LAGRANGE TRIANGLE OF QUARKS*

W. Królikowski

Institute of Theoretical Physics, Warsaw University
Hoża 69, 00–681 Warszawa, Poland

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An approximate model is proposed for a system of three Schrödinger particles of equal masses, interacting mutually through a universal two-body potential. They are assumed to form during their motion a (generally) varying equilateral triangle corresponding to Lagrange's exact triangle solution of the classical three-body problem. The resulting wave equation is formally a two-body Schrödinger equation (in the centre-of-mass frame). This is applied to three constituent quarks in the nucleon. The presented model, called "Lagrange triangle of Schrödinger particles", may be considered as a nonrelativistic approximation to the much more complicated "Lagrange triangle of Dirac particles" constructed by the author a decade ago.

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1. Introduction

Since Lagrange's golden epoch in analytic mechanics a few special solutions are known for the classical (nonrelativistic) three-body problem with Newtonian gravitational attraction, first of all, the highly symmetrical solution where three interacting bodies (particles) of arbitrary masses form, during the motion, an equilateral triangle of (generally) varying size and orientation in a plane [1]. Then, in the centre-of-mass frame, the particles describe three coplanar conics, all with the same eccentricity and one common focus located at their centre of mass.

As is known, Lagrange's exact solutions appear also in the context of the restricted three-body problem, where two particles circle around each other, and are treated as two moving external centres of gravitational attraction for a light particle remaining in the same plane. Then, this third

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particle, if situated at (or close to) one of two stable libration points usually denoted byL_4 and L_5, gives exactly (or approximately) two Lagrange's triangle solutions. In the context of the solar system dominated by the Sun and Jupiter as two heaviest bodies, there are two familiar groups of asteroids called Trojans, located in proximity of the Lagrange points L_4 and L_5 (their total number exceeds 160, *ca.* 2/3 and 1/3 of them being grouped around L_4 and L_5, respectively).

In the case of universal static two-body interactions *other* than gravitational (as *e.g.* the static color attractions between classical quarks forming pairwise color-antitriplet states), Lagrange's triangle solution still exists, but requires the three masses to be *equal.*

A decade ago the classical triangle solution inspired us to propose an approximate quantum model for a highly symmetrical configuration of three Dirac particles such *e.g.*as three quarks in the nucleon (we called this model *Lagrange triangle of Dirac particles*) [2]. Because of its three Dirac bispinor indices the model is still very complicated, unless some simplifying assumptions concerning Dirac's degrees of freedom are made [2]. In order to make our idea more operable, we discuss in the present note a nonrelativistic quantum model that may be called *Lagrange triangle of Schrödinger particles.* Then, the model is perturbed by spin-orbit and spin-spin interactions introduced on the level of Pauli approximation.

The Reader may find in Ref. [3] a recent interesting proposal of application of the classical Lagrange points L_4 and L_5 to the quantum physics of semiclassical Rydberg atomic states.

2. Schrödinger equation for Lagrange triangle

In the case of equal masses $m_1 = m_2 = m_3$ $(\equiv m)$, the centre-of-mass and relative coordinates and their canonical momenta are

$$\vec{R} = \tfrac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3)\ ,\ \vec{r} = \vec{r}_1 - \vec{r}_2\ ,\ \vec{\rho} = \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2) - \vec{r}_3\ , \tag{1}$$

and

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3\ ,\ \vec{p} = \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2)\ ,\ \vec{\pi} = \tfrac{1}{3}(\vec{p}_1 + \vec{p}_2 - 2\vec{p}_3)\ , \tag{2}$$

respectively. The orbital angular momentum and kinetic energy are given as

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \vec{R} \times \vec{P} + \vec{r} \times \vec{p} + \vec{\rho} \times \vec{\pi}\ , \tag{3}$$

and

$$T = \sum_i \frac{\vec{p}_i^{\,2}}{2m_i} = \frac{\vec{P}^{\,2}}{2M} + \frac{\vec{p}^{\,2}}{2\mu} + \frac{\vec{\pi}^{\,2}}{2\nu}\ , \tag{4}$$

where

$$M = m_1 + m_2 + m_3 = 3m \,, \; \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{1}{2}m \,,$$
$$\nu = \frac{(m_1 + m_2)m_3}{m_1 + m_2 + m_3} = \frac{2}{3}m \,. \tag{5}$$

In the centre-of-mass frame

$$\vec{P} = 0 \,, \; \vec{p} = \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2) \,, \; \vec{\pi} = \vec{p}_1 + \vec{p}_2 = -\vec{p}_3 \,. \tag{6}$$

The equilateral-triangle condition

$$|\vec{r}_1 - \vec{r}_2| = |\vec{r}_2 - \vec{r}_3| = |\vec{r}_3 - \vec{r}_1| \;\; (\equiv r) \tag{7}$$

implies

$$\vec{r} \cdot \vec{\rho} = 0 \,, \; r = \frac{2}{\sqrt{3}}\rho \,, \tag{8}$$

where $r = |\vec{r}|$ and $\rho = |\vec{\rho}|$. If supplemented by the requirement of coplanarity of motion, it leads in the centre-of-mass frame, where $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$, to the equalities

$$\vec{p}_1^{\,2} = \vec{p}_2^{\,2} = \vec{p}_3^{\,2} \; (\equiv \vec{\pi}^{\,2}) \,, \tag{9}$$

or

$$\vec{p} \cdot \vec{\pi} = 0 \,, \; \vec{p}^{\,2} = \tfrac{3}{4}\vec{\pi}^{\,2} \,, \tag{10}$$

and

$$\vec{r}_1 \times \vec{p}_1 = \vec{r}_2 \times \vec{p}_2 = \vec{r}_3 \times \vec{p}_3 \;\; (\equiv \tfrac{2}{3}\vec{\rho} \times \vec{\pi}) \,, \tag{11}$$

or

$$\vec{r} \times \vec{p} = \vec{\rho} \times \vec{\pi} \,. \tag{12}$$

Thus, the kinetic and potential energies of our triangle take the forms

$$T_{\rm LT} = 3\frac{\vec{\pi}^{\,2}}{2m} = 4\frac{\vec{p}^{\,2}}{2m} \,, \; V_{\rm LT} = 3V\left(\frac{2}{\sqrt{3}}\rho\right) = 3V(r) \,, \tag{13}$$

where $V(|\vec{r}_i - \vec{r}_j|) \;\; (i \neq j)$ is a universal static two-body interaction between particles. Hence, the Schrödinger equation $(T_{\rm LT} + V_{\rm LT})\psi = E_{\rm LT}\psi$ reads

$$\left[4\frac{\vec{p}^{\,2}}{2m} + 3V(r)\right]\psi(\vec{r}) = 3E\psi(\vec{r}) \,, \;\; E_{\rm LT} = 3E \,, \tag{14}$$

with $\vec{p} = -i\partial/\partial\vec{r}$. Here, $\langle\psi|\psi\rangle = 1$ (in its centre-of-mass frame) when states of the triangle are bound. The wave equation (14) *defines* formally the quantum dynamical system that we propose to call Lagrange triangle of Schrödinger particles. Note that this equation can be rewritten in the form of a *two-body* Schrödinger equation (in the centre-of-mass frame):

$$\left[\frac{\vec{p}^{\,2}}{2\mu_{\rm eff}} + V(r)\right]\psi(\vec{r}) = E\psi(\vec{r})\,, \tag{15}$$

where

$$\mu_{\rm eff} = \tfrac{3}{4}m\,, \tag{16}$$

the constant $\mu_{\rm eff} = \frac{1}{2}m_{\rm eff}$ being *formally* the reduced mass of a system of two particles of equal masses $m_{\rm eff} = \frac{3}{2}m = \frac{1}{2}m_{\rm LT}$. This equation gives, of course, the radial equation

$$\left\{\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + 2\mu_{\rm eff}\Big[E_{nl} - V(r)\Big]\right\} r\psi_{nl}(r) = 0\,, \tag{17}$$

when we insert $\psi(\vec{r}) = \psi_{nlm_l}(\vec{r}) = \psi_{nl}(r)Y_{lm_l}(\hat{r})$ with $\hat{r} = \vec{r}/r$. Then, $E = E_{nl}$. Here, the orbital and magnetic quantum numbers l and m_l pertain to the orbital angular momentum of our triangle which, therefore, is identified as

$$\vec{L}_{\rm LT} = \vec{r}\times\vec{p}\,. \tag{18}$$

It may be expressed also as $\vec{\rho}\times\vec{\pi}$ due to Eq. (12).

3. Lagrange triangle of quarks

Take for each pair of three constituent quarks in the nucleon, for instance, a Cornell-type potential equal to the static one-gluon-exchange vector potential plus a linear confining scalar potential *i.e.*,

$$V(r_{ij}) = -\frac{2}{3}\frac{\alpha_{\rm st}}{r_{ij}} + \frac{1}{2}\kappa^2 r_{ij} + \frac{1}{2}C\,, \tag{19}$$

where $r_{ij} = |\vec{r}_{ij}|$, $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ and $\alpha_{\rm st} = g^2_{\rm st}$ $(\hbar = 1 = c)$. Then, we get for our triangle the potential

$$V_{\rm LT}(r) = 3V(r) = 3\left(-\frac{2}{3}\frac{\alpha_{\rm st}}{r} + \frac{1}{2}\kappa^2 r + \frac{1}{2}C\right)\,. \tag{20}$$

Notice that in the potential (19) there is an effective strong coupling constant $-\frac{2}{3}\alpha_{\rm st}$ instead of $-\frac{4}{3}\alpha_{\rm st}$ as it appears in the case of the original Cornell potential for quarkonia (*c.f.* the second Ref. [4], p. 265).

If nonstatic corrections of the order of $(\vec{p}_i/m_i)^2 = \vec{v}_i^2$ with $m_1 = m_2 = m_3$ $(\equiv m)$ are taken into account, the sum of three static central interactions (19) is perturbed in the nucleon by the kinetic-energy correction

$$-\frac{1}{8m^3}\left(\vec{p}_1^{\,4} + \vec{p}_2^{\,4} + \vec{p}_3^{\,4}\right) \tag{21}$$

plus the sum of three Breit–Fermi noncentral interactions [4]

$$\begin{aligned} V_{ij}(\vec{r}_{ij}) =& \frac{2}{3}\alpha_{\rm st}\left\{\pi\delta^3(\vec{r}_{ij}) + \frac{1}{2m^2 r_{ij}}\left[\vec{p}_i\cdot\vec{p}_j + (\hat{r}_{ij}\cdot\vec{p}_i)(\hat{r}_{ij}\cdot\vec{p}_j)\right]\right. \\ &+ \frac{1}{2m^2 r_{ij}^3}\left[(\vec{r}_{ij}\times\vec{p}_i)\cdot(\vec{S}_i + 2\vec{S}_j) - (\vec{r}_{ij}\times\vec{p}_j)\cdot(\vec{S}_j + 2\vec{S}_i)\right] \\ &\left.+ \frac{8\pi}{3m^2}\delta^3(\vec{r}_{ij})\,\vec{S}_i\cdot\vec{S}_j + \frac{1}{m^2 r_{ij}^3}\left[3(\hat{r}_{ij}\cdot\vec{S}_i)(\hat{r}_{ij}\cdot\vec{S}_j) - \vec{S}_i\cdot\vec{S}_j\right]\right\} \\ &- \frac{1}{2}\kappa^2\left\{\frac{1}{2m^2}\left[r_{ij}(\vec{p}_i^{\,2} + \vec{p}_j^{\,2}) - i\hat{r}_{ij}\cdot(\vec{p}_i + \vec{p}_j)\right]\right. \\ &\left.+ \frac{1}{2m^2 r_{ij}}\left[(\vec{r}_{ij}\times\vec{p}_i)\cdot\vec{S}_i - (\vec{r}_{ij}\times\vec{p}_j)\cdot\vec{S}_j\right]\right\}. \end{aligned} \tag{22}$$

Here,

$$\vec{S}_i = \tfrac{1}{2}\vec{\sigma}_i \tag{23}$$

are Pauli spins of quarks and $\hat{r}_{ij} = \vec{r}_{ij}/r_{ij}$. In the centre-of-mass frame, after some calculations, the sum of three spin-dependent parts $V_{ij}^{\rm spin}(\vec{r}_{ij})$ of the interactions (22) reduces for our triangle to the form

$$\begin{aligned} V_{\rm LT}^{\rm spin}(\vec{r}) =& 3V^{\rm spin}(\vec{r}) = \frac{2}{3}\alpha_{\rm st}\left\{\frac{3}{m^2 r^3}(\vec{r}\times\vec{p})\cdot\vec{S}\right. \\ &\left.+ \frac{4\pi}{3m^2}\delta^3(\vec{r})\left(\vec{S}^2 - \frac{9}{4}\right) + \frac{1}{m^2 r^3}\left[3(\hat{r}\cdot\vec{S})^2 - \vec{S}^2\right]\right\} \\ &- \frac{1}{2}\kappa^2\frac{1}{m^2 r}(\vec{r}\times\vec{p})\cdot\vec{S}, \end{aligned} \tag{24}$$

if this sum is used *under* the sign of expectation value $\langle V_{\rm LT}^{\rm spin}(\vec{r})\rangle_{nlsjm_j}$ in the unperturbed state $\psi_{nlsjm_j}(\vec{r})$. Such a state is equal to the Clebsch–Gordan superposition of the products $\psi_{nlsjm_j}(\vec{r})\chi_{sm_s}$ satisfying the spin-independent Schrödinger equation (14) or (15). In Eq. (24)

$$\vec{S}_{\rm LT} = \vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3 \tag{25}$$

is the spin of our triangle, while $m \equiv m_q$ denotes the constituent mass of quarks in the nucleon (we put $m_q = m_u = m_d$).

Of course, the eigenvalues of $\vec{S}^2$ are $s(s+1)$, where $s = 1/2, 3/2$ correspond to $j = |l-s|, \ldots, l+s$ with $l = 0, 1, 2, \ldots$. In the case of $l = 0$, due to $Y_{00}(\hat{r}) = 1/\sqrt{4\pi}$ and $\chi^\dagger_{sm_s}\chi_{sm_s} = 1$, the first-order perturbative spin correction to the energy level $3E_{n0}$ caused by the interaction (24) is

$$3E^{(1)}_{n0ss} = \langle V^{\rm spin}_{\rm LT}(\vec{r})\rangle_{n0ssm_s} = \mp\frac{\alpha_{\rm st}}{3m^2}|\psi_{n0}(0)|^2 \quad \text{for} \quad s = \begin{cases} 1/2 \\ 3/2 \end{cases}, \tag{26}$$

where $\psi_{n0}(r)$ satisfies the radial equation (17) with $l = 0$ (in the case of $l = 0$ we have $\psi_{nlsjm_j}(\vec{r}) = \psi_{nlm_l}(\vec{r})\chi_{sm_s}$ with $j = s$, $m_j = m_s$ and $\psi_{n00}(\vec{r}) = (1/\sqrt{4\pi})\psi_{n0}(r)$). In order to evaluate $|\psi_{n0}(0)|^2$ we may use the exact formula following from Eq. (17):

$$|\psi_{n0}(0)|^2 = 2\mu_{\rm eff}\int_0^\infty r^2\,dr\frac{dV(r)}{dr}|\psi_{n0}(r)|^2 = \frac{3m}{2}\left\langle\frac{dV(r)}{dr}\right\rangle_{n00}, \tag{27}$$

with $\mu_{\rm eff} = (3/4)\,m$ (*cf.* the first Ref. [4], p. 147). For the potential (20) we get

$$\left\langle\frac{dV(r)}{dr}\right\rangle_{n00} = \frac{2\alpha_{\rm st}}{3}\left\langle\frac{1}{r^2}\right\rangle_{n00} + \frac{1}{2}\kappa^2, \tag{28}$$

since $\langle\psi_{nlm_l}|\psi_{nlm_l}\rangle = 1$. Thus, from Eqs. (26), (27) and (28) we obtain

$$M_{n0ss} = 3m + 3E_{n0} \mp \frac{\alpha_{\rm st}}{3m}\left(\alpha_{\rm st}\left\langle\frac{1}{r^2}\right\rangle_{n00} + \frac{3\kappa^2}{4}\right) \quad \text{for} \quad s = \begin{cases} 1/2 \\ 3/2 \end{cases}, \tag{29}$$

where

$$M_{nlsj} = 3m + 3E_{nl} + 3E^{(1)}_{nlsj} \tag{30}$$

is the perturbed mass corresponding to the unperturbed state $\psi_{nlsj}(\vec{r})$ of our triangle of three constituent quarks in the nucleon.

The mass spectrum (29) implies the following mass relations:

$$M_{n\,0\,3/2\,3/2} + M_{n\,0\,1/2\,1/2} = 6m + 6E_{n0}, \tag{31}$$

$$M_{n\,0\,3/2\,3/2} - M_{n\,0\,1/2\,1/2} = \frac{2\alpha_{\rm st}}{3m}\left(\alpha_{\rm st}\left\langle\frac{1}{r^2}\right\rangle_{n00} + \frac{3\kappa^2}{4}\right), \tag{32}$$

and

$$\begin{aligned} M_{n+1\,0\,ss} - M_{n\,0\,ss} &= 3(E_{n+1\,0} - E_{n\,0}) \\ &\mp \frac{\alpha^2_{\rm st}}{3m}\left(\left\langle\frac{1}{r^2}\right\rangle_{n+1\,0\,0} - \left\langle\frac{1}{r^2}\right\rangle_{n\,0\,0}\right) \quad \text{for} \quad s = \begin{cases} 1/2 \\ 3/2 \end{cases}. \end{aligned} \tag{33}$$

Now, apply the relations (31) and (32) to the nucleon-type baryons [5]:

$$1\,{}^2S_{1/2} : N(939)\ ,\ M_N = \ 939\,\mathrm{MeV}\,,$$
$$1\,{}^4S_{3/2} : \Delta(1232)\,,\ M_\Delta = 1232\,\mathrm{MeV}$$

and

$$2\,{}^2S_{1/2} : N(1440)\,,\ M_N^* = 1440\,\mathrm{MeV}\,,$$
$$2\,{}^4S_{3/2} : \Delta(1600)\,,\ M_\Delta^* = (1550\ \text{ to }\ 1700)\,\mathrm{MeV} \sim 1600\,\mathrm{MeV}\,,$$

where $M_\Delta^* = 1600\,\mathrm{MeV}$ is the mass recommended for $\Delta(1600)$ in Ref. [5]. Then

$$m + E_{10} = 362\,\mathrm{MeV}\,, \tag{34}$$
$$m + E_{20} = (498\ \text{ to }\ 523)\,\mathrm{MeV} \sim 507\,\mathrm{MeV}\,, \tag{35}$$
$$\frac{2\alpha_{\mathrm{st}}^2}{3m}\left\langle\frac{1}{r^2}\right\rangle_{100} + \frac{\alpha_{\mathrm{st}}\kappa^2}{2m} = 293\,\mathrm{MeV}\,, \tag{36}$$
$$\frac{2\alpha_{\mathrm{st}}^2}{3m}\left\langle\frac{1}{r^2}\right\rangle_{200} + \frac{\alpha_{\mathrm{st}}\kappa^2}{2m} = (110\ \text{ to }\ 260)\,\mathrm{MeV} \sim 150\,\mathrm{MeV}\,. \tag{37}$$

From Eqs (36) and (37):

$$\frac{2\alpha_{\mathrm{st}}^2}{3m}\left(\left\langle\frac{1}{r^2}\right\rangle_{200} - \left\langle\frac{1}{r^2}\right\rangle_{100}\right) = -(183\ \text{ to }\ 33)\,\mathrm{MeV} \sim -143\,\mathrm{MeV}\,. \tag{38}$$

In Eqs. (35), (37) and (38) the values corresponding to $M_\Delta^* = 1600$ MeV are indicated by the sign $\sim$. Note the minus sign at the rhs of Eq. (38), consistent with the simple intuition.

If the Dirac magnetic moments $e_q/2m_q$ are accepted for quarks, then — in the framework of Schrödinger equation corrected by the Pauli spin coupling — the experimental proton magnetic moment $2.79e/2M_p$ implies for u quark the constituent mass $m_u = M_p/2.79 = 336$ MeV, where $2e_u + e_d = e$ and $M_p = 938$ MeV (we put $m_u = m_d (\equiv m_q)$). Then,

$$m \equiv m_q = 336\,\mathrm{MeV}\,, \tag{39}$$

and so we can infer from Eqs. (34) and (35) that

$$E_{10} = 26\,\mathrm{MeV}\ ,\ E_{20} = (162\ \text{ to }\ 187)\,\mathrm{MeV} \sim 171\,\mathrm{MeV}\,. \tag{40}$$

Note the positive sign of the "binding energies" (40) which shows that the confining part of the Cornell-type potential (20) prevails over its Coulombic part, as far as the unperturbed energy spectrum $E = E_{nl}$ is concerned. In fact, this spectrum may be expressed as follows:

$$E_{nl} = \left\langle \frac{\vec{p}^2}{2\mu_{\rm eff}} \right\rangle_{nlm_l} - \frac{2}{3}\alpha_{\rm st} \left\langle \frac{1}{r} \right\rangle_{nlm_l} + \frac{1}{2}\kappa^2 \langle r \rangle_{nlm_l} + \frac{1}{2}C$$
$$= -\frac{1}{3}\alpha_{\rm st} \left\langle \frac{1}{r} \right\rangle_{nlm_l} + \frac{3}{4}\kappa^2 \langle r \rangle_{nlm_l} + \frac{1}{2}C \,, \qquad (41)$$

due to the virial theorem which in this case reads

$$2\left\langle \frac{\vec{p}^2}{2\mu_{\rm eff}} \right\rangle_{nlm_l} = \left\langle r\frac{dV(r)}{dr} \right\rangle_{nlm_l} = -\left\langle -\frac{2}{3}\frac{\alpha_{\rm st}}{r} \right\rangle_{nlm_l} + \langle \frac{1}{2}\kappa^2 r \rangle_{nlm_l} \,. \qquad (42)$$

Evidently, in the case of $\kappa^2 > 0$ when the confinement works, the expectation values $\langle \frac{1}{r} \rangle_{nlm_l}$ and $\langle r \rangle_{nlm_l}$ in Eq. (41) should not be exactly Coulombic:

$$\left\langle \frac{1}{r} \right\rangle_{nlm_l} = \frac{(2\alpha_{\rm st}/3)\mu_{\rm eff}}{n^2} = \frac{\alpha_{\rm st} m}{2n^2}, \; \langle r \rangle_{nlm_l} = \frac{3n^2 - l(l+1)}{2(2\alpha_{\rm st}/3)\mu_{\rm eff}} = \frac{3n^2 - l(l+1)}{\alpha_{\rm st} m}, \qquad (43)$$

though in such a case $(3/4)\kappa^2 \langle r \rangle_{nlm_l}$ would be large in comparison with $(1/3)\alpha_{\rm st}\langle 1/r \rangle_{nlm_l}$ for $(2/27)\alpha_{\rm st}^3 m^2/\kappa^2$ small enough. In fact, for Coulombic expectation values (43) the formula (41) would give

$$E_{10} = -\frac{\alpha_{\rm st}^2 m}{6} + \frac{9\kappa^2}{4\alpha_{\rm st} m} + \frac{C}{2} \simeq \left(-\frac{\alpha_{\rm st}^2}{18} + \frac{27\,\kappa^2}{4\alpha_{\rm st}\,{\rm GeV}^2} + \frac{C}{2{\rm GeV}} \right) {\rm GeV}$$

with $3m \simeq 1$ GeV, resulting into 0.026 GeV if Eq. (40) was used. For $\alpha_{\rm st}$ and κ^2 as estimated later on in Eqs. (49) and (52), this would imply

$$26\,{\rm MeV} \simeq E_{10} \simeq \left(-67 + 483 + \frac{C}{2\,{\rm MeV}} \right) {\rm MeV} \;\; {\rm or} \;\; \frac{C}{2} \simeq -390\,{\rm MeV}\,,$$

showing a large negative $C/2$.

4. Coulombic approximation for spin structure

At any rate, if for the expectation value $\left\langle \frac{1}{r^2} \right\rangle_{n00}$ of the function $\frac{1}{r^2}$ (quadratically singular at $r = 0$) the Coulombic dependence on n held approximately,

$$\left\langle \frac{1}{r^2} \right\rangle_{n+1\,0\,0} : \left\langle \frac{1}{r^2} \right\rangle_{n\,0\,0} \simeq \frac{1}{(n+1)^3} : \frac{1}{n^3} \qquad (44)$$

(at least when $n = 1$), then Eq. (38) would give

$$\frac{2\alpha_{st}^2}{3m}\left\langle\frac{1}{r^2}\right\rangle_{100} \simeq (209 \text{ to } 37.7)\,\text{MeV} \sim 163\,\text{MeV}. \tag{45}$$

From Eqs. (36) and (45):

$$\frac{\alpha_{st}\kappa^2}{m} \simeq (168 \text{ to } 511)\,\text{MeV} \sim 260\,\text{MeV}. \tag{46}$$

If the Coulombic expectation value

$$\left\langle\frac{1}{r^2}\right\rangle_{nlm_l} = \frac{2(2\alpha_{st}/3)^2\mu_{\text{eff}}^2}{(2l+1)n^3} = \frac{\alpha_{st}^2 m^2}{2(2l+1)n^3} \tag{47}$$

was used approximately, then Eq. (45) would take the form

$$\frac{\alpha_{st}^4 m}{3} \simeq (209 \text{ to } 37.7)\,\text{MeV} \sim 163\,\text{MeV}. \tag{48}$$

With $3m \simeq 1$ GeV (as it follows from Eq. (39)) the estimation (48) would imply that

$$\alpha_{st} \simeq 1.2 \text{ to } 0.76 \sim 1.1\,. \tag{49}$$

Then, Eqs. (45) and (46) would give, respectively,

$$\left\langle\frac{1}{r^2}\right\rangle_{100} \simeq (0.076 \text{ to } 0.032)\,\text{GeV}^2 \sim 0.067\,\text{GeV}^2, \tag{50}$$

or

$$\left\langle\frac{1}{r^2}\right\rangle_{100}^{1/2} \simeq (0.28 \text{ to } 0.18)\,\text{GeV} \sim 0.26\,\text{GeV}, \tag{51}$$

and

$$\kappa^2 \simeq (0.048 \text{ to } 0.22)\,\text{GeV}^2 \sim 0.079\,\text{GeV}^2, \tag{52}$$

or

$$\kappa \simeq (0.22 \text{ to } 0.47)\,\text{GeV} \sim 0.28\,\text{GeV}. \tag{53}$$

Of course, the figure (50) follows also directly from the Coulombic expectation value (47).

The values $\langle 1/r^2\rangle_{100}^{1/2}$ and κ, as estimated in Eqs. (51) and (53), are both of the same order of magnitude $O(0.1)$ GeV, and become roughly equal for the recommended value $M_\Delta^* = 1600$ MeV [5]. Note that the asymptotic behaviour of the wave function in the case of Cornell-type potential (20) is

$\psi_{nlm_l}(\vec{r}) \sim \exp\left[-(\kappa\sqrt{m/3})r^{3/2}\right]$ at $r \to \infty$, what suggests for our system the confining radius $\simeq (3/\kappa^2 m)^{1/3}$. Thus, with $3m \simeq 1\,\mathrm{GeV}$ and $M^*_\Delta = 1600\,\mathrm{MeV}$

$$\text{confining radius} \simeq \left(\frac{3}{\kappa^2 m}\right)^{1/3} \simeq \left(\frac{3\,\mathrm{GeV}}{\kappa}\right)^{2/3} \mathrm{GeV}^{-1} \simeq 4.9\,\mathrm{GeV}^{-1}\,, \tag{54}$$

and so, it is of the same order of magnitude as the relative effective Compton wave length $1/\mu_{\rm eff} = (4/3)(1/m) \simeq 4\mathrm{GeV}^{-1}$. Note that with $3m \simeq 1\,\mathrm{GeV}$ and the estimation (49) for $\alpha_{\rm st}$ the Coulombic expectation values (43) would give the numbers

$$\left\langle\frac{1}{r}\right\rangle^{-1}_{100} \simeq \frac{6}{\alpha_{\rm st}}\,\mathrm{GeV}^{-1} \simeq 5.5\,\mathrm{GeV}^{-1}\,,\ \langle r\rangle_{100} \simeq \frac{9}{\alpha_{\rm st}}\,\mathrm{GeV}^{-1} \simeq 8.2\,\mathrm{GeV}^{-1}\,, \tag{55}$$

the first of them being an effective "Bohr radius".

Of course, a systematic numerical discussion of the Schrödinger equation (15) dependent on four free parameters $\mu_{\rm eff} = (3/4)\,m$, $\alpha_{\rm st}$, κ^2 and C,

$$\left(\frac{\vec{p}^{\,2}}{2\mu_{\rm eff}} - \frac{2}{3}\frac{\alpha_{\rm st}}{r} + \frac{1}{2}\kappa^2 r + \frac{1}{2}C\right)\psi(\vec{r}) = E\psi(\vec{r})\,, \tag{56}$$

is the only direct way to check (in the case of potential (20)) the applicability of our Lagrange triangle of quarks to the physics of nucleon-type baryons and/or Ω-type baryons. The latter contain three strange quarks which, as heavier, are more comfortable than u and d quarks for nonrelativistic approximations. Alternatively, harmonic-oscillator potential may be considered. We intend to proceed with such a program.

