QUANTUM VORTEX CONFIGURATIONS*

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(Received July 3, 1996)

Quantum vortex configurations associated with representations of the group of area- or volume-preserving diffeomorphisms are obtained by geometric quantization techniques. This article reviews some of the mathematical results and physical predictions, providing a current perspective. A brief discussion of vortex creation and annihilation field operators is included.

PACS numbers: 03.65. Bz

1. Introduction

Quantized vortex motion is of interest in the theory of superfluidity, type II superconductivity, defects, and surface phenomena, and in fundamental discussions of flux tubes, anyons, and monopoles. Here I shall review an approach to quantization of vorticity in an ideal, incompressible fluid, based on coadjoint orbits of diffeomorphism groups. This work was begun in collaboration with Menikoff and Sharp at Los Alamos National Laboratory [1–3], and some of the perspectives presented are based on my continuing work with Sharp [4, 5].

The paper is organized as follows. For purposes of background and contrast with our approach, I begin by mentioning in Sec. 2 some simpler theories where classical point vortices moving in the plane are quantized [6–9]. Here the models incorporate only finitely many degrees of freedom in describing the vortex systems.

In Sec. 3 results from the diffeomorphism group approach are summarized. These include the outcomes of geometric quantization, and consequent predictions as to what properties quantum vortex configurations in

* Presented at the II German–Polish Symposium "New Ideas in the Theory of Fundamental Interactions", Zakopane, Poland, September 1995.

ideal two- and three-dimensional fluids can have. In two dimensions, it turns out that pure point vortices cannot occur at all. Thus the situation is already quite different from the usual framework in which the motion of classical point vortices is quantized, and existence of the quantum vortices is assumed from the start. Nevertheless ideal point vortex *dipoles* in the plane do exist in the diffeomorphism group approach, as well as spinning vortex dipoles. Furthermore one-dimensional *filaments* (string-like configurations) of vorticity in the plane occur, as well as extended, non-uniform vortex patches. Such vortex configurations may obey the intermediate statistics of "anyons". In three-dimensional space, the classically occurring one-dimensional filaments of vorticity are forbidden in the quantum theory. Here, however, two-dimensional *surfaces* of vorticity (tubes, ribbons, knotted tubes and ribbons, ribbons with twists, *etc.*) are quantum-mechanically permitted. New internal degrees of freedom are thus expected. These results in three-space are consistent with the perspective adopted by Owczarek [10, 11] in recent work.

Sec. 4 describes some ideas as to how the creation and annihilation of vortex configurations should be described. The goal is to treat systems with large numbers of (possibly knotted) vortices, as would be needed in realistic models of superfluidity [12, 13]. The approach presented here is modeled on work with Sharp in which we obtain creation and annhihilation fields obeying q-commutation relations as intertwining operators in a hierarchy of N-anyon diffeomorphism group representations [4, 5]. It is the subject of our continuing research.

2. Point vortices in the plane

The classical hydrodynamical equations of motion for an ideal, incompressible fluid (Euler's equations) have vortex solutions. Thus the most straightforward — in a sense, "naive" — way to quantize vortex motion for a planar fluid is to begin with point vortices, and to quantize their classical dynamics. As that dynamics can be described by a Hamiltonian function, quantization may be accomplished by replacing the Hamiltonian and the positional vortex coordinates with self-adjoint operators, whose commutators respect the corresponding Poisson brackets.

Now a point vortex at the origin in the plane, having vorticity κ, is associated with a velocity field $\boldsymbol{v}$ with radial component $v_r = 0$ and tangential component $v_\theta = \kappa/2\pi r$. Then $\nabla \times \boldsymbol{v} = \kappa\,\Delta^{(2)}(\boldsymbol{r})$. We think of such a vortex as occurring physically in an infinitesimally thin film of ideal, incompressible, inviscid superfluid. Consider a pair of point vortices of this sort, located at the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in $\boldsymbol{R}^2$. Let $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ be the relative positional coordinates, with Cartesian components (x, y) and polar components

