D_{3h} INTRINSIC SYMMETRY VERSUS LABORATORY REFERENCE FRAME*,**

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In the paper the physical quantum numbers labelling the irreducible representations of the point symmetry D_{3h} have been derived. Both sets of labels, obtained in the intrinsic and laboratory frames, have been considered. This symmetry is related to the octupole deformation of α_{33}-type which leads to the exotic octupole states of nuclei. The derivations are based on a group algebra formalism.

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1. Introduction

The point group symmetries and their influence on the shell structure in the systems that can be treated using a mean field formalism have been studied by Hamamoto and collaborators [1]. These authors have considered systems without spin-orbit interactions and therefore more adequate for the atomic and cluster physics rather than nuclear physics. In particular it followed from their study that the symmetry represented by the combinations $Y_{32} + Y_{3-2}$, giving rise to the T_d point group, leads to high degeneracies of the individual levels.

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Also a C_4 symmetry, one other type of the point group symmetries that may become of importance for the nuclear physics applications has been discussed recently [2, 3].

A theoretical analysis based on the nuclear average field concept has shown new types of nucleon correlations. These correlations imply the existence of exotic forms of the equilibrium deformations and consequently new shape symmetries [4–7].

All these results encourage us to consider the different types of possible point symmetries in nuclei and their consequences. One of the most important problem is the labelling of the irreducible representations (IR) of point symmetry groups by the physical quantum numbers and a relation between the labels in intrinsic and laboratory frames.

In the paper we consider the both problems for D_{3h} point symmetry group.

2. $\mathbf{D_{3h}}$ symmetry

Nuclear surface can be described in the usual form [8] which for the special case of octupole deformation of α_{33}-type [9] can be expressed by the equation

$$R(\vartheta,\phi) = R_o c(\{\alpha\})[1 + \alpha_{33}(Y_{33}(\vartheta,\phi) - Y_{3-3}(\vartheta,\phi))]\,. \tag{1}$$

In this case one can expect exotic symmetry represented by the point group D_{3h} [9, 6]. This 12 elements group is the simple product of two groups:

$$D_{3h} = D_3 \times \Sigma_{\rm h}\,, \tag{2}$$

where

$$D_3 = \{e, C_3, C_3^2, \mathcal{U}_2, \mathcal{U}_2', \mathcal{U}_2''\} \tag{3}$$

is generated by two operations: C_3 which denotes a rotation around z-axis about the angle $2\pi/3$ and $\mathcal{U}_2$ which is interpreted as a rotation around x-axis about the angle π ($\mathcal{U}_2' = C_3\mathcal{U}_2C_3^2$, $\mathcal{U}_2'' = C_3^2\mathcal{U}_2C_3$) and $\Sigma_{\rm h}$ denotes the group of reflections σ_h in respect to a horizontal plain [10].

3. The group algebra QM(O(3), D_{3h})

As a tool for further considerations we use the appropriate form of group algebras. This excellent formalism allows to solve our problem determining the representations of the symmetry group D_{3h} in the intrinsic frame and translate the results to the laboratory one. The elementary theory of group algebras for finite groups one can find *e.g.*, in [7]. On the other hand, a lot of information about group algebras for continuous groups is contained in [11]. For our purposes we need to introduce a combined structure of both algebras: for discrete and continuous groups. We will denote this algebra by $QM(\mathrm{O}(3),D_{3h})$, where O(3) is 3 dimensional orthogonal group consisted of the rotational group SO(3) and two-element group of spatial inversion $\mathcal{C}_i$, O(3)=SO(3)$\times\mathcal{C}_i$. Because we do not know the convenient references, we describe here, in a few words, the structure of this algebra.

The algebra $QM(G,G')$, where to save typing we have introduced the abbreviations $G=\mathrm{O}(3)$ and $G'=D_{3h}$, consist of the formal sums:

$$S = u + \breve{\alpha}\,, \tag{4}$$

where, u is a function belonging to the space of square integrable functions $L^2(G)$ and

$$\breve{\alpha} = \sum_{g\in G} \alpha(g) g\,. \tag{5}$$

In the equation (5) $\alpha(g)$ is a complex function on the group G. Sum and multiplication of elements of this type by a complex numbers are defined in the usual way, but multiplication of the elements is defined by the following relations ($u,v \in L^2(G)$ and $g,g_1,g_2 \in G'$):

$$(u\circ v)(g'') = \int_G dg' u(g')v(g'^{-1}g'')\,, \tag{6a}$$

$$g\circ u(g') = u(g^{-1}g')\,, \tag{6b}$$

$$u(g')\circ g = u(g'g^{-1})\,, \tag{6c}$$

$$g_1\circ g_2 = g_1 g_2;\quad \text{multiplication in } G'\,, \tag{6d}$$

and the distributive law in respect to the addition. Another available operation in this algebra is involution $\sharp$, an analog of Hermitian conjugation, defined by:

$$\left(u(g) + \sum_{g'\in G}\alpha(g')g'\right)^{\sharp} = u^\star(g^{-1}) + \sum_{g'\in G}\alpha^\star(g')g'^{-1}\,. \tag{7}$$