Appendix

The radial part of the Schrödinger equation (56), *viz.*

$$\left\{\frac{1}{2\mu_{\rm eff}}\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right] + \frac{2}{3}\frac{\alpha_{\rm st}}{r} - \frac{1}{2}\kappa^2 r + E_{nl} - \frac{1}{2}C\right\} r\psi_{nl}(r) = 0\,, \tag{A.1}$$

where $\mu_{\rm eff} = (3/4)\,m$, can be rewritten in the following scale-invariant form:

$$\left[\frac{d^2}{d\xi^2} - \frac{l(l+1)}{\xi^2} + \frac{\lambda}{\xi} - \xi + \varepsilon_{nl} - \delta\right]\xi\varphi_{nl}(\xi) = 0\,, \tag{A.2}$$

with

$$\xi \equiv (\kappa^2 \mu_{\text{eff}})^{1/3} r \,, \; \lambda \equiv \frac{4}{3}\alpha_{\text{st}} \left(\frac{\mu_{\text{eff}}}{\kappa}\right)^{2/3} ,$$
$$\delta \equiv \left(\frac{\mu_{\text{eff}}}{\kappa^4}\right)^{1/3} C \,, \; \varepsilon_{nl} \equiv 2 \left(\frac{\mu_{\text{eff}}}{\kappa^4}\right)^{1/3} E_{nl} \,, \tag{A.3}$$

and

$$\varphi_{nl}(\xi) \equiv \frac{1}{\kappa\sqrt{\mu_{\text{eff}}}} \psi_{nl}(r) \,, \; \int_0^\infty d\xi |\xi \varphi_{nl}(\xi)|^2 \equiv \int_0^\infty dr |r\psi_{nl}(r)|^2 = 1 \,. \tag{A.4}$$

In the special case when $\alpha_{\text{st}} \to 0$ and $l = 0$, Eq. (A.2) becomes the Airy equation

$$\left(\frac{d^2}{d\eta^2} - \eta\right) w_{n0}\,(\eta + \varepsilon_{n0} - \delta) = 0 \,, \tag{A.5}$$

with

$$\eta \equiv \xi - \varepsilon_{n0} + \delta \,, \; w_{n0}(\xi) \equiv \xi \varphi_{n0}(\xi) \,. \tag{A.6}$$

Thus,

$$w_{n0}(\eta + \varepsilon_{n0} - \delta) = N_n \text{Ai}(\eta) \,, \tag{A.7}$$

where

$$\begin{aligned} \text{Ai}(\eta) &\equiv \frac{1}{\pi} \int_0^\infty du \cos(u\eta + \frac{1}{3}u^3) \\ &\equiv -i\frac{\sqrt{\eta}}{3} \left[J_{-1/3}(i\frac{2}{3}\eta^{3/2}) + J_{1/3}(i\frac{2}{3}\eta^{3/2}) \right] \\ &\equiv \frac{\sqrt{\eta}}{\pi\sqrt{3}} K_{1/3}(\frac{2}{3}\eta^{3/2}) \end{aligned} \tag{A.8}$$

is the Airy function [6]. Then, the regularity condition at $\xi = 0$ for the radial wave function $\xi\varphi_{nl}(\xi)$ requires the equation

$$0 = w_{n0}(0) = \text{Ai}(-\varepsilon_{n0} + \delta) \tag{A.9}$$

to be satisfied for ε_{n0}. This shows that

$$\varepsilon_{n0} = -\eta_n + \delta \,, \tag{A.10}$$

where η_n are zeros of the Airy function: $\text{Ai}(\eta_n) = 0$ $(n = 1, 2, 3, \ldots)$. As is known, η_n are negative:

$$\eta_1 = -2.3381 \,, \; \eta_2 = -4.0879 \,, \; \eta_3 = -5.5206 \,, \; \eta_4 = -6.7867 \,, \ldots \tag{A.11}$$

(*cf.* Ref. [6] and the first Ref. [4], p. 142). Thus, due to Eq. (A.3), the energy spectrum in the case of $\alpha_{\rm st} \to 0$ and $l = 0$ is given as follows:

$$E_{n0} = \frac{1}{2}\mu_{\rm eff}\left(\frac{\kappa}{\mu_{\rm eff}}\right)^{4/3} |\eta_n| + \frac{1}{2}C \,. \tag{A.12}$$

Hence, we calculate

$$\kappa^2 \simeq 0.034\,{\rm GeV}\,,\ \tfrac{1}{2}C \simeq -168\,{\rm MeV}\,, \tag{A.13}$$

if $3m \simeq 1\,{\rm GeV}$ (*i.e.*, $4\mu_{\rm eff} \simeq 1\,{\rm GeV}$) and the values (40) are used for E_{10} and E_{20}. The figure (A.13) for κ^2 implies the confining radius (*cf.* Eq. (54)) 10% larger than the estimation (54).

It is interesting to note that in the complex plane each two of the triplet of solutions ${\rm Ai}(z)$, ${\rm Ai}(ze^{2\pi i/3})$, ${\rm Ai}(ze^{-2\pi i/3})$ to the Airy equation $(d^2/dz^2 - z)w = 0$ are independent, determining the third [6] through the "equilateral-triangle relation"

$${\rm Ai}(z) + e^{2\pi i/3}{\rm Ai}(ze^{2\pi i/3}) + e^{-2\pi i/3}{\rm Ai}(ze^{-2\pi i/3}) = 0\,, \tag{A.14}$$

where, of course, $z + ze^{2\pi i/3} + ze^{-2\pi i/3} = 0$. In the case of Eq. (A.5) ${\rm Re}z = \eta$, ${\rm Im}z = 0$. Also the solution [6]

$${\rm Bi}(z) \equiv e^{2\pi i/6}{\rm Ai}(ze^{2\pi i/3}) + e^{-2\pi i/6}{\rm Ai}(ze^{-2\pi i/3}) \tag{A.15}$$

is independent of ${\rm Ai}(z)$, giving

$${\rm Ai}(ze^{\pm 2\pi i/3}) = \tfrac{1}{2}e^{\pm 2\pi i/3}\left[{\rm Ai}(z) \mp i{\rm Bi}(z)\right] \tag{A.16}$$

from the "equilateral-triangle relation". One can establish a correspondence between this triplet of solutions to the Airy equation in the complex plane and three vectors $\vec{r}_1 - \vec{r}_2$, $\vec{r}_3 - \vec{r}_1$, $\vec{r}_2 - \vec{r}_3$ in the plane of motion forming three sides of our equilateral triangle of quarks. Then,

$$|z| = |(\kappa^2\mu_{\rm eff})^{1/3}\,|\vec{r}_1 - \vec{r}_2| - \varepsilon_{n0} + \delta|\,, \tag{A.17}$$

and

$$|ze^{\pm 2\pi i/3}| = \left|(\kappa^2\mu_{\rm eff})^{1/3}\left\{\begin{matrix}|\vec{r}_3 - \vec{r}_1|\\ |\vec{r}_2 - \vec{r}_3|\end{matrix}\right\} - \varepsilon_{n0} + \delta\right|\,, \tag{A.18}$$

as $\eta = \xi - \varepsilon_{n0} + \delta$ and $\xi = (\kappa^2\mu_{\rm eff})^{1/3}r$ with $r = |\vec{r}_1 - \vec{r}_2|$.

