

BOREL QUANTIZATION: KINEMATICS AND DYNAMICS*

H.-D. DOEBNER AND P. NATTERMANN

Institute for Theoretical Physics A
Technical University Clausthal
38678 Clausthal-Zellerfeld, Germany
E-mail: `ashdd,aspn@pt.tu-clausthal.de`

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In this contribution we review results on the kinematics of a quantum system localized on a connected configuration manifold and compatible dynamics for the quantum system including external fields and leading to non-linear Schrödinger equations for pure states.

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1. Introduction

Physics starts in general with the notion of space and time. In a non-relativistic theory physical objects are understood to be localized in space and to move in time. In the case of classical mechanics these objects are represented by points in a configuration manifold M and its building blocks are geometrical objects living M : Borel sets on M or equivalently functions on M may be taken to describe the localization of the system, and vector-fields and their flows on M can be used to describe the displacements, *i.e.* the possible movements of the system on M . In this picture the functions and vectorfields serve to describe the kinematics of the classical system; endowed with the natural algebraic (semi-direct sum) structure they form the *kinematical algebra* $\mathcal{S}(M)$. In symplectic mechanics this algebra occurs as the algebra of functions on phase space that are affine in momentum.

The dynamics of such a classical system — the introduction of time — is given by a second order differential equations on M , *i.e.* geometrically,

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by vectorfields on the tangent bundle TM fulfilling a certain flip condition. If M is (Pseudo-)Riemannian this condition can be transported to the cotangent bundle and yields a natural restriction on the time evolution of functions on M .

A quantization of the classical theory will therefore involve two steps:

First, it requires a representation of the kinematical algebra by self-adjoint operators in a separable Hilbert-space. Starting from ideas of Segal [1] and Mackey [2] the *Quantum Borel Kinematics* was developed [3, 4] and classified unitarily inequivalent (local and differentiable) representations of the kinematical algebra $\mathcal{S}(M)$; we review the results in Section 2.1. The phase space description establishes a relation of this Borel Quantization to Geometric Quantization (Section 2.2). This relation will be used in Section 2.3 to generalize the scheme to include external fields.

Secondly, there should be an analogue of the classical condition on the time evolution of functions on M for their quantized counterparts. This relation will be established in Section 3.3. As we will see in Section 3.2 this condition leads to nonlinear Schrödinger equations, if pure states evolve into pure states.

2. Kinematics

2.1. Quantum Borel kinematics

As mentioned in the introduction the idea of Quantum Borel kinematics [3, 4] is to describe the quantization of the *localization* and the *displacement* of a system on a smooth connected configuration-manifold M .

The localization is characterized classically by Borel sets $B \in \mathcal{B}(M)$ and is “quantized” by a projection valued measure

$$E : \mathcal{B}(M) \rightarrow \text{Proj}(\mathcal{H}), \quad (1)$$

on a separable Hilbert space \mathcal{H} . Obviously, these projection valued measures provide a representation of the infinite dimensional algebra $C^\infty(M)$ of smooth functions on M via the spectral integral

$$Q : C^\infty(M) \ni f \mapsto Q(f) := \int_M f(m) dE_m, \quad (2)$$

on a domain $\vartheta_f = \{\psi \in \mathcal{H} \mid \int_M |f(m)|^2 d(\psi, E_m \psi) < \infty\}$.

The classical displacements of the system on M are described by the flow Φ_s^X of complete smooth vectorfields $X \in X_c(M)$. Borel sets are displaced by

$$B'_s = \left\{ \Phi_s^X(m) \mid m \in B \right\} \equiv \Phi_s^X(B) \in \mathcal{B}(M). \quad (3)$$

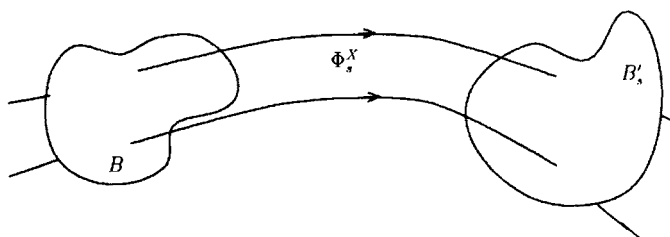


Fig. 1. Displacement of Borel sets

A quantization of these displacements is achieved by a representation of the flows Φ_s^X by one parameter groups of unitary operators on \mathcal{H} :

$$V_s^X = \exp \left(\frac{i}{\hbar} P(X) \right) \quad (4)$$

with generators $P(X)$ depending on the vectorfield. The representation should be consistent with the quantization of the localization of the system, *i.e.* for any given $X \in X_c(M)$ we require (E, V^X) to be a *system of imprimitivity* [2]:

$$V_s^X E(B) V_{-s}^X = E \left(\Phi_s^X(B) \right). \quad (5)$$

Using fundamental results on the structure of these systems of imprimitivity [5] one can show that for any given $X \in X_c(M)$ on a common dense domain \mathcal{D}_X for all $f, g \in C^\infty(M)$ and $\alpha \in \mathbb{R}$ [4]:

$$Q(f) + \alpha Q(g) = Q(f + \alpha g), \quad (6)$$

$$[Q(f), Q(g)] = 0, \quad (7)$$

$$[P(X), Q] = \frac{\hbar}{i} Q(\mathcal{L}_X f). \quad (8)$$

Thus it is natural to assume that the map $P : X_c(M) \rightarrow \mathcal{L}(\mathcal{H})$ respects also the algebraic structure of $X_c(M)$, *i.e.* for all $X, Y \in X_c(M)$ and $\alpha \in \mathbb{R}$ such that $X + \alpha Y, [X, Y] \in X_c(M)$, respectively, we require

$$P(X) + \alpha P(Y) = P(X + \alpha Y), \quad (9)$$

$$[P(X), P(Y)] = \frac{\hbar}{i} P([X, Y]), \quad (10)$$

together with (6)–(8) on a common invariant domain \mathcal{D} . Note, that $X_c(M)$ contains the closed Lie-subalgebra of vectorfields with compact support $X_0(M)$ correspond to a Lie-algebra homomorphism.

If we extend the representation of complete vectorfields to *all* vectorfields on M (thereby possibly loosing the selfadjointness of $\mathbf{P}(X)$), the pair (\mathbf{Q}, \mathbf{P}) forms a symmetric irreducible representation of the *kinematical algebra*

$$\mathcal{S}(M) = X(M) \otimes_s C^\infty(M) \tag{11}$$

with commutator (\mathcal{L}_X denotes the Lie derivative)

$$\left[(X_1, f_1), (X_2, f_2) \right]_{\mathcal{S}(M)} := \left([X_1, X_2]_{X(M)}, \mathcal{L}_{X_1} f_2 - \mathcal{L}_{X_2} f_1 \right). \tag{12}$$

With further assumptions on the homogeneity of \mathbf{E} (leading to spinless particles), on the locality of $\mathbf{P}(X)$ and on ϑ (leading to finite differential operators $\mathbf{P}(X)$ w.r.t. a differentiable structure on $M \times \mathbb{C}$) the Borel quantizations (\mathbf{Q}, \mathbf{P}) have been classified in [3, 4]: Unitarily inequivalent representations can be labeled by elements of

$$\boxed{\pi_1(M)^* \times \mathbb{R}}, \tag{13}$$

where $\pi_1(M)^*$ denotes the group of characters of the fundamental group of M . Furthermore, the Hilbert space can be realized as the space of square integrable sections of a flat Hermitean line bundle $(\eta, \langle \cdot, \cdot \rangle, \nabla)$ with respect to a smooth measure μ on M ,

$$\mathcal{H} \simeq L^2(\eta, \langle \cdot, \cdot \rangle, \mu). \tag{14}$$

In this realization the representation of the kinematical algebra reads for all sections $\sigma \in \vartheta \subset L^2(\eta, \langle \cdot, \cdot \rangle, \mu)$ (χ_B denotes the characteristic function of the set B , div_μ the divergence w.r.t. the measure μ)

$$\mathbf{E}(B)\sigma = \chi_B \cdot \sigma \tag{15}$$

$$\mathbf{Q}(f)\sigma = f \cdot \sigma \tag{16}$$

$$\mathbf{P}(X)\sigma = \frac{\hbar}{i} \nabla_X \sigma + \left(c + \frac{\hbar}{2i} \right) \operatorname{div}_\mu X \cdot \sigma. \tag{17}$$

In (13) the elements of $\pi_1(M)^*$ classify the inequivalent Hermitean line bundles η with flat Hermitean connection ∇ and hence a differentiable structure, whereas $c \in \mathbb{R}$ is an additional parameter independent of the topology of M . For a trivial bundle η the Hilbert space (14) is isomorphic to $L^2(M, \mu)$, and (17) transforms to $(\psi \in \vartheta \subset L^2(M, \mu))$:

$$\mathbf{P}(X)\psi = \frac{\hbar}{i} \mathcal{L}_X \psi + \omega(X)\psi + \left(c + \frac{\hbar}{2i} \right) \operatorname{div}_\mu X \cdot \sigma, \tag{18}$$

with a closed real differential one-form $\omega \in Z^1(M)$.

