TIME-REVERSAL AND IRREVERSIBILITY*

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The time reversal and irreversibility in conventional quantum mechanics are compared with those of the rigged Hilbert space quantum mechanics. We discuss the time evolution of Gamow and Gamow-Jordan vectors and show that the rigged Hilbert space case admits a new kind of irreversibility which does not appear in the conventional case. The origin of this irreversibility can be traced back to different initial-boundary conditions for the states and observables. It is shown that this irreversibility does not contradict the experimentally tested consequences of the time-reversal invariance of the conventional case but instead we have to introduce a new time reversal operator.

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1. Introduction

Irreversibility in the title refers to intrinsic irreversibility for quantum physical systems on the microphysical level; this means there exist microphysical "states" ψ whose time evolution (generated by an essentially self-adjoint semibounded Hamiltonian H) $\psi(t) = \text{“e}^{-iHt/\hbar}\text{”}\psi(0)$ has a preferred direction, $t \geq 0$, [1]. Time reversal in the title refers to the existence of an operator A_T which is usually viewed as associating to every state vector

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$\phi(t)$ a state $\phi'(-t) \equiv A_T^{-1}\phi(-t)$ at the negative time $-t$ (relative to a distinguished point of time $t = 0$). Irreversibility and time-reversibility thus appear to be in conflict with each other. Here we want to discuss the resolution of this conflict, based on the empirical fact, that (due to the boundary and initial conditions) for a given state one can usually not (experimentally) prepare its time-reversed state, and that the experimentally tested time-reversal invariance (like, *e.g.*, reciprocity relations) refers to the relations of A_T with the observables and not the action of A_T on the states.

2. Irreversibility, initial-boundary conditions, the time-evolution semigroup and Gamow vectors

Standard (Hilbert Space) quantum mechanics admits only reversible time evolution because time evolution is represented by a group (generated by a self-adjoint Hamiltonian). In contrast to this mathematical theory, there is ample empirical evidence of intrinsically irreversible time evolution of microphysical systems, *e.g.*, the decay of quasistationary states or resonances. Truly stationary states of such quantum physical systems like stable elementary particles are rare. Most relativistic or non-relativistic elementary particles are decaying states (weakly or electromagnetically) or hadron resonances. Empirically, stability or the values of the lifetime does not appear to be a criterion for elementarity. Stable particles are not qualitatively different from quasistable particles, but only quantitatively different by a zero or negligible value of the width Γ. (A particle decays if it can decay and it is stable if selection rules for some quantum numbers prevent it from decaying.)

Resonances have a preferred direction of time (arrow of time). If one takes the point of view that resonances are autonomous quantum-physical entities and decaying particles are *not* less fundamental than stable particles, then one needs a mathematical theory which includes semigroup time evolution. Further, if both stable and quasistable states should be described on the same footing and, since there are state vectors for stable states, there should also be state vectors for quasistable states. The state vector of a resonance, however, needs to have irreversible time evolution.

The standard way in which irreversibility is introduced in quantum theory is through the master equation [2]

$$\frac{\partial \rho(t)}{\partial t} = L\rho(t)\,, \tag{1}$$

where $\rho(t)$ describes the state of the system S, the Liouville operator L is given, *e.g.*, by [2, 3]

$$L\rho(t) = -\frac{i}{\hbar}[H, \rho(t)] + \delta H \varrho\,. \tag{2}$$

For $\delta H = 0$, (1) with (2) is the irreversible time evolution of the isolated quantum system (von Neumann equation). The term $\delta H \varrho$ represents some complicated external effects upon the non-isolated system. With this term (1) is the standard way of describing extrinsic irreversibility due to the effect of an external reservoir R (*e.g.*, measuring apparatus) upon the system. This irreversible time evolution is described by a semigroup $\rho(t) = \Lambda(t)\rho(0)$ generated by the Liouvillian L, $\Lambda(t) = \mathrm{e}^{Lt}$, $t \geq 0$. Equation (1) has also been applied to the time evolution of such microphysical systems as the $K_L - K_S$ meson system [3].

That a fundamental concept like irreversibility should be caused by extrinsic influences has been considered unsatisfactory by many people working on irreversibility and statistical physics. According to Prigogine's ideas [4], irreversibility should be intrinsic to the dynamics and should have its origin in the resonances (Poincaré resonances) rather than being caused by merely external effects of a quantum reservoir or the irreversible act of a measurement apparatus. This requires also a dynamical semigroup which, however, should be generated by the Hamiltonian H, $\rho(t) = \mathrm{e}^{-iHt}\rho(0)\mathrm{e}^{iHt}$, and not by a Liouvillian like (2).

The idea of intrinsic irreversibility and the empirical facts of resonances can both be accommodated by a new mathematical theory which is similar to the standard (von Neumann) quantum mechanics (nonrelativistic or relativistic) but uses a different mathematical idealization [5].

The interpretation of this new quantum theory is, like the Hilbert space idealization, based on the Copenhagen interpretation of quantum mechanics, but it makes a much more distinct separation between the *state* and the *observable*. The state is defined by a preparation apparatus that prepares the state and is described mathematically by a statistical operator (density matrix) or a state vector. The observable is defined by a registration apparatus that measures its values in the state and is mathematically described by self-adjoint operators and their projectors (in place of the projection operator $\mid \psi\rangle\langle\psi \mid$ one can also take the vector ψ up to a phase factor to describe this observable).

The mathematical formulation of this new quantum theory uses also a linear topological space, but instead of von Neumann's Hilbert space it uses the Gelfand triplet (also called rigged Hilbert space (RHS)).

Rigging the Hilbert space may turn many people away from this subject because it may appear to some as an unnecessary mathematical complication (or even a disreputable practice).

This is really not the case, because on the level of the mathematical rigor employed by the physicist the RHS formulation of quantum mechanics is like Dirac's bra- and ket-formalism. When physicists talk about the Hilbert space they mostly mean a pre-Hilbert space, *i.e.*, a linear space Ψ with a

"scalar product", denoted by (ψ, F) or $\langle \psi \mid F \rangle$ without worrying about its topological completion. The Hilbert space of mathematicians is a much more complicated structure, its elements being represented not by functions but by classes of functions whose elements differ on a set of Lebesgue measure zero, a mathematically complicated and physically useless concept (because the apparatus resolution is described by smooth functions). The RHS is the same linear space Ψ only with different topological completions: one completes Ψ with respect to a topology that is stronger than the topology given by the $\mathcal{H}$-space norm (*e.g.*, one uses a countable number of norms) to obtain the space $\Phi \subset \mathcal{H}$ and considers in addition the topological dual to Φ, *i.e.*, the space of *continuous* antilinear functionals of Φ denoted by $\Phi^\times$. Then one obtains the triplet

$$\begin{array}{rccccc} \text{Gelfand triplet: } \Psi \subset & \Phi & \subset \mathcal{H} = \mathcal{H}^\times \subset & \Phi^\times & \\ & \in & & \in & \\ \text{with elements "bra" and "ket"} & \langle \phi \mid & & \mid F \rangle & . \\ \text{or "ket" and "bra"} & \mid \phi \rangle & & \langle F \mid & \end{array} \tag{3}$$

A widespread example for Φ is the Schwartz space.

The vectors $\phi \in \Phi$ (in their form as kets $\mid \phi \rangle$ or bras $\langle \phi \mid$) represent physical quantities connected with the experimental apparatuses (*e.g.*, state ϕ defined by a preparation apparatus or an observable $\mid \psi \rangle\langle \psi \mid$ defined by a registration apparatus (detector) fulfill $\phi, \psi \in \Phi$), the vectors $\langle F \mid$ or $\mid F \rangle \in \Phi^\times$ represent quantities connected with the microphysical system (*e.g.*, "scattering states" $\mid E \rangle$ or decaying states $\mid E - i\Gamma/2 \rangle$). $\mathcal{H}$ itself does not have any special physical meaning.