Now we can come back to the main problem. Let us consider an even– even nucleus (for half–integer total angular momentum one needs to extend G' to the double–group).

First we introduce special elements of the algebra $QM(G,G')$ connected with subgroups of D_{3h} group. Let $H\subset D_{3h}$ and $\chi_H^{(\nu)}$ denotes the irreducible characters of H [7, 6] then

$$\breve{\chi}_H^{(\nu)} = \sum_{h\in H}\chi_H^{(\nu)}(h)^\star h\,. \tag{8}$$

For the most important subgroups of D_{3h}, from Eq. (8), we have

$$\check{\chi}^{(\sigma)}_{\Sigma_h} = e + \sigma\sigma_h, \tag{9}$$

where the character $\check{\chi}^{(\sigma)}_{\Sigma_h}(\sigma_h) = \sigma = \pm 1$. For the group $\mathcal{C}_2(\mathcal{U}_2)$ generated by $\mathcal{U}_2$ we get $\check{\chi}^{(\mu_\perp)}_{\mathcal{C}_2(\mathcal{U}_2)}(\mathcal{U}_2) = \mu_\perp = \pm 1$.

$$\check{\chi}^{(\mu_\perp)}_{\mathcal{C}_2(\mathcal{U}_2)} = e + \mu_\perp \mathcal{U}_2. \tag{10}$$

And for the group C_3 generated by C_3 rotations, (where $\epsilon = e^{-i2\pi/3}$ and the character $\check{\chi}^{(\mu)}_{C_3}(C_3) = \epsilon^\mu$, $\mu = 0, \pm 1$) we obtain

$$\check{\chi}^{(\mu)}_{C_3} = e + \epsilon^{-\mu} C_3 + \epsilon^{\mu} C_3^2. \tag{11}$$

The structure of the IR of the group D_3 is more complicated. For the representations $A1$, $A2$ (both one dimensional) and for two dimensional representation $E1$ we obtain, respectively

$$\check{\chi}^{(A1)}_{\mathrm{D}_3} = \sum_{g \in \mathrm{D}_3} g\,, \tag{12a}$$

$$\check{\chi}^{(A2)}_{\mathrm{D}_3} = e + C_3 + C_3^2 - (\mathcal{U}_2 + \mathcal{U}_2' + \mathcal{U}_2'')\,, \tag{12b}$$

and

$$\check{\chi}^{(E1)}_{\mathrm{D}_3} = 2e - C_3 - C_3^2\,. \tag{12c}$$

After some algebraical considerations one can derive the following element of our group algebra

$$\mathcal{X}^{(\mu,\mu_\perp)} = \check{\chi}^{(\mu)}_{C_3} \circ \check{\chi}^{(\mu_\perp)}_{\mathcal{C}_2(\mathcal{U}_2)} \circ \check{\chi}^{(E1)}_{\mathrm{D}_3} \tag{13}$$

which together with the elements (operators in a Hilbert space of quantum states) (9–12) allows to write down the appropriate basis for the IR of the symmetry group D_{3h}. The results are summarized in Table I. In that table, the label μ can take only two values ± 1.

After straightforward, but lengthy calculations one can see that for labelling of IR for D_{3h} we need a set of 3 quantum numbers $\Gamma = (|\mu|, \mu_\perp, \sigma)$, where $|\mu| = 0, 1$, $\mu_\perp$ and $\sigma = \pm 1$. $|\mu| = 0$ and $|\mu| = 1$ corresponds to one dimensional and two dimensional irreducible representations, respectively. For two dimensional case $\mu_\perp$ is irrelevant and can be chosen either $\mu_\perp = 1$ or $\mu_\perp = -1$. The label $\mu = \pm|\mu|$ distinguishes the appropriate basic vectors within the representation.