2.2. Relation to geometric quantization

The quantization method known as *geometric quantization* usually starts with a general classical phase space, *i.e.* a symplectic manifold (P, ω) . Given a configuration space M , the natural choice is the cotangent bundle $P := T^*M$ with canonical one-form θ and symplectic form $\omega = d\theta$. A “full quantization”, *i.e.* an irreducible selfadjoint representation of the Poisson-algebra $(C^\infty(P), \{.,.\})$ defined by the symplectic structure fails in general¹. Depending on the polarization chosen only a sub-algebra of “quantizable observables” is represented irreducibly.

If, for instance, a complex polarization on $P = \mathbb{R}^2 = T^*\mathbb{R}$ is chosen, the sub-algebra $Q(P)$ of polynomials in x, p of max. second order can be represented irreducibly. This leads to a quantization of the one-dimensional harmonic oscillator including the Hamiltonian of the system. Thus the dynamics of the particular system is also quantized.

In general for $P = T^*M$ and the vertical polarization the set $L(P)$ of functions linear in the momenta is used. $L(P)$ is isomorphic to the kinematical algebra, $L(P) \simeq \mathcal{S}(M)$,

$$\mathcal{S}(M) \ni (f, X) \mapsto Q_f + P_X \in L(P), \quad (19)$$

where the functions Q_f and P_X are defined as

$$Q_f(\alpha) := f(\pi_{T^*M}(\alpha)), \quad P_X(\alpha) := \alpha(X_{\pi_{T^*M}(\alpha)}), \quad \forall \alpha \in T^*M, \quad (20)$$

with Poisson brackets

$$\{Q_f, Q_g\} = 0, \quad \{P_X, Q_g\} = Q_{\mathcal{L}_X g}, \quad \{P_X, P_Y\} = P_{[X, Y]}. \quad (21)$$

In geometric quantization only the “standard” representation ($c = 0$, $\nabla \hat{=} \mathcal{L}$) of this algebra is considered.

Note that so far only the kinematics of the system has been quantized and a dynamics has to be introduced by an additional argument. In geometric quantization this is generally achieved by choosing another suitable polarization in which the Hamiltonian is quantizable and the use of the BKS-kernel (see *e.g.* [7]). Here we will proceed differently in Section 3.

¹ For a counterexample of a symplectic manifold providing a full quantization see [6].

2.3. Borel Quantization with external fields

To describe the interaction with external (magnetic) fields on M we utilize the phase space picture (see previous section). We introduce external fields in terms of closed two-forms $\phi \in A^2(M)$ on M by changing the symplectic form [7] on T^*M :

$$\omega_e := d\theta + e\pi^*\phi, \tag{22}$$

where $\pi : T^*M \rightarrow M$ is the projection of the cotangent bundle and e is a coupling constant (charge). Using the Poisson bracket $\{.,.\}_e$ induced by this structure we obtain commutation relations different from (21):

$$\{Q_f, Q_g\}_e = 0, \{P_X, Q_g\}_e = Q_{\mathcal{L}_X f}, \{P_X, P_Y\}_e = P_{[X,Y]} + eQ_{\phi(X,Y)}. \tag{23}$$

A Borel quantization of this algebra leads to the same operators (15)–(17) on $L^2(\eta, \langle ., . \rangle, \mu)$, but the Hermitean line bundle η is not longer flat, the external field comes in as a curvature two-form of the bundle [8],

$$R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = \frac{ie}{\hbar}\phi(X, Y). \tag{24}$$

However, such line bundles with curvature R exist — due to a geometric obstruction — if and only if

$$\left[\frac{1}{2\pi i}R\right] \in H^2(M, \mathbb{Z}), \tag{25}$$

i.e. the integral of ϕ over any singular two-cycle has to be an integer multiple of $2\pi\hbar$ (see *e.g.* [9]). Hence, only “quantized” values of the external field are admissible.

2.4. Examples

To illustrate the method of Borel Quantization we give two simple examples.

1. Euclidean space $M = \mathbb{R}^n$. [4, 10]
The classification (13) reduces to \mathbb{R} since

$$\pi_1(\mathbb{R}^n) = 0; \tag{26}$$

in global coordinates $\vec{x} = (x^1, \dots, x^n)$ the vectorfields are (using summation convention)

$$X = X^j(\vec{x}) \frac{\partial}{\partial x^j}. \tag{27}$$

The Hermitean line bundle η and the connection ∇ are trivial and

$$P(X) = \frac{\hbar}{2i} \left(X^j(\vec{x}) \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^j} X^j(\vec{x}) \right) + c \frac{\partial X^j(\vec{x})}{\partial x^j} \quad (28)$$

on $L^2(\mathbb{R}^n, d^n x)$. Note that the extra term does not influence the linear and angular momenta

$$P_j := P \left(\frac{\partial}{\partial x^j} \right) = \frac{\hbar}{i} \frac{\partial}{\partial x^j}, \quad (29)$$

$$L_{jk} := P \left(x^j \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^j} \right) = \frac{\hbar}{i} \left(x^j \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^j} \right). \quad (30)$$

Since \mathbb{R}^n is simply connected, any external field ϕ is admissible.

2. The n -torus $M = T^n = \underbrace{S^1 \times \dots \times S^1}_{n \text{ times}}$ [11, 12].

The classification of inequivalent QBK is

$$U(1)^n \times \mathbb{R}, \quad (31)$$

since

$$\pi_1(T^n) = \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{n \text{ times}}. \quad (32)$$

However, the Hermitean line bundles η are trivial, so that in local coordinates $\vec{\varphi} = (\varphi_1, \dots, \varphi_n)$

$$X = X^j(\vec{\varphi}) \frac{\partial}{\partial \varphi^j}, \quad (33)$$

$$P(X) = \frac{\hbar}{i} \frac{1}{2} \left(X^j(\vec{\varphi}) \frac{\partial}{\partial \varphi^j} + \frac{\partial}{\partial \varphi^j} X^j(\vec{\varphi}) \right) + c \frac{\partial X^j(\vec{\varphi})}{\partial \varphi^j} + \theta_j X^j(\vec{\varphi}) \quad (34)$$

on $L^2(T^n, d^n \phi)$ where θ_j can be chosen to be a constant $\theta_j \in [0, 2\pi)$. Hence we have a “topological influence” on the “angular” momentum operators:

$$J_j := P \left(\frac{\partial}{\partial \varphi^j} \right) = \frac{\hbar}{i} \frac{\partial}{\partial \varphi^j} + \theta_j, \quad (35)$$

leading to a Aharonov-Bohm-type effect.

The topology of T^n also restricts the possible choice of the external field ϕ . For constant external fields $\phi(X, Y) = \phi_0$, for instance, condition (24) implies a “modified Dirac quantization condition”

$$\phi_0 = \frac{\hbar}{2\pi e} n, \quad n \in \mathbb{Z}. \quad (36)$$

3. Dynamics

3.1. Generalized first Ehrenfest relation

In order to find conditions for the evolution of the quantum system, let us take a look at the classical system first (see [13, 14]).

Classical dynamics on M (as a configuration manifold) is in general described by a second order differential equation, *i.e.* a vectorfield \tilde{D} on the tangent bundle TM fulfilling the flip condition

$$T\pi_{TM} \circ \tilde{D} = \text{id}_{TM}, \quad (37)$$

or in local coordinates (\vec{x}, \vec{v}) of TM ,

$$\begin{aligned} \tilde{D}_{(x,v)} &\equiv (\dot{\vec{x}}, \dot{\vec{v}}) = (\vec{v}, \vec{F}(\vec{x}, \vec{v})) \\ &\Rightarrow \ddot{\vec{x}} = \vec{F}(\vec{x}, \vec{v}). \end{aligned}$$

If M is (Pseudo-) Riemannian with metric g^2 there is a natural isomorphism $g^\flat : TM \rightarrow T^*M$ with inverse g^\sharp . Using this isomorphism we can define a dynamical vectorfield $D := Tg^\flat \circ \tilde{D} \circ g^\sharp$ on T^*M and the condition (37) turns into

$$T\pi_{T^*M} \circ D = g^\sharp. \quad (38)$$

Let Φ_t be the flow of D on T^*M and

$$t \mapsto \alpha_t := \Phi_t(\alpha_0) \quad (39)$$

be the *classical time evolution* of the classical state $\alpha_0 \in T^*M$ then from (20) and (38) we get the following condition for the time evolution of the (quantizable) observable Q_f [4, 13, 14]:

$$\boxed{\frac{d}{dt}Q_f(\alpha_t) = P_{\text{grad}_g f}(\alpha_t).} \quad (40)$$

We use this formula for a quantum analogue, *i.e.* a condition for the time evolution of the quantum mechanical states \mathcal{Z} ,

$$t \mapsto \mathcal{Z}_t := \Phi_t^{QM}(\mathcal{Z}_0), \quad (41)$$

and require that in the mean quantum operators behave under the time evolution of the quantum states \mathcal{Z}_t like the classical observables under the

² For an n particle system g could be inherited from the geometry of the space manifold and the mass matrix of the particles.

time evolution of the classical states α_t , *i.e.* a generalization of the first Ehrenfest relation of quantum mechanics for all $f \in C^\infty(M)$

$$\boxed{\text{Exp}_{\mathcal{Z}_t}(\mathbf{Q}(f)) = \text{Exp}_{\mathcal{Z}_t}(\mathbf{P}(\text{grad}_g f))} \quad (42)$$

Now there are unitarily inequivalent representations of the operator $\mathbf{P}(X)$ leading to different conditions (42) on the time evolution of the quantum mechanical state. Indeed, there is *no unitary* time evolution of the states satisfying (42) except for $c = 0$. This is basically due to the algebraic (multiplicative) structure of $C^\infty(M)$.

We have two alternative ways to obtain evolution equations satisfying (42) for $c \neq 0$.

The first way is to look for evolutions of density matrices, *i.e.* of positive trace-class operators on \mathcal{H} with trace 1

$$t \mapsto \mathbf{W}_t := \Phi_t^{QM}(\mathbf{W}_0) \in \mathcal{T}_1^+(\mathcal{H}), \quad (43)$$

fulfilling

$$\boxed{\text{Tr}(\mathbf{Q}(f)\mathbf{W}_t) = \text{Tr}(\mathbf{P}(\text{grad}_g f)\mathbf{W}_t)} \quad (44)$$

By the usual interpretation of $\mathbf{W} \in \mathcal{T}_1^+(\mathcal{H})$ as statistical mixtures, Φ^{QM} has to be *linear*.

For completely positive, norm-continuous³ there is a classification of their generators given by Lindblad [15]. Though we are not necessarily looking for norm-continuous evolutions, there are Lindblad-type evolution equations satisfying (44) [13, 14]; the details of this way will be explained elsewhere.

The second alternative is to restrict (42) to pure states,

$$t \mapsto \sigma_t \in L^2(\eta, \langle \cdot, \cdot \rangle, \mu), \quad (45)$$

fulfilling

$$\boxed{\langle \sigma_t | \mathbf{Q}(f) \sigma_t \rangle = \langle \sigma_t | \mathbf{P}(\text{grad}_g f) \sigma_t \rangle} \quad (46)$$

If the line bundle is trivializable, this leads formally to nonlinear Schrödinger equations for wavefunctions $\psi_t \in L^2(M, \mu)$, as we will see in the next section.

³ Actually, norm-continuity is a strong restriction. For unitary evolutions it corresponds to bounded Hamilton-operators!

3.2. Nonlinear Schrödinger equations

For trivial line bundles η , $L^2(\eta, \langle \cdot, \cdot \rangle, \mu) \simeq L^2(M, \mu)$, and for wave functions $\psi_t \in L^2(M, \mu)$ condition (42) reads

$$\frac{d}{dt} \langle \psi_t | \mathbf{Q}(f) \psi_t \rangle = \langle \psi_t | \mathbf{P}(\text{grad}_g f) \psi_t \rangle \tag{47}$$

$$\Leftrightarrow \frac{d}{dt} \int_M f(m) \rho_t(m) d\nu_g(m) = \int_M f(m) (-j_t(m) + c \Delta_g \rho_t(m)) d\nu_g(m), \tag{48}$$

where

$$\begin{aligned} \rho_t(m) &:= \psi_t(m) \bar{\psi}_t(m), \\ j_t^\omega(m) &:= \frac{\hbar}{i} \Big(\bar{\psi}_t(m) (\text{grad}_g \psi_t)(m) - \psi_t(m) (\text{grad}_g \bar{\psi}_t)(m) \Big) + \rho_t(m) g^\sharp \omega \end{aligned} \tag{49}$$

are the probability distribution and the probability current, respectively. Since (48) has to hold for all $f \in C^\infty(M)$, we get a Fokker-Planck-type equation

$$\dot{\rho}_t + \text{div}_g^\omega j_t = c \Delta_g \rho_t. \tag{50}$$

restricting the evolution equation of the pure state ψ_t (Schrödinger equation) to

$$i\hbar \frac{\partial}{\partial t} \psi_t = \left(-\frac{\hbar^2}{2} \Delta_g^\omega + V \right) \psi_t + i \frac{\hbar c}{2} \frac{\Delta_g \rho_t}{\rho_t} \psi_t + R[\psi_t] \psi_t, \tag{51}$$

where $\Delta_g^\omega := (\text{div}_g + \frac{i}{2\hbar} \omega) \circ (\text{grad}_g + \frac{i}{2\hbar} g^\sharp \omega)$, $R[\cdot]$ is some real-valued functional on \mathcal{H} and V is a real potential on M .

If we assume for a fairly general model [16, 17] that $R[\cdot]$ is “similar” to the imaginary nonlinearity $\Delta_g \psi / \psi$ we get

$$R[\psi] = \sum_{j=1}^5 R_j[\psi],$$

where

$$\begin{aligned} R_1[\psi] &:= \frac{\text{div}_g j^\omega}{\rho}, & R_2[\psi] &:= \frac{\Delta_g \rho}{\rho}, & R_3[\psi] &:= \frac{g(j^\omega, j^\omega)}{\rho^2}, \\ R_4[\psi] &:= \frac{d\rho \cdot j^\omega}{\rho^2}, & R_5[\psi] &:= \frac{d\rho \cdot \text{grad}_g \rho}{\rho^2}. \end{aligned} \tag{52}$$

4. Final remarks

We have shown how nonlinear quantum mechanical evolution equations arise from Borel quantization on a connected, (Pseudo-)Riemannian configuration manifold; they are natural generalizations of the Doebner–Goldin equations on $M = \mathbb{R}^3$ [16, 17] to more general manifolds. Some of the properties (see *e.g.* the contributions in [18]) of the Doebner–Goldin equations in \mathbb{R}^3 are also valid on more general Riemannian manifolds M .

In particular, there is a sub-family linearizable [19] by ⁴

$$\psi \mapsto N_{(\Lambda, \gamma)}[\psi] := \psi^{\frac{1}{2}(1+\Lambda+i\gamma)} \bar{\psi}^{\frac{1}{2}(1-\Lambda+i\gamma)}. \quad (53)$$

Obviously, this transformation leaves the probability density invariant. As in non-relativistic quantum mechanics all measurements can in principle be reduced to positional ones (see *e.g.* [20, 21]), $N_{(\Lambda, \gamma)}$ was thus called a *nonlinear gauge transformation* and one can identify gauge equivalent classes among the equations in (51) [22].

Because of some confusion in the context of nonlinear quantum theories (and superluminal communications therein) we emphasize finally that equation (51) describes only the nonlinear time evolution of pure states. Mixtures of these pure states have to be identified according to the set of observables. A description of this idea has already been given by Mielnik in [21]: taking positions as *primitive observables* and generating the set of all observables by combining the primitive observables with the time evolutions (under different external conditions such as V and ϕ) one defines a mixture as an equivalence class of probability measures on the set of pure states w.r.t. the observables. By construction, the so-defined mixtures are consistent with the time evolution of pure states and it is evident for a nonlinear time evolution of pure states that the mixtures are *not* represented by trace class operators in $\mathcal{T}_1^+(\mathcal{H})$ (see also [23] for a discussion of observables in a nonlinear theory).

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⁴ Note that for $\Lambda \neq \pm 1$ the linearization is well-defined only for non-vanishing wave-functions.

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ERRATUM

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H.-D. Doebner, P. Nattermann, Borel Quantization: Kinematics and Dynamics, *Acta Phys. Pol.* **B27**, 2327 (1996).

There is a misprint in equations (42), (44), and (46) on page 2335, a time derivative on the left hand side is missing. These formulas should read:

$$\boxed{\frac{d}{dt} \text{Exp}_{Z_t} (Q(f)) = \text{Exp}_{Z_t} (P(\text{grad}_g f))} . \quad (42)$$

$$\boxed{\frac{d}{dt} \text{Tr} (Q(f) W_t) = \text{Tr} (P(\text{grad}_g f) W_t)} . \quad (44)$$

$$\boxed{\frac{d}{dt} \langle \sigma_t | Q(f) \sigma_t \rangle = \langle \sigma_t | P(\text{grad}_g f) \sigma_t \rangle} . \quad (46)$$