A general observable is now represented by a bounded operator A in Φ (but in general by an unbounded $\overline{A}$ or $A^\dagger$ in $\mathcal{H}$) and corresponding to the triplet (3) one has now a triplet of operators

$$A^\dagger|_\Phi \subset A^\dagger \subset A^\times. \tag{4}$$

In here $A^\dagger$ is the Hilbert space adjoint of A (if A is essentially self adjoint then $A^\dagger = \overline{A}$), $A^\dagger|_\Phi$ denotes its restriction to the space Φ, and the operator $A^\times$ in $\Phi^\times$ is the conjugate operator of A defined by

$$\langle A\phi \mid F \rangle = \langle \phi \mid A^\times F \rangle \quad \text{for all } \phi \in \Phi \text{ and all } \mid F \rangle \in \Phi^\times. \tag{5}$$

By this definition $A^\times$ is the extension of the operator $A^\dagger$ to the space $\Phi^\times$ (and not the extension of the operator A which is most often used in mathematics). A very important point is that the operator $A^\times$ is only defined for an operator A which is continuous=bounded in Φ, then $A^\times$ is a continuous (but not bounded) operator in $\Phi^\times$. It is impossible in quantum

mechanics (empirically) to restrict oneself to continuous=bounded operators $\overline{A}$ in $\mathcal{H}$, but one can restrict oneself to algebras of observables $\{A, B \ldots\}$ described by continuous operators in Φ. Then $A^\times, B^\times \ldots$ are defined and continuous in $\Phi^\times$. If A in (5) is not self-adjoint then $A^\dagger \mid_\Phi$ need not be a continuous operator in Φ even if A is, but one can still define the conjugate $A^\times$ which is continuous in $\Phi^\times$.

A generalized eigenvector $F \in \Phi^\times$ of an operator A is defined by

$$\langle A\phi \mid F\rangle = \langle \phi \mid A^\times F\rangle = \omega\langle \phi \mid F\rangle \text{ for all } \phi \in \Phi\,, \tag{6}$$

where the complex number ω is called the generalized eigenvalue. This is also written as

$$A^\times \mid F\rangle = \omega \mid F\rangle. \tag{7}$$

For an essentially self-adjoint operator $A^\dagger = \overline{A}$ (= closure of A) this is often also written as

$$A \mid F\rangle = \omega \mid F\rangle \tag{8}$$

especially if one suppresses the mathematical subtleties and acts as if one has just a linear scalar product space Ψ.

Calculating just in the pre-Hilbert space Ψ — as physicists usually do — the RHS formulation is really not more difficult than the Hilbert space formulation. One just has to use a slightly more general set of rules for these calculations. This has always been done in the Dirac formalism of bra's and ket's. In addition to the rules of the Dirac formalism, the RHS provides a mathematical justification for additional rules of mathematical manipulations. The most important of these are:

1. the eigenvectors of self-adjoint observables A (*i.e.* with $A^\dagger = \overline{A}$) in (8) can be complex
2. the time evolution for some of the solutions of the Schrödinger equation can be given by a semigroup and not by a reversible unitary group
3. some vectors can decay exponentially (as envisioned by Gamow).

Dynamical equations (laws of nature) are the same in both the Hilbert space and the RHS formulations, namely given by the Schrödinger equation

$$i\hbar \frac{\partial \mid \phi(t)\rangle}{\partial t} = H \mid \phi(t)\rangle\,. \tag{9}$$

or the von Neumann equation (2) with $\delta H = 0$. But in the RHS formulation different initial and boundary conditions than in the Hilbert space formulation allow for a greater variety of solutions; (this goes back to Dirac (kets $\mid E\rangle$), Gamow (exponentially decaying "state" vectors $\mid E - i\Gamma/2\rangle$) and Peierls (purely outgoing boundary conditions). These new vectors are

in the rigged Hilbert space, $|E\rangle, |E - i\Gamma/2\rangle \in \Phi^\times \supset \mathcal{H} \supset \Phi$, but not in the Hilbert space $\mathcal{H}$. Distinct initial-boundary conditions for state vectors (*e.g.*, *in*-states ϕ^+ of a scattering experiment) and observables $|\psi^-\rangle\langle\psi^-|$ (*e.g.*, so-called *out*-states ψ^- of a scattering experiment) lead to two different rigged Hilbert spaces [1], whose precise mathematical properties had been defined earlier [6]:

$$\Phi_- \subset \mathcal{H} \subset \Phi_-^\times, \qquad \text{for ensembles or states,} \tag{10}$$

$$\Phi_+ \subset \mathcal{H} \subset \Phi_+^\times, \qquad \text{for observables or effects.} \tag{11}$$

The Hilbert space $\mathcal{H}$ is the same in both RHS's (10) and (11) and Φ of (3) is $\Phi = \Phi_- + \Phi_+$ with $\Phi_- \cap \Phi_+ \neq \emptyset$.

In (10), Φ_- describes the possible state vectors (preparation apparatus, *e.g.*, ϕ^{in} or ϕ^+ of a scattering experiment) and Φ_+ in (11) describes the possible observables (*e.g.*, $|\psi^{\text{out}}\rangle\langle\psi^{\text{out}}|$ or $|\psi^-\rangle\langle\psi^-|$ of a scattering experiment).

For the typical scattering experiment the physical meaning of $\Phi_- \ni \phi^+$ is depicted in Fig. 1. The *in*-state ϕ^+ (precisely the state which evolves from the prepared *in*-state ϕ^{in} outside the interaction region where $V = H - H_0$ is zero) is determined by the accelerator. The so called *out*-state ψ^- (or ψ^{out}) is determined by the detector; $|\psi^{\text{out}}\rangle\langle\psi^{\text{out}}|$ is therefore the observable which the detector registers and not a state. In the conventional formulation one describes both the ϕ^{in} and the ψ^{out} by any vectors of the Hilbert space. In reality the ϕ^{in} (and ϕ^+) and ψ^{out} (and ψ^-) are subject to different initial and boundary conditions and should therefore be described by different sets of vectors.

These distinct initial-boundary conditions for state vectors and observable vectors are stated as an "arrow of time" in the form [1]:

> The state $\phi(t) \in \Phi_-$ must be prepared before an observable $|\psi\rangle\langle\psi|$ (with $\psi \in \Phi_+$) can be measured in that state; *i.e.*, if $t = 0$ is the time before which the preparation is completed and after which the registration begins, then $\phi(t)$ must be prepared by a time $t \leq 0$. (12)

The property of spaces in (10) and (11) can be derived from a mathematical formulation of the "arrow of time" (12) using the Paley-Wiener theorem [1]. It turns out that Φ_- is the space of well behaved Hardy class vectors from below and Φ_+ is the space of well behaved Hardy class vectors from above [7]. These are the same mathematical properties that had been obtained earlier [6, 8] from the existence conditions for Gamow vectors. The notation $\phi^+ \in \Phi_-$ and $\psi^- \in \Phi_+$ of opposite sub- and superscripts for vectors and spaces has no significance but is just a consequence of the fact that

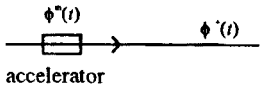

Fig. 1a. Preaparation of $\phi^{\text{in}}(t)$

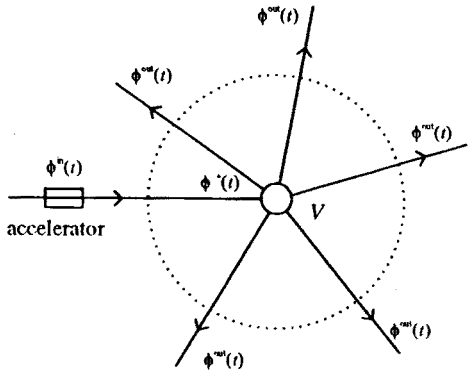

Fig. 1b. Preparation of $\phi^{\text{out}}(t)$

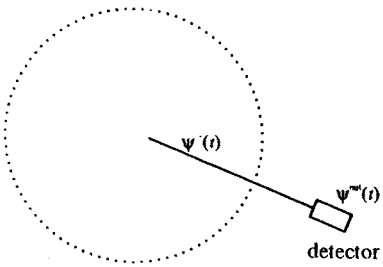

Fig. 1c. Registration of $\psi^{\text{out}}(t)$

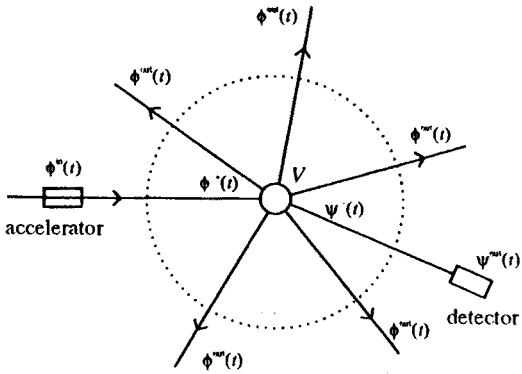

Fig. 1d. Combination of preparation and registration

the nomenclature in physics (scattering theory for ϕ^+, ψ^-) and mathematics (theory of Hardy class functions for Φ_- and Φ_+) had been developed independently.

The semi-group of time evolution, and therewith irreversibility on the microphysical level, is a mathematical consequence of the bi-partition of the rigged Hilbert space into the two rigged Hilbert spaces (10) and (11) and therewith of the dichotomy of state and observables and their "arrow of time" (12).

In conventional quantum mechanics in Hilbert space the time evolution of a state

$$W(t) = U^\dagger(t)W(0)U(t) = \mathrm{e}^{-iHt/\hbar}W(0)\mathrm{e}^{iHt/\hbar}, \qquad -\infty < t < \infty \quad (13)$$

is given by a group

$$U^{\dagger}(t) = \mathrm{e}^{-iHt/\hbar} \qquad -\infty < t < +\infty\,. \tag{14}$$

Therefore for every statistical operator (or density matrix) $W(t)$ one obtains (by calculation) a state operator $W^{\mathrm{neg}}(t) \equiv W(-t)$.

In the rigged Hilbert spaces (10), (11) we have the two extensions of the Hilbert space operator $U^{\dagger}(t)$:

$$\text{the conjugate of } U\,|_{\Phi_-}\colon\ U^{\dagger}(t) \subset U_-^{\times} = \mathrm{e}^{-iH^{\times}t/\hbar};\ \text{for } t \leq 0\,, \tag{15}$$

$$\text{the conjugate of } U\,|_{\Phi_+}\colon\ U^{\dagger}(t) \subset U_+^{\times} = \mathrm{e}^{-iH^{\times}t/\hbar};\ \text{for } t \geq 0\,, \tag{16}$$

where $U_{\pm}^{\times}$ ($H^{\times}$) denote the extensions of the unitary (self-adjoint) operators $U^{\dagger}(t)$ ($H^{\dagger} = H$) to the spaces $\Phi_{\pm}^{\times}$. It turns out that, mathematically, $U_-^{\times}$ in $\Phi_-^{\times}$ can only be defined by (5) for values of the parameters $t \leq 0$, since for $t > 0$ U is not continuous in Φ_-. By the same arguments $U_+^{\times}$ in $\Phi_+^{\times}$ can only be defined for values of the parameters $t \geq 0$. This is the mathematical strategy by which the semigroup time evolution is obtained. In the physical interpretation of the mathematical theory it is based on the "arrow of time" (12). Now one can no more define for every state $|\phi^-(t)\rangle\langle\phi^-(t)|$, $\phi^- \subset \Phi_+$ a state $W^{\mathrm{neg}}(t) \equiv |\phi^-(-t)\rangle\langle\phi^-(-t)|$, which seems to reflect the experimental situation better (time reversal transforms *out*-states of scattering experiment which are highly correlated spherical waves, into highly correlated incoming spherical waves that go into outgoing uncorrelated plane waves). But an experiment in which highly correlated incoming spherical waves go into uncorrelated plane waves is practically impossible to set up.

Summarizing, if one wants an irreversible time evolution on the microphysical level one needs a mathematical idealization (*i.e.*, a topological completion of the linear algebraic space) which uses not Hilbert space but the rigged Hilbert space. This quantum theory in rigged Hilbert space has the following properties:

I. It has Dirac kets (scattering states) $|\ E\rangle$ and an algebra of observables.

II. It has vectors, called Gamow vectors which we also denote by kets as $|\psi^G\rangle = |z_R^-\rangle\sqrt{2\pi\Gamma}$, that have the following properties which make them ideally suited for the description of resonance states in quantum theory:

1. They are generalized eigenvectors of Hamiltonians H (which we always assume to be (essentially) self-adjoint and bounded from below) with generalized eigenvalues $z_R = E_R - i\Gamma/2$,

$$H^{\times}\left|\psi^G\right\rangle = z_R\left|\psi^G\right\rangle\,, \tag{17}$$

where E_R and Γ are respectively interpreted as the resonance energy and width.

2. They satisfy the following exponential decay law for $t \geq 0$ only:

$$\begin{aligned} W^G(t) &= \text{“}\mathrm{e}^{-iHt/\hbar}\text{”} \left|\psi^G\right\rangle \left\langle\psi^G\right| \text{“}\mathrm{e}^{iHt/\hbar}\text{”}, \\ &= \mathrm{e}^{-i(E_R - i\,\Gamma/2)t/\hbar} \left|\psi^G\right\rangle \left\langle\psi^G\right| \mathrm{e}^{i(E_R + i\,\Gamma/2)t/\hbar}, \\ &= \mathrm{e}^{-\Gamma t/\hbar}\, W^G(0)\,. \end{aligned} \tag{18}$$

3. They have a Breit–Wigner energy distribution.
4. They obey an exact Golden Rule of which Fermi's Golden Rule is the Born approximation.
5. They are associated with a pole at z_R in the second sheet of the analytically continued S-matrix. They are derived as the functionals of the pole term of the S-matrix.

In the absence of a vector description of resonances in the Hilbert space formulation, the pole of the S-matrix has commonly been taken as the definition of a resonance. Since in the RHS formulation the Gamow vectors are derived from the pole term of the S-matrix [8], these vectors $|\, z_R^-\rangle \in \Phi_+^\times$ define decaying resonance states as autonomous microphysical entities. (There are also Gamow vectors $|z_R^{*\,+}\rangle$, $z_R^* = E_R + i\Gamma/2$ associated with the pole at z_R^*, which have an exponentially growing semi-group evolution for $-\infty < t \leq 0$).

3. Gamow–Jordan vectors — a mathematical actuality and a physical possibility

The mathematical definition of Dirac kets was given in 1966 [9], the Gamow vectors were introduced about 1976 [10, 8]; a generalization of Gamow vectors to higher order poles of the S-matrix was given in 1995 [11]. An $\mathcal{N}$-th order S-matrix pole at the complex energy $z_\mathcal{N} = E_\mathcal{N} - i\gamma_\mathcal{N}$ has $\mathcal{N}$ Gamow vectors of order $0, 1, \ldots k \ldots (\mathcal{N}-1)$:

$$|z_\mathcal{N}^-\rangle^{(0)},\ |z_\mathcal{N}^-\rangle^{(1)}, \ldots, |z_\mathcal{N}^-\rangle^{(k)}, \ldots, |z_\mathcal{N}^-\rangle^{(\mathcal{N}-1)} \tag{19}$$

associated with it. The k-th order Gamow vector $|z_\mathcal{N}^-\rangle^{(k)}$ is a Jordan vector of degree $(k+1)$, *i.e.*it fulfills the eigenvalue equations [12]

$$\begin{aligned} &(H^\times - z)^k |z_\mathcal{N}^-\rangle^{(k)} = 0\,, \\ &H^\times |z_\mathcal{N}^-\rangle^{(k)} = z_\mathcal{N} |z_\mathcal{N}^-\rangle^{(k)} + |z_\mathcal{N}^-\rangle^{(k-1)} \quad \text{for} \quad k = 0, 1, \ldots, (\mathcal{N}-1)\,. \end{aligned} \tag{20}$$

These equations are, like the eigenvector equation for Dirac kets and for Gamow vectors (= Gamow vectors of order 0 = Jordan vectors of degree

1), understood as generalized eigenvector equations (*i.e.*, functionals) over the space Φ_+:

$$\begin{aligned}\langle H\psi^-|z_{\mathcal{N}}^-\rangle^{(k)} &\equiv \langle\psi^-|H^\times|z_{\mathcal{N}}^-\rangle^{(k)}, \\ &= z_{\mathcal{N}}\langle\psi^-|z_{\mathcal{N}}^-\rangle^{(k)} + \langle\psi^-\mid z_{\mathcal{N}}^-\rangle^{(k-1)} \text{ for all } \psi_- \in \Phi_+. \quad (21)\end{aligned}$$

This means $|z_{\mathcal{N}}^-\rangle^{(k)} \in \Phi_+^\times$, and the $\mathcal{N}$-th order S-matrix pole is associated to a $\mathcal{N}$-dimensional subspace $\mathcal{M}_{z_{\mathcal{N}}} \subset \Phi_+^\times$, spanned by the $|z_{\mathcal{N}}^-\rangle^{(k)}$, $k = 0, 1, \ldots, (\mathcal{N}-1)$, *i.e.*, to the set of all

$$|z_{\mathcal{N}}^-\} = \sum_{k=0}^{\mathcal{N}-1} |z_{\mathcal{N}}^-\rangle^{(k)} c_k, \quad c_k \in \mathcal{C}. \tag{22}$$

On $\mathcal{M}_{z_{\mathcal{N}}} \subset \Phi_+^\times$ the Hamiltonian $H^\times$ (*i.e.*, the extension of the self-adjoint operator $H^\dagger$ to $\Phi^\times$) is not diagonalizable, but can only be brought into the normal form of a Jordan block:

$$H_{\mathcal{N}}^\times \Longleftrightarrow \begin{pmatrix} z_{\mathcal{N}} & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & z_{\mathcal{N}} & & & & \vdots \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 1 & z_{\mathcal{N}} \end{pmatrix}. \tag{23}$$

From this follows that the matrix representation of the time evolution operator (16) on the $\mathcal{N}$-dimensional eigenspace $\mathcal{M}_{z_{\mathcal{N}}}$ is given by

$$\langle\psi^-|\mathrm{e}^{-itH^\times}|z_{\mathcal{N}}^-\rangle^{(k)} =$$

$$\begin{pmatrix} \mathrm{e}^{-z_{\mathcal{N}}t} & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{(-it)}{1!}\mathrm{e}^{-z_{\mathcal{N}}t} & \mathrm{e}^{-z_{\mathcal{N}}t} & 0 & & & \vdots \\ \frac{(-it)^2}{2!}\mathrm{e}^{-z_{\mathcal{N}}t} & \frac{(-it)}{1!}\mathrm{e}^{-z_{\mathcal{N}}t} & \mathrm{e}^{-z_{\mathcal{N}}t} & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \frac{(-it)^{\mathcal{N}-1}}{(\mathcal{N}-1)!}\mathrm{e}^{-z_{\mathcal{N}}t} & \frac{(-it)^{\mathcal{N}-2}}{(\mathcal{N}-2)!}\mathrm{e}^{-z_{\mathcal{N}}t} & \cdots & \cdots & \frac{(-it)}{1!}\mathrm{e}^{-z_{\mathcal{N}}t} & \mathrm{e}^{-z_{\mathcal{N}}t} \end{pmatrix}$$

$$\times \begin{pmatrix} \langle\psi^-|z_{\mathcal{N}}^-{}^{(0)} \\ \langle\psi^-|z_{\mathcal{N}}^-{}^{(1)} \\ \vdots \\ \\ \langle\psi^-|z_{\mathcal{N}}^-{}^{(\mathcal{N}-1)} \end{pmatrix} \tag{24}$$

which can also be written as

$$e^{-itH^\times} \mid z_{\mathcal{N}}^-\rangle^{(k)} = e^{-itz_{\mathcal{N}}} \left\{ \mid z_{\mathcal{N}}^+\rangle^{(k)} + \frac{(-it)}{1!} \mid z_{\mathcal{N}}^+\rangle^{(k-1)} + \cdots \right.$$

$$\left. + \frac{(-it)^l}{l!} \mid z_{\mathcal{N}}^+\rangle^{(k-l)} + \cdots + \frac{(-it)^k}{k!} \mid z_{\mathcal{N}}^+\rangle^{(0)} \right\} \quad \text{for } t \geq 0 \text{ only}. \quad (25)$$

This means that whereas the zeroth order Gamow state only decays exponentially with time, the k-th order Gamow vector $\mid z_{\mathcal{N}}^-\rangle^{(k)}$ evolves into a superposition of lower order Gamow vectors. After a long time (relative to the scale set by $\hbar/\Gamma$) the most significant term is the zeroth order Gamow vector $\mid z_{\mathcal{N}}^-\rangle^{(0)}$, whose time dependence is given by $e^{-itz_{\mathcal{N}}}t^k$.

4. The complex basis vector expansion and some of its consequences in physical applications

The most important result of the new mathematical theory of quantum physics in the rigged Hilbert space is the complex eigenvector expansion. This is the generalization of the elementary basis vector expansion of a 3-dimensional vector, $\boldsymbol{x} = \sum_{i=1,2,3} \boldsymbol{e}_i(\boldsymbol{e}_i \cdot \boldsymbol{x}) = \sum \boldsymbol{e}_i \cdot x_i$ to the expansion of vectors $\phi^+ \in \Phi_-$ using as basis vectors the generalized eigenvectors $|z_{R_i}^-\rangle$ and $|z^-\rangle^{(k)}$ of self-adjoint operators H with complex eigenvalues z_{R_i}, and z, respectively.

Earlier developments towards this generalization were the fundamental theorem of linear algebra which states that for every self-adjoint operator H in a n-dimensional Euclidian space $\mathcal{H}_n$ there exists an orthonormal basis $e_i \ldots e_n$ in $\mathcal{H}_n$ of eigenvectors $He_i = E_i e_i$. *I.e.*, $f \in \mathcal{H}_n$ can be written $f = \sum_{i=1}^n e_i(e_i, f)$. This theorem generalizes to the infinite dimensional Hilbert space $\mathcal{H}$, but only for self-adjoint operators H which are completely continuous (also called compact operators which include Hilbert-Schmidt, nuclear, traceclass operators). For an arbitrary self-adjoint operators H one has to go outside the space to find a complete basis system of eigenvectors (generalized).

The *first* step in this direction is the Dirac basis vector expansion which in mathematical terms is called the nuclear spectral theorem. It states that for every $\phi \in \Phi$

$$\phi = \int_0^{+\infty} dE \, |E^+\rangle\langle {}^+E|\phi^+\rangle + \sum_n |E_n)(E_n|\,\phi) \quad \text{for } \phi \in \Phi. \quad (26)$$

In here, $|E_n)$ are the discrete eigenvectors of the exact Hamiltonian $H = K + V$, (describing the bound states) $H|E_n) = E_n|E_n)$, and $|e^+\rangle$ are the

generalized eigenvectors (Dirac kets) of H corresponding to the continuous spectrum (describing scattering states). The integration extends over the spectrum of H: $0 \leq E < \infty$; and in place of the $|e^+\rangle$ one could also have chosen the $|e^-\rangle$, if the *out*-wavefunctions are more readily available.

The *second* step is the "complex basis vector expansion" For every $\phi^+ \in \Phi_-$ (a similar expansion holds also for every $\psi^- \in \Phi_+$) one obtains for the case of a finite number of resonances poles at the positions z_{R_i}, $i = 1, 2, \cdots N$, the following basis system expansion:

$$\phi^+ = \int_0^{-\infty_{II}} d\omega\, |\omega^+\rangle\langle{}^+\omega|\phi^+\rangle + \sum_{i=1}^{N} |z_{R_i}^-\rangle\, 2\pi\Gamma_i\, \langle{}^+z_{R_i}|\phi^+\rangle + \sum_n |E_n)(E_n|\phi) \quad \text{for} \quad \phi^+ \in \Phi_-\,, \tag{27}$$

where $|z_{R_i}^-\rangle\sqrt{2\pi\Gamma_i} = \psi^{G_i} \in \Phi_+^\times$ are Gamow kets (17) representing decaying states (18).

If we assume that there are two decaying states $R_1 = S$ and $R_2 = L$ and no bound states, then the pure state (prepared by the experimental apparatus) has according to (27) the following representation in terms of the Gamow vectors $\psi_L^G = |z_L^-\rangle\sqrt{2\pi\Gamma_L}$, $\psi_S^G = |z_S^-\rangle\sqrt{2\pi\Gamma_S}$ and the remaining part which we call ϕ_{bg}^+:

$$\phi^+ = \psi_L^G\, b_L + \psi_S^G\, b_S + \int_0^{-\infty_{II}} dE\, |E^+\rangle\langle{}^+E|\phi^+\rangle. \tag{28}$$

In here b_L and b_S are some complex numbers that depend upon the "normalization" of the Gamow vectors $\psi_{L,S}^G$ (and of ϕ^+), and upon some phase convention. All the vectors in the generalized basis system expansion are (generalized) eigenvectors of the exact Hamiltonian, and, in particular, the Gamow vectors $\psi_{L,S}^G$ are eigenvectors of the exact Hamiltonian H, with complex eigenvalue $(E_L - i\,\Gamma_L/2)$ and $(E_S - i\,\Gamma_S/2)$, respectively.

We now apply the time evolution operator to equation (28). Since the ψ_i^G are elements of $\Phi_+^\times$ we can only apply the operator $U_+^\times(t)$ of (16) to it and we obtain:

$$\begin{aligned}\phi^+(t) &\equiv e^{-iH^\times t}\phi^+ \\ &= e^{-i(E_L - i\,\Gamma_L/2)t}\,\psi_L^G b_L + e^{-i(E_S - i\,\Gamma_S/2)t}\,\psi_S^G b_s + \phi_{bg}^+(t)\,;\ t \geq 0\,.\end{aligned} \tag{29}$$

Since the time evolution semigroup (16) has the restriction $t \geq 0$, the same restriction must be used for (29). $\phi_{bg}^+(t)$ is the time evolved background

term

$$\phi^+_{\rm bg}(t) \equiv \int_0^{-\infty_{II}} dE\, {\rm e}^{-iEt}\, |E^+\rangle\langle {}^+E|\phi^+\rangle\,. \tag{30}$$

These equations are understood as a functional equation over all $\psi^- \in \Phi_+$. This means that $\phi^+(t) \in \Phi_- \subset \Phi_+^\times$ can be used to obtain $\langle\psi^-|\phi^+(t)\rangle$, whose modulus square is the probability to find the time evolved state by a detector that detects the observable $|\psi^-\rangle\langle\psi^-|$ for any $\psi^- \in \Phi_+$, but *not* for a $\psi^- \in \Phi_-$.

The result (29) means that the time evolution of a superposition of two (or more) Gamow states does not regenerate one Gamow state from the other, or from the background $\phi^+_{\rm bg}(t)$. In particular, if the state ϕ^+ can be prepared such that at some time $t_0 \geq 0$ the background term $\phi^+_{\rm bg}(t)$ is practically zero, then it will remain zero for all $t > t_0$, and the two Gamow states will evolve separately with their separate exponential laws without regenerating each other:

$$\phi^+(t) \approx {\rm e}^{-iE_L t}\, {\rm e}^{-(\Gamma_L/2)t}\, \psi^{\rm G}_L\, b_L \;+\; {\rm e}^{-iE_S t}\, {\rm e}^{-(\Gamma_S/2)t}\, \psi^{\rm G}_S\, b_S\,. \tag{31}$$

Approximations like (31) have been used for the time evolution of a two-resonance system (like the K_L–K_S-system with $\phi^+(t)$ representing the K^0 state [13]) in theories with "effective Hamiltonians" given by 2×2 complex diagonizable matrix. These effective theories are usually legitimized by the Wigner–Weisskopf approximation [14]. In our irreversible quantum theory the expression (29) is exact and it justifies to some extent the effective theory (31). And (29) shows that the problem of "deviation from the exponential decay law" or "vacuum regeneration of K_S from K_L" [15] arises from the artifacts of the Hilbert space mathematics and can be overcome in the exact theory using the rigged Hilbert space.

However there is an extra term in (29) which we called $\phi^+_{\rm bg}$ and which is not taken into consideration in any of the finite dimensional effective theories of complex Hamiltonians, in particular not in the Lee, Oehme, Yang theory [13] of the neutral Kaon system. This term, which comes from the integral along the negative real axis in the second sheet of the S-matrix, can be shown to be also decaying, *i.e.*, $|\langle\psi^-|\phi^+_{\rm bg}(t)\rangle| \to 0$ for $t \to \infty$ for every $\psi^- \in \Phi_+$, but it decays more slowly than the exponential [16]. Thus if one takes for ϕ^+ the $|K^0\rangle$ state prepared *e.g.*, by the reaction $p\pi^- \to \Lambda K^0$ and for the observables $|\psi^-\rangle\langle\psi^-|$ projectors on the $\pi^+\pi^-$ space one obtains according to the *exact* equation (29) also a term $|\langle\pi^+\pi^-|\phi^+_{\rm bg}(t)\rangle|$. This term vanishes more slowly than to the rapidly disappearing ${\rm e}^{-i\Gamma_S t}|\langle\pi^+\pi^-|K^0_1\rangle|$. This may provide an alternative mechanism to explain the $\pi\pi$ decay mode of the prepared K^0 long after the $K^0_S = K^0_1$ has vanished.

The *third* step outside the space to obtain the basis of eigenvectors is the *general complex basis vector* expansion. It includes in addition to the ordinary Gamow kets (17) also higher order Gamow kets (21) which occur when (and if) the S-matrix has poles of order $\mathcal{N} > 1$. Instead of writing down the general expansion we restrict ourselves here to the special case that there are no bound states, there are two resonances at z_{R_1} and z_{R_2} and there is one second order pole at $z_d = E_d - i\gamma_d$. Then the following basis system expansion holds for $\phi_+ \in \Phi_-$:

$$\phi^+ = |z_d^-\rangle^{(0)}(-2\pi i a_{-2})^{(1)}\langle^+ z_d|\phi^+\rangle - |z_d^-\rangle^{(1)}(-2\pi i a_{-2})^{(0)}\langle^+ z_d|\phi^+\rangle + \sum_{i=1}^{2}|z_{R_i}^-\rangle(-2\pi i a_{-1}^{(i)})\langle^+ z_{R_i}|\phi^+\rangle + \int_0^{-\infty_{II}} d\omega|\omega^+\rangle\langle^+\omega|\phi^+\rangle\,, \tag{32}$$

the a_{-2}, $a_{-1}^{(i)}$ are the expansion coefficients in the Laurent series expansion of the S-matrix at the poles z_d, z_{R_1} and z_{R_2}, respectively.

The important distinction to (27) is that this basis system contains Jordan vectors and the Hamiltonian is not diagonal but can only attain the Jordan normal form:

$$\begin{pmatrix} \langle\psi^-|H^\times|z_d^-\rangle^{(0)} \\ \langle\psi^-|H^\times|z_d^-\rangle^{(1)} \\ \langle\psi^-|H^\times|z_{R_1}^-\rangle \\ \langle\psi^-|H^\times|z_{R_2}^-\rangle \\ \langle\psi^-|H^\times|\omega^+\rangle \end{pmatrix} = \begin{pmatrix} z_d & 0 & & & \\ 1 & z_d & & & \\ & & z_{R_1} & & \\ & & & z_{R_2} & \\ & & & & (\omega) \end{pmatrix} \begin{pmatrix} \langle\psi^-|z_d^-\rangle^{(0)} \\ \langle\psi^-|z_d^-\rangle^{(1)} \\ \langle\psi^-|z_{R_1}^-\rangle \\ \langle\psi^-|z_{R_2}^-\rangle \\ \langle\psi^-|\omega^+\rangle \end{pmatrix}. \tag{33}$$

The time evolution of the basis vectors on the r.h.s. of (33) is again given by the semigroup (16), *i.e.*, they have an arrow of time. However, now in addition to the exponential dependence the time evolution operator also transforms according to (24) inside the two dimensional eigenspace $\mathcal{M}_{z_d}$ with an additional linear time dependence. That second order poles of the S-matrix will introduce an additional linear time dependence in the decay law has been known for long time [17], only it was not clear what the vector was that evolved in this way. This vector $|z_d^-\rangle^{(1)}$ has now been defined. In addition the new result (25) shows that the different values of k get mixed up by the time evolution.

Whereas there is no doubt that ordinary, 0-order, Gamow vectors will be the suitable vectors to describe resonance states because of their properties II.1 ... II.5 above we have no idea what the physical meanining of the higher order Gamow vectors may be, if any. In contrast to the fact that ordinary Gamow states have been identified in abundance, *e.g.*, through their Breit-Wigner profile in scattering experiments, or the exponential decay law [18],

there is no convincing evidence for the existence of higher order poles in nature [19]. The k-th order Gamow states and their time evolution (25) are completely new and unusual. Their effect should also be so overwhelming that the meager evidence for higher order poles of the S-matrix which has been discussed in the past (A_2-splitting in particle physics, ^{8}Be in nuclear physics) would not be able to account for it. It is possible that there does not exist anything in nature that is described by higher order Gamow vectors and first order resonance poles is all there is. But since there is no theoretical reason against higher order poles of the S-matrix and these higher order Gamow "states" emerge naturally from the poles, it is worthwhile to investigate their properties further [23]. The only place that we can think to look for effects of these higher order Gamow states are the high-multiplicity events in high energy hadronic and nuclear collisions. That a quantum mechanically rather pure initial state of two hadrons can result in a high multiplicity event could have its origin in the highly impure "resonance" state associated with the $\mathcal{N}$-dimensional subspace of higher order Gamow kets.

5. Reversed time evolution and time reversal transformation

An irreversible time evolution on the microphysical level immediately leads to the question as to the time reversal transformation A_T. In the usual reversible time evolution (13) one always has with a state $W(t)$ also a state $W(-t)$ (or with the state vector $\phi(t)$ also a state vector $\phi(-t) = \mathrm{e}^{-i(-2t)H}\phi(t)$). The time reversed state defined by $W^T(t) \equiv A_T^{-1} W(t) A_T$ or $\phi^T = A_T^{-1}\phi$ can therefore be identified with the negative time state:

$$W'(-t) \equiv W^T(t) = W(-t); \quad \phi^T(t) = \phi(-t) \quad \phi'(-t) = \phi(-t)\,, \tag{34}$$

or in terms of the wave function since A_T is antiunitary:

$$\phi'(\boldsymbol{x}, -t) \equiv \phi^T(\boldsymbol{x}, t) = \phi^*(\boldsymbol{x}, t)\,. \tag{35}$$

For irreversible time evolution one has the two semigroups (15) and (16):

$$U^\dagger(t)|_{\Phi_-} \subset U_-^\times(t) \quad \text{in the space of states} \quad \Phi_- \subset \Phi_-^\times \quad \text{for } t \leq 0 \tag{36}$$

and

$$U^\dagger(t)|_{\Phi_+} \subset U_+^\times(t) \quad \text{in the space of observables} \quad \Phi_+ \subset \Phi_+^\times \quad \text{for } t \geq 0\,. \tag{37}$$

Therefore a state vector at the negative of the time t, *i.e.*, $\phi(-t) = \phi(|t|)$ cannot be obtained from $\phi(0)$ by this semigroup transformation. Thus there

is in general no negative time *state* $W(-t)$ (or $\phi(-t)$) which the time reversed state $W^T(t)$ (or $\phi^T = A_T^{(-1)}\phi$) could be identified with. In particular one cannot have the standard requirement (34), A_T cannot be the operator that transforms every conceivable state $W(t)$ into $W(-t)$. The operator $U(|t|)|_{\Phi_-}$ is not continuous operator from Φ_- to Φ_- but transforms out of the space of states Φ_- into the space Φ_+.

A mathematically consistent resolution of the problem with the time reversal operator, therefore, would be to define

$$A_T : \Phi_- \to \Phi_+;\ \Phi_+ \ni \psi^- = A_T\phi^+,\ \phi^+ \in \Phi_- . \tag{38}$$

This is indeed the solution suggested by conventional scattering theory where the *in*-states ϕ^+ or (ϕ^{in}) are the time reversed of the so called *out*-states ψ^- or (ψ^{out}). (The "*out*-states" ψ^{out} are actually observables and not states because they are specified by the detector whereas states are specified by the preparation apparatus (accelerator)). This solution is based on the standard A_T transformation properties of the eigenkets of the exact Hamiltonian H

$$A_T|E^{\pm}, \eta\rangle = \alpha|E^{\mp}, \eta_T\rangle;\quad A_T^2 = (-1)^{2j}1 , \tag{39}$$

which are defined by the Lipmann–Schwinger equation

$$|E^{\pm}, \eta\rangle = |E, \eta\rangle + \frac{1}{E - H \pm i\varepsilon}V|E, \eta\rangle \tag{40}$$

(η are the degeneracy quantum numbers which include angular momentum (spin) j and $|E, \eta\rangle$ are the eigenkets of $(H - V) = H_0$).

However (38) would mean that A_T transforms observables into states (and vice versa) and would therefore lead back to the identification of the set of observables with the set of states. Though $\Phi_+ \cap \Phi_- \neq \emptyset$ (zero vector), which means that there are vectors $\phi \in \Phi = \Phi_+ + \Phi_-$ which can represent states as well as observables, in general one cannot postulate that every observable $|\psi\rangle\langle\psi|$ can be prepared as a state. *E.g.*, in a typical scattering experiment the "*out*-states" represent highly correlated spherical waves whereas the prepared *in*-states are typically two uncorrelated plane waves (*e.g.*, two colliding monochromatic beams). The time reversal of this experiment would require a preparation apparatus that prepares highly correlated (with fixed phase relationship) incoming spherical waves that would be scattered into two uncorrelated plane waves. An apparatus that would accomplish this is impossible (or highly improbable) to build, at least in this world. Thus, not for every preparable state W can one also prepare a state which would be described by its time reversal transformed $W^T = A_T^{-1}WA_T$

(for another example see, *e.g.* chapter 13 of ref. [13]). This means that neither of the two quantities equated in the standard theory by (34) may have a physical meaning in terms of a preparation procedure.

The division of Φ into Φ_- (for states) and Φ_+ (for observables) that we obtained from the arrow of time is not contradicted by the physics of time reversal (because one can build a rotated, a translated or even a parity transformed preparation apparatus but one cannot build a time reversal transformed preparation apparatus). But it is just in contradiction with the standard theoretical description (38) for the time reversal operator. Therefore, if irreversible processes on the microphysical level are to be described, we need a time reversal operator more general than the one conventionally used in non-relativistic quantum mechanics and relativistic field theory. Wigner has already provided such a time reversal operator [20] which has also been mentioned a few times in the literature [21, 22]. But so far only the A_T with the standard property has found acceptance in physics.

The time reversal operator A_T is not defined by its action on states like (34) and (35), but by its relation to the observables [10, 22]. In general, the quantum mechanical operator A_T representing time reversal

$$T\begin{pmatrix} t \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} -t \\ \boldsymbol{x} \end{pmatrix} = -gx\,, \qquad g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

is an element of the (co)representation [20] of space-time transformations. Space-time transformations (*i.e.*, the extended (by time reversal and space reflection) Poincaré group for relativistic space time and the extended Galilei group for non-relativistic space-time $\begin{pmatrix} t \\ \boldsymbol{x} \end{pmatrix}$) were represented by unitary (and antiunitary for A_T) operators in the Hilbert space. The time reversal operator A_T is therefore defined (not by its action on the states) by its relation to the other symmetry operators like the space reflection U_P and the restricted space time transformations $U\left(\begin{pmatrix} t \\ \boldsymbol{x} \end{pmatrix}, \Lambda\right)$. An example of such a relation is [22]

$$A_T U\left(\begin{pmatrix} t \\ \boldsymbol{x} \end{pmatrix}, \Lambda\right) A_T^{-1} = U\left(\begin{pmatrix} -t \\ \boldsymbol{x} \end{pmatrix}, g\Lambda g\right). \tag{41}$$

From this one obtains the relation of A_T to the observables, which are the generators of $U(x, \Lambda)$. Examples of these relations are

$$A_T P_i A_T^{-1} = -P_i, \quad A_T J_i A_T^{-1} = -J_i, \quad A_T U_P A_T^{-1} = \varepsilon_T \varepsilon_I U_P, \tag{42}$$

$$A_T H A_T^{-1} = H, \quad A_T H_0 A_T^{-1} = H_0, \quad A_T S A_T^{-1} = S. \tag{43}$$

In here the generators P_i, H, J_i represent momentum, energy, angular momentum, respectively. The S-operator is a complicated function of the interaction Hamiltonian $V = H - H_0$ and U_P is the unitary and hermitian parity operator normalized to $U_P^2 = 1$. The quantities

$$\varepsilon_T = A_T^2, \qquad \varepsilon_I = (U_P A_T)^2 \equiv A_I^2,$$

are real phase factors which define the 4 different extensions of the restricted space-time symmetry transformations by space inversion $P = g$, time inversion $T = -g$ and space-time inversion $I = PT = -1$. (At this level where we have not talked about any charges, U_P could also be interpreted as representing the usual CP). Of the 4 possible extensions $(\varepsilon_T, \varepsilon_I) = (\pm 1, \pm 1)$ the almost exclusive choice [22, 10] for $(\varepsilon_T, \varepsilon_I)$ is:

$$\varepsilon_T = (-1)^{2j} \qquad \varepsilon_I = (-1)^{2j} \qquad \text{where } j \text{ is the spin}. \tag{44}$$

With this choice the only possibility for A_T is (38) which in the interpretation requires to identify the set of states with the set of observables (*i.e.*, no arrow of time) and to assign to every $W(t)$ a $W^T(t) \equiv A_T^{-1} W(t) A_T$ fulfilling (34). This is in contradiction to the experience that at least for some states it is highly improbable to also prepare their time reversed states (*cf.* remark above and ch. 13 Ref. [13]).

A way out would be to give up either irreversible time evolution or the time reversal operator. But since time reversal invariance, defined by (42) and (43) has consequences which can be tested experimentally, *e.g.*, reciprocity relations, it is useful to retain the notion of A_T also if one includes irreversible time evolution. We therefore want to explore the three other possibilities for $(\varepsilon_T, \varepsilon_I)$ which do not fulfill (44), *i.e.*, the other extensions of the space time symmetry groups provided by Wigner [20]. All three unconventional extensions involve *time-reversal doubling* of the representation spaces. This will introduce a further label r in addition to the quantum numbers which we called η in (40). For η we will choose angular momentum (spin) j, its component j_3 and other intrinsic quantum numbers n, which we do not specify further: $\eta = j_3, j, n$. Thus the basis vectors are denoted by $|e^{\pm}, j_3, j, n; r\rangle$. The four possible cases, of which the standard case (44) is given in the first row, are listed in the following table.

TABLE I

Extensions of the space-time symmetry groups by P and T

Characterization of the P and T extensions		Representation of	Representation of
ε_T	ε_I	U_P	A_T
$(-1)^{2j}$	$(-1)^{2j}$	1	C
$-(-1)^{2j}$	$(-1)^{2j}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}$
$(-1)^{2j}$	$-(-1)^{2j}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$
$-(-1)^{2j}$	$-(-1)^{2j}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}$

In this table C is the well known operator:

$$\begin{aligned} C|E, j_3, j, n; r\rangle &= \alpha(r)(-1)^{j-j_3}|E, -j_3, j, n; r\rangle \\ &= \sum_{j_3'} \alpha(r)|E, j_3', j, n; r\rangle C^{(j)}_{j_3' j_3}, \end{aligned} \tag{45}$$

where $\alpha(r)$ is a phase factor and the matrix $C^{(j)}_{\mu\nu}$ is given by

$$C^{(j)}_{\mu\nu} = (-1)^{j-\mu}\delta_{\mu,-\nu} \quad (-j \leq \mu, \nu \leq +j)\,. \tag{46}$$

The index $r(= + \text{ or } -)$ labels two subspaces $\mathcal{H}(r)$ in which all the other known observables B are identical, *i.e.*, B and U_g, where g are continuous space-time transformations not containing P and T, are given by

$$B = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}; \quad U_g = \begin{pmatrix} U_g & 0 \\ 0 & U_g \end{pmatrix}. \tag{47}$$

The index r thus also labels the rows and columns of the operator matrices in the Table I and in (47).

In the conventional case (44) the label r is not needed and A_T is given by (ignoring all the unspecified quantum numbers n)

$$A_T \mid E, j_3, j\rangle = (-1)^{j+j_3}\alpha' \mid E, -j_3, j\rangle \tag{48}$$

which we also write (suppressing from now on the quantum numbers j_3, j)

$$A_T \mid E\rangle = \alpha \mid E\rangle. \tag{49}$$

The exact eigenvectors $\mid E^{\pm}\rangle$ which are related to the $\mid E\rangle$ by (the formal solution of) the Lippmann–Schwinger equation (40), have the standard A^T transformation property (39)

In the conventional case (48), (39) we have *one* Hilbert space $\mathcal{H}$, *one* RHS $\Phi = \Phi_+ + \Phi_- \subset \mathcal{H} \subset \Phi^\times$; $\Phi_+ \cap \Phi_- \neq \emptyset$ and *one* pair of RHS's of Hardy class type (10), (11). The operator A_T can only be defined as in (38), *i.e.*:

$$A_T : \Phi_\pm \to \Phi_\mp; \quad A_T^\times : \Phi_\pm^\times \to \Phi_\mp^\times \tag{50}$$

which means that the two spaces Φ_- and Φ_+ are A_T transforms of each other. In our earlier discussion of the scattering experiment we have already concluded that this cannot be possible for empirical reason. Thus, if one has a quantum mechanical arrow of time, then the time reversal operator cannot be defined in the standard way with $A_T^2 = +1$ (or $A_T^2 = +(-1)^{2j}$).

Of the three unconventional cases the second and the third line of Table I gives the cases in which A_T transforms between parity eigenspaces of opposite (relative) parity. In these cases the label r can be given by the relative parity and is therefore also not needed. We therefore choose the case in the fourth line of the Table I characterized by ($\varepsilon_T = -(-1)^{2j}$, $\varepsilon_I = -(-1)^{2j}$). In this case the action of A_T is given by

$$A_T \mid E, r\rangle = \alpha(r) \mid E, -r\rangle; \quad \alpha^*(r)\alpha(-r) = \varepsilon_T = (-1)(-1)^{2j} \tag{51}$$

and the action of A_T upon the exact energy eigenvectors $\mid e^{\pm}, r\rangle$ is given by

$$A_T \mid E^{\pm}, r\rangle = \alpha(r) \mid E^{\mp}, -r\rangle\,. \tag{52}$$

In this new case we have two RHS's labeled by the index r, $\Phi^r \subset \mathcal{H}^r \subset \Phi^{r\times}$ and two pairs of the RHS's of Hardy classes, in place of the one pair (10) and (11):

$$\Phi_+^r \subset \mathcal{H} \subset \Phi_+^{r\times} \text{ and } \Phi_-^r \subset \mathcal{H} \subset \Phi_-^{r\times}, \quad r = \pm \tag{53}$$

$$\text{for any } \phi^+ \in \Phi_-^r \text{ we have a } \psi^- \equiv A_T\phi^+ \in \Phi_+^{-r}, \tag{54}$$

$$\text{for any } \psi^- \in \Phi_+^r \text{ we have a } \phi^+ \equiv A_T\psi^- \in \Phi_-^{-r}. \tag{55}$$

From this we conclude that the operator A_T maps the space $\Phi_\pm^r$ (continuously, one to one and) onto the space $\Phi_\mp^{-r}$

$$A_T : \Phi_\pm^r \to \Phi_\mp^{-r} \quad r = +, -\,. \tag{56}$$

The conjugate operator which is defined as the extension of the adjoint operator $A_T^\dagger : \mathcal{H}^r \to \mathcal{H}^{-r}$ according to

$$A_T^\dagger|_{\Phi^r} \subset A_T^\dagger \subset A_T^\times \text{ in } \Phi^r \subset \mathcal{H}^r \subset \Phi^{r\times}, \tag{57}$$

is then a (continuous, one to one) mapping between the corresponding dual spaces

$$A_T^\times : \Phi_\pm^{r\times} \to \Phi_\mp^{-r\times} \quad r = +, - \ . \tag{58}$$

Thus an operator A_T, which is compatible with our physical interpretation of the spaces Φ_+ and Φ_- has indeed been given by Wigner in [20] for the case $\varepsilon_T = \varepsilon_I = -1 (= -(-1)^{2j})$. In this case A_T (and $A_T^\dagger$) transforms — according to (56) — from the space Φ_+^r $(r = +)$, which contains vectors representing properties of the outgoing scattering products of our real experiment, into the space Φ_-^{-r}, which contains *in*-state vectors of scattering experiment which we cannot prepare (*e.g.*, incoming spherical waves with fixed phase relations). Vice versa, the space Φ_-^r (containing vectors that represent real preparable *in*-states) is mapped by A_T onto Φ_+^{-r} (containing properties which we cannot observe).

The same arguments apply according to (58) to the microphysical resonance states. The exponentially decaying Gamow vector $\psi^{\mathrm{G}} = \mid z_R, r^- \rangle \in \Phi_+^{r\times}$, $z_R = E_R - i\Gamma/2$, is mapped into a vector $\mid z_R^*, -r^+ \rangle \in \Phi_+^{-r\times}$ which exponentially decreases into the *negative* time direction. And the Gamow state of our resonance scattering experiment, $\tilde{\psi}^{\mathrm{G}} = \mid z_R^*, r^+ \rangle \sqrt{2\pi\Gamma} \in \Phi_+^{r\times}$, which exponentially grows from $t = -\infty$ to $t = 0$ (the time when the preparation is completed and the registration begins) is mapped by $A_T^\times$ into a vector $\mid z_R, -r^- \rangle \in \Phi_+^{-r}$ which like the $\mid z_R^*, -r^+ \rangle$ cannot be detected in our scattering experiment.

Thus mathematically, due to the time reversal doubling, we have two arrows of time pointing in opposite directions. For $r = +$ we have two semigroups (15), (16) both evolving into the same direction of time. For $t \leq 0$ we have the semigroup $U_-^\times = \mathrm{e}^{-iH^\times t}$ (of growth) and for $t \geq 0$ we have the semigroup $U_+^\times = \mathrm{e}^{-iH^\times t}$ (of decay). These provide our arrow of time. The RHS's (53) with $r = -$ describe the time-reversal image of our physical experiments; this time-reversed experiment we will find impossible to prepare.

One can show that like in the conventional case also in this new case with $(\varepsilon_T = -(-1)^j,\ \varepsilon_I = -(-1)^j)$ we have

$$\mid E, r^+ \rangle = \mid E, r^- \rangle S(E) = \mid E, r^- \rangle \mathrm{e}^{2\delta(E)} \quad \text{for} \quad r = \pm, \tag{59}$$

(where $\delta(E)$ is the phase shift and $S(E)$ the S-matrix). This is the consequence of "time reversal invariance" defined by (42) and (43). This means that the two spaces Φ_-^r (describing states) and the two spaces Φ_+^r (describing observables) with different values of r, $r = +$ and $r = -$, are not intermingled by the dynamics given by H or the S-operator. The experimentally tested consequences of time reversal invariance like reciprocity relation remain intact separately for each value of r.

In conclusion, we have seen that the quantum mechanical arrow of time and irreversible time evolution on the microphysical level (as exemplified by all quantum mechanical resonance states) are not in contradiction to time reversal invariance as defined by (42) and (43). However, for quantum physical systems with irreversible time evolutions (resonances) the time-reversal operator A_T is not the standard operator with $A_T^2 = (-1)^{2j}$. The price that we have to pay for describing irreversible time evolution and time reversal invariance in a consistent way is the doubling of the spaces. One pair of spaces, (53) with $r = +$, contains microphysical states that became and decay in our time direction. The other, (53) with $r = -$, contains microstates that became and decay in the opposite time direction. Time-reversal invariance, as defined by (42) and (43) for the observables, does not lead to a time symmetry for the states, like (34) and (35). This is in agreement with the empirical facts that some conceivable time-reversed states are highly improbable and practically impossible to prepare [13]. Theoretically, the time symmetry of the observables given by (42) and (43) can be broken for the states in two different ways leading to two arrows of time, $r = +$ and $r = -$. We belive that the principle, if any, that selects the one arrow over the other lies outside the scope of the theory

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