TABLE I

Basis for D_{3h} irreducible representations

$A1$	$\check{\chi}^{(A1)}_{\mathrm{D}_3} \circ \check{\chi}^{(1)}_{\Sigma_\mathrm{h}}$	$A2$	$\check{\chi}^{(A2)}_{\mathrm{D}_3} \circ \check{\chi}^{(1)}_{\Sigma_\mathrm{h}}$
$B1$	$\check{\chi}^{(A1)}_{\mathrm{D}_3} \circ \check{\chi}^{(-1)}_{\Sigma_\mathrm{h}}$	$B2$	$\check{\chi}^{(A2)}_{\mathrm{D}_3} \circ \check{\chi}^{(-1)}_{\Sigma_\mathrm{h}}$
$E1$	$X^{(\mu,\mu_\perp=1)} \circ \check{\chi}^{(1)}_{\Sigma_\mathrm{h}}$	$E2$	$X^{(\mu,\mu_\perp=1)} \circ \check{\chi}^{(-1)}_{\Sigma_\mathrm{h}}$

4. The physical quantum numbers for D_{3h} symmetry

The physical quantum numbers are introduced by the orthogonal group O(3). They are: the total angular momentum J, its projection K onto internal axis and the parity κ related to the group C_i. Because the symmetry D_{3h} is referred to the intrinsic frame a description of the irreducible representations cannot depend on the third component of the total angular momentum M.

The matrix elements of irreducible representations of the orthogonal group O(3) can be written as products of usual Wigner functions $D^J_{MK}(\Omega)$ for SO(3) group and the characters $\chi^{(\kappa)}_{C_i}(s)$ of the inversion group isomorphic to Σ_h:

$$D^{(\kappa,J)}_{MK}(s\Omega) = \chi^{(\kappa)}_{C_i}(s) D^J_{MK}(\Omega)\,. \tag{14}$$

To find relations between the internal labels $\Gamma = (|\mu|, \mu_\perp, \sigma)$ and μ, and the quantum numbers in the laboratory frame one can calculate

$$D^{(\kappa,J)\star}_{MK} \circ (C_3 \circ \check{\chi}^{(\mu)}_{C_3}) = \chi^{(\mu)}_{C_3}(C_3)[D^{(\kappa,J)\star}_{MK} \circ \check{\chi}^{(\mu)}_{C_3}]\,, \tag{15a}$$

$$[D^{(\kappa,J)\star}_{MK} \circ C_3] \circ \check{\chi}^{(\mu)}_{C_3}) = e^{-i2\pi/3K}[D^{(\kappa,J)\star}_{MK} \circ \check{\chi}^{(\mu)}_{C_3}]\,. \tag{15b}$$

Comparing both equations we get the following relation between the internal label μ and the quantum number K:

$$e^{-i2\pi/3K} = \chi^{(\mu)}_{C_3}(C_3)\,. \tag{16}$$

In similar way one can get other relations:

$$(-1)^K \kappa = \chi^{(\sigma)}_{\Sigma_\mathrm{h}}(\sigma_h)\,, \tag{17}$$

and for $|\mu| = 0$, in addition, we have:

$$(-1)^J = \chi^{(\mu_\perp)}_{C_2(\mathcal{U}_2)}(\mathcal{U}_2)\,. \tag{18}$$

The equations (16–18) allow to classify the spectra of the hamiltonians invariant under D_{3h} symmetry using the measurable quantum numbers J, K and parity κ. The results are shown in the Table II.

TABLE II

Quantum numbers for D_{3h} irreducible representations

IR	$\mu\mu_\perp\sigma$	J	K^κ
$A1$	$0+1+1$	even	0^+ $\pm3^-$ $\pm6^+$ $\pm9^-$ $\pm12^+$
$B1$	$0+1-1$	even	0^- $\pm3^+$ $\pm6^-$ $\pm9^+$ $\pm12^-$
$A2$	$0-1+1$	odd	0^+ $\pm3^-$ $\pm6^+$ $\pm9^-$ $\pm12^+$
$B2$	$0-1-1$	odd	0^- $\pm3^+$ $\pm6^-$ $\pm9^+$ $\pm12^-$
	$-1+1$		-4^+ -1^- 2^+ 5^- 8^+
$E1$		even, odd	
	$+1+1$		-5^- -2^+ 1^- 4^+ 7^-
	$-1-1$		-4^- -1^+ 2^- 5^+ 8^-
$E2$		even, odd	
	$+1-1$		-5^+ -2^- 1^+ 4^- 7^+

This way we have obtained a classification of nuclear states for nuclei with D_{3h} internal symmetry.

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