(r, θ). According to Euler's equations each vortex moves as if transported by the velocity field of the other. This means that

$$\dot{\boldsymbol{r}} = \frac{\kappa}{\pi|r|}\hat{\theta}, \tag{1}$$

where $\hat{\theta}$ is the unit vector in the direction of the polar angle and κ is the vorticity at each point. Equivalently

$$\dot{x} = -\frac{\kappa y}{\pi(x^2+y^2)}, \quad \dot{y} = \frac{\kappa x}{\pi(x^2+y^2)}. \tag{2}$$

Evidently the motion is described by a Hamiltonian, whose formula is

$$H = -\frac{\kappa}{\pi}\ln|\boldsymbol{r}| = -\frac{\kappa}{2\pi}\ln(x^2+y^2). \tag{3}$$

It is easy to check that

$$\{x, H\} = \dot{x}, \quad \{y, H\} = \dot{y}, \tag{4}$$

where the Poisson bracket is defined by

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}. \tag{5}$$

When this classical motion is quantized, x and y are represented by canonically conjugate operators, while H is proportional to the logarithm of the quantum-mechanical harmonic oscillator Hamilton [6, 7]. Alternatively, following Leinaas and others, indistinguishability of the vortices can be put in by quantizing only observables that are manifestly symmetric under the vortex exchange $\boldsymbol{r} \to -\boldsymbol{r}$. Thus for a pair of vortices, the three-dimensional Poisson algebra generated by x^2, y^2, and $xy + yx$ is represented by self-adjoint operators.

The quantum theory of three point vortices can be described by representing the Lie algebra of the special linear group $SL(2, \boldsymbol{R})$, or its semidirect product with the group of translations in the plane; *i.e.*, the area-preserving affine transformations [8, 9]. Approaches such as these seek to simplify the problem from the outset by selecting only finitely many coordinates that can (partially) describe the collective motion of the fluid; these are represented by operators in the quantum theory. The procedures are in a way analogous to "first quantization"–vortices, like particles, are taken as given and moving classically; then that classical motion is quantized.

In contrast Menikoff, Sharp, and I seek to quantize the underlying fluid velocity field directly, by representing an infinite-dimensional Lie group (the diffeomorphism group) and its corresponding Lie algebra (the algebra of vector fields). This leads to a more realistic, though still highly idealized, model. The "particle content" of the quantum theory is given by the vortex configurations in appropriate coadjoint orbits, regarded as excitations in the underlying field (*i.e.*, quanta). In that sense, our procedures are more analogous to a "second quantized" field theory. I shall now describe some aspects of this model.

3. Diffeomorphism groups and vorticity

As suggested by Marsden and Weinstein [14], take the classical configuration space for an incompressible fluid in the plane to be the group manifold of the group of area-preserving, C^∞ diffeomorphisms of $\boldsymbol{R}^2$. Thus a diffeomorphism of a region — a smooth, invertible transformation whose inverse is also smooth — is identified with a configuration of fluid in the region. The group operation is, of course, composition of diffeomorphisms. This group is called $G = SDiff(\boldsymbol{R}^2)$, where S stands for "special". For the case of $\boldsymbol{R}^3$, G is the group of volume-preserving diffeomorphisms. The property of preserving the area or volume means that the Jacobian of the diffeomorphism is identically one. For important technical reasons, we also restrict the diffeomorphisms ϕ to those having the property that $\phi(\boldsymbol{x}) \to \boldsymbol{x}$ (rapidly in all derivatives) as $|\boldsymbol{x}| \to \infty$; *i.e.*, our configurations describe a fluid stationary at infinity.

The group elements are just the general coordinate transformations of the space in which fluid is located that respect the incompressibility of the fluid. The action of G on its own group manifold permits its interpretation as a symmetry group. As a manifold G is infinite-dimensional, and its cotangent bundle $T^*(G)$ can be taken to be the classical phase space.

Marsden and Weinstein describe the classical hydrodynamics of an ideal incompressible fluid using the noncanonical Lie–Poisson bracket assoicated with the Lie algebra of G. This Lie algebra is the set $g = sVect(\boