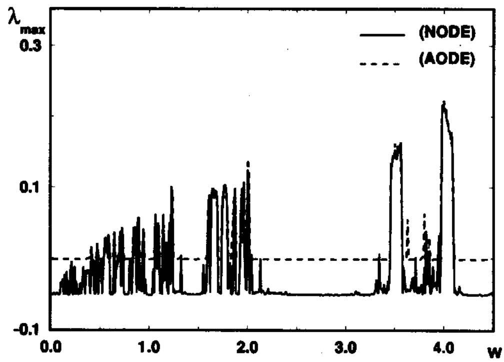

Fig. 3. Maximal Lyapunov exponents λ_{max} for NODE and AODE.

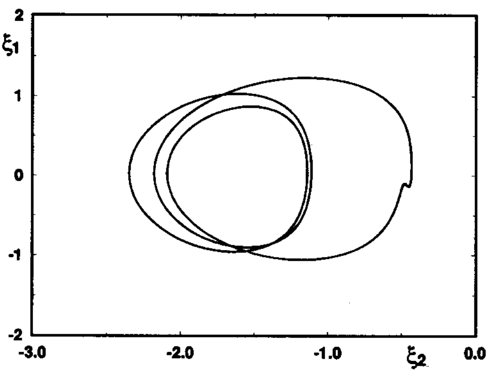

Fig. 4. A limit cycle for NODE (the time $150 < t < 170$ and $w = 1.0$). The value of the MLE equals $\lambda_{\max} = -0.0513$. The respective Lyapunov spectrum is given by $(-0.0718, -0.0724)$.

Maximal Lyapunov exponent method

We have used the sets for NODE and AODE to calculate the MLE directly from Eq. (11). Fig. 3 shows the results and one immediately notes the good agreement between the two cases of the solution of ODE for $\lambda_{\max} \geq 0$. However, in the AODE case, we lose all information about the rate at which the perturbed system becomes indistinguishable from the attractor (fixed point or limit cycle). This information is given by negative exponents, but in the kind of systems considered here the minimal value of MLE in the autonomized version of ODE is zero. For example, a limit cycle for negative Lyapunov exponents in the NODE case is plotted in Fig. 4.

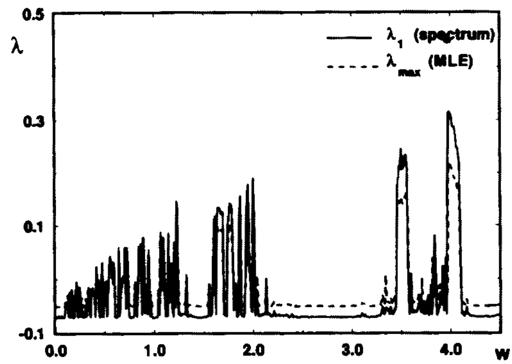

Fig. 5. Comparison of the first (biggest) Lyapunov exponent λ_1 from the spectrum and the maximal Lyapunov exponent λ_{max} calculated directly from Eq. (11) (both for NODE).

In contrast, Fig. 5 shows the biggest Lyapunov exponent λ_1 in the spectrum and the MLE — both obtained for NODE. Differerences appear due to the different methods of calculation. Precisely, in the calculation of the whole spectrum of Lyapunov exponents we used Gram–Schmidt method of reorthonormalization [9]. These differences were expected but it is easy seen from Fig. 5 that both methods lead to exactly the same regions of chaos. The behaviour of both λ_1 and $\lambda_{\max}$ versus the parameter w is the same (with a small scaling factor).

4. Concluding remarks

The main aim of this paper was to compare two approaches (autonomous and nonautonomous version of ODE) for the computation of the spectrum of Lyapunov exponents as well as the MLE for a given continuous flow. As an example, we have considered a system well known in nonlinear optics with Kerr nonlinearity pumped by an external, time-dependent field. We are well aware that AODE is much more popular in theoretical and numerical investigations of dynamical systems. However, on performing calculations for the NODE and AODE we have found certain advantages of the NODE approach. We close with the following remarks:

(1) The computation procedure of the Lyapunov exponent is simpler for the NODE case [Eqs (5) and (8)] than for the AODE case [Eqs (6) and (9)]. The NODE approach involves two equations less and no terms of the form $\frac{\partial f_i}{\partial x_{n+1}} \delta x_{n+1}$.

(2) To calculate the Lyapunov exponents in the case of AODE one needs the derivate of the function $f_i(t)$ in Eq. (9). If $f_i(t)$ is nondifferentiable in certain points, the calculation procedure of Lyapunov exponents for the AODE case cannot be used. However, we are able to solve the problem in the NODE case. For example, we have recently investigated a system generating the second-harmonic of light with an external field of rectangular pulses [14, 15].

(3) For the autonomous equations and the autonomized version of nonautonomous equations the maximal Lyapunov exponent λ_{max} always satisfies the condition $\lambda_{max} \geq 0$. This condition is not satisfied for nonautonomous system. As we mentioned earlier, the magnitude of the MLE provides information about the dynamics of the attractors. The MLE is a measure of the rate at which the system provides or refuses to provide information in the course of its evolution [9]. The average rate at which information contained in the transients is lost can be determined from the negative exponent and is only possible to obtain by NODE calculation of the MLE. Of course, there are no such problems when considering the whole spectrum of the system.

(4) Fig. 3 shows that for NODE, in chaotic regions, MLE is less that λ_1 (the biggest exponent from the spectrum) whereas in regions of order MLE is greater that λ_1. However, both methods lead to exactly the same regions of chaos and order.

(5) The comparison of the differences between the NODE and AODE versions of Lyapunov analysis for the same system can be a very useful tool for checking numerical algorithms and procedures.

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