Comparing Eqs. (41) and (A.12) we can infer in the case of $\alpha_{\rm st} \to 0$ that

$$\langle r\rangle_{n00} = \tfrac{2}{3}(\kappa^2\mu_{\rm eff})^{-1/3}|\eta_n|\,, \tag{A.19}$$

what gives for $\langle r\rangle_{100}$ the figure 6% smaller than the Coulombic number (55), if $3m \simeq 1$ GeV (*i.e.*, $4\mu_{\rm eff} \simeq 1$ GeV) and κ^2 has the value (A.13). The result (A.19) follows also directly from Eq. (A.12) through the Feynman–Hellmann theorem,

$$\frac{\partial}{\partial\lambda}\langle\psi(\lambda)|H(\lambda)|\psi(\lambda)\rangle = \langle\psi(\lambda)\left|\frac{\partial H(\lambda)}{\partial\lambda}\right|\psi(\lambda)\rangle \tag{A.20}$$

for a parameter-dependent energy-eigenvalue problem

$$H(\lambda)\psi(\lambda) = E(\lambda)\psi(\lambda)\,,\ \langle\psi(\lambda)|\psi(\lambda)\rangle = 1\,. \tag{A.21}$$

In fact, for our hamiltonian

$$H(\kappa^2) \equiv \frac{\vec{p}^2}{2\mu_{\rm eff}} + \frac{1}{2}\kappa^2 r + \frac{1}{2}C \tag{A.22}$$

it gives

$$\langle r\rangle_{n00} = 2\frac{\partial}{\partial\kappa^2}\langle H(\kappa^2)\rangle_{n00} = 2\frac{\partial E_{n0}(\kappa^2)}{\partial\kappa^2} = \frac{2}{3}(\kappa^2\mu_{\rm eff})^{-1/3}|\eta_n|\,, \tag{A.23}$$

where Eq. (A.12) is applied (the constants κ^2 and C are independent). Notice, by the way, that for the second derivative the theorem

$$\frac{\partial^2}{\partial\lambda^2}\langle\psi(\lambda)|H(\lambda)|\psi(\lambda)\rangle = \left\langle\psi(\lambda)\left|\frac{\partial^2 H(\lambda)}{\partial\lambda^2}\right|\psi(\lambda)\right\rangle + 2\left\langle\frac{\partial\psi(\lambda)}{\partial\lambda}|E(\lambda) - H(\lambda)|\frac{\partial\psi(\lambda)}{\partial\lambda}\right\rangle \tag{A.24}$$

holds.

Now, in the general case of arbitrary $\alpha_{\rm st} \geq 0$ and $l = 0, 1, 2, \ldots$, rewrite Eq. (A.2) in the hamiltonian-type form

$$h(\lambda)\varphi(\xi,\lambda) = \varepsilon(\lambda)\varphi(\xi,\lambda) \ \text{ with } \ h(\lambda) \equiv -\frac{1}{\xi}\frac{d^2}{d\xi^2}\xi + \frac{l(l+1)}{\xi^2} - \frac{\lambda}{\xi} + \xi + \delta\,, \tag{A.25}$$

where

$$\varepsilon(\lambda) \equiv \varepsilon_{nl}(\lambda)\,,\ \varphi(\xi,\lambda) \equiv \varphi_{nl}(\xi,\lambda)\,,\ \int_0^\infty \xi^2\, d\xi|\varphi(\xi,\lambda)|^2 = 1\,,$$
$$w(\xi,\lambda) \equiv w_{nl}(\xi,\lambda) \equiv \xi\varphi_{nl}(\xi,\lambda)\,. \tag{A.26}$$

Hence,

$$\varepsilon(\lambda) = \langle h(\lambda)\rangle_\lambda \equiv \int_0^\infty \xi^2\, d\xi\, \varphi^*(\xi,\lambda) h(\lambda) \varphi(\xi,\lambda)\,, \tag{A.27}$$

and then

$$\frac{\partial\varepsilon(\lambda)}{\partial\lambda} = \left\langle \frac{\partial h(\lambda)}{\partial\lambda}\right\rangle_\lambda = -\left\langle \frac{1}{\xi}\right\rangle_\lambda = -\int_0^\infty \xi\, d\xi |\varphi(\xi,\lambda)|^2 \tag{A.28}$$

from the counterpart of Feynman–Hellmann theorem. Thus,

$$\varepsilon(\lambda) = \varepsilon(0) - \int_0^\lambda d\lambda' \left\langle \frac{1}{\xi}\right\rangle_{\lambda'} = \varepsilon(0) - \lambda \left\langle \frac{1}{\xi}\right\rangle_0 - \frac{1}{2}\lambda^2 \left\langle \frac{1}{\xi}\right\rangle'_0 - \frac{1}{6}\lambda^3 \left\langle \frac{1}{\xi}\right\rangle''_0 - \ldots\,, \tag{A.29}$$

where

$$\left\langle \frac{1}{\xi}\right\rangle_\lambda = \left\langle \frac{1}{\xi}\right\rangle_0 + \lambda \left\langle \frac{1}{\xi}\right\rangle'_0 + \frac{1}{2}\lambda^2 \left\langle \frac{1}{\xi}\right\rangle''_0 + \ldots\,. \tag{A.30}$$

In the spectrum (A.29) the term $\varepsilon(0)$, equal to $-\eta_n + \delta$ in the case of $l = 0$ according to Eq. (A.10), is an eigenvalue of the operator $h(0)$. The correction given by the integral over λ is caused by the Coulombic operator $h(\lambda) - h(0) \equiv -\lambda/\xi$ whose expectation value is evaluated in an exact eigenstate $\varphi(\xi,\lambda)$ of $h(\lambda)$. This expectation value can be expanded in powers of $\lambda \sim \alpha_{\rm st}/\kappa^{2/3}$ with coefficients calculated by means of $\varphi(\xi,0)$ and all derivatives of $\varphi(\xi,\lambda)$ with respect to λ at the point $\lambda = 0$. In particular, the approximate form

$$\varepsilon(\lambda) \simeq \varepsilon(0) - \lambda\langle \tfrac{1}{\xi}\rangle_0 \quad \text{with} \quad \varepsilon(0) = -\eta_n + \delta \tag{A.31}$$

corresponds to the first-order perturbative formula with respect to the Coulombic operator $-\lambda/\xi$ treated as a perturbation in Eq. (A.25).

From Eqs. (A.25) and (A.28) we can also write

$$\frac{\partial\varepsilon(\lambda)}{\partial\lambda} = \frac{1}{\lambda}\langle h(\lambda) - h(0)\rangle_\lambda = \frac{1}{\lambda}\left[\varepsilon(\lambda) - \langle h(0)\rangle_\lambda\right]\,, \tag{A.32}$$

and so the following linear inhomogeneous first-order differential equation for $\varepsilon(\lambda)$:

$$\lambda\frac{\partial\varepsilon(\lambda)}{\partial\lambda} - \varepsilon(\lambda) = -\langle h(0)\rangle_\lambda\,. \tag{A.33}$$

Its general solution expressed in terms of its inhomogeneity $\langle h(0)\rangle_\lambda$ reads

$$\varepsilon(\lambda) = \frac{\lambda}{\lambda_0}\varepsilon(\lambda_0) - \lambda \int_{\lambda_0}^{\lambda} d\lambda' \, \frac{\langle h(0)\rangle_{\lambda'}}{\lambda'^2} \,, \tag{A.34}$$

where the value $\varepsilon(\lambda_0)$ at an initial point λ_0 is *a priori* an arbitrary constant with respect to λ. Expanding $\langle h(0)\rangle_\lambda$ in powers of $\lambda \sim \alpha_{\rm st}/\kappa^{2/3}$ we obtain

$$\langle h(0)\rangle_\lambda \equiv \langle \varphi(\lambda)|h(0)|\varphi(\lambda)\rangle = \varepsilon(0) + \tfrac{1}{2}\lambda^2 \langle h(0)\rangle_0'' + \dots \,, \tag{A.35}$$

where making use of the series

$$\varphi(\xi,\lambda) = \varphi(\xi,0) + \lambda\varphi'(\xi,0) + \tfrac{1}{2}\lambda^2\varphi''(\xi,0) + \dots \tag{A.36}$$

we calculate

$$\langle h(0)\rangle_0' = 0 \;\; , \;\; \langle h(0)\rangle_0'' = 2\langle \varphi'(0)|h(0) - \varepsilon(0)|\varphi'(0)\rangle \;\; , \;\; \dots . \tag{A.37}$$

Then, from Eq. (A.34)

$$\varepsilon(\lambda) = \varepsilon(0) + \frac{\lambda}{\lambda_0}\left[\varepsilon(\lambda_0) - \varepsilon(0)\right] - \frac{1}{2}\lambda(\lambda - \lambda_0)\langle h(0)\rangle_0'' + \dots . \tag{A.38}$$

Hence, for $\lambda_0 \to 0$

$$\begin{aligned} \varepsilon(\lambda) &= \varepsilon(0) + \lambda\varepsilon'(0) - \tfrac{1}{2}\lambda^2\langle h(0)\rangle_0'' + \dots \\ &= \varepsilon(0) - \lambda\left\langle \tfrac{1}{\xi}\right\rangle_0 - \tfrac{1}{2}\lambda^2\langle h(0)\rangle_0'' + \dots . \end{aligned} \tag{A.39}$$

Note from the counterpart of theorem (A.24) and Eqs. (A.28) and (A.37) that

$$-\left\langle \frac{1}{\xi}\right\rangle_0' = \varepsilon''(0) = 2\langle \varphi'(0)|\varepsilon(0) - h(0)|\varphi'(0)\rangle = -\langle h(0)\rangle_0'' \,, \tag{A.40}$$

as it should be. But, it is not necessary to take in Eq. (A.38) the limit $\lambda_0 \to 0$ where the initial value $\varepsilon(\lambda_0)$ corresponds to the purely confining case. In contrast, in the opposite limit $\lambda_0 \to \infty$ the initial value $\varepsilon(\lambda_0)$ is related to the purely Coulombic case, if $\lambda_0 \to \infty$ means $\kappa_0^2 \to 0$ while $\alpha_{\rm st}$ and $\mu_{\rm eff}$ are fixed. In fact, in this limit we get

$$E(\lambda_0) \to -\frac{(2\alpha_{\rm st}/3)^2\mu_{\rm eff}}{2n^2} = -\frac{\alpha_{\rm st}^2 m}{6n^2} \;\; , \;\; \varepsilon(\lambda_0) \equiv \frac{9}{8}\frac{\lambda_0^2}{\alpha_{\rm st}^2\mu_{\rm eff}}E(\lambda_0) \to -\infty \,, \tag{A.41}$$

when we use Eqs. (A.1) and (A.3). This shows that in the nearly Coulombic case the effect of confining potential manifests itself through an asymptotic expansion in powers of $1/\lambda_0$.

Finally, it is interesting to notice that, because the Schrödinger equation (15) for our triangle is formally a two-body wave equation, the number n_l of bound states having a given orbital angular momentum l (in the case of a central potential $V(r)$) obeys the Bargmann inequality [7]:

$$n_l < \frac{2\mu_{\rm eff}}{2l+1} \int\limits_0^\infty r dr |V(r)| \tag{A.42}$$

(then, obviously $V(r) \to 0$ at $r \to \infty$). Thus, if the integral in Eq. (A.42) is finite, our triangle has certainly a finite total number $\sum_l n_l$ of bound states, since all n_l are finite and

$$n_l = 0 \;\; \text{for} \;\; 2l+1 > 2\mu_{\rm eff} \int\limits_0^\infty r dr |V(r)| \, . \tag{A.43}$$

Of course, in the case of our triangle of quarks with the Cornell-type potential (20) the integral in the Bargmann inequality is divergent to ∞ (both because of the Coulombic term and confining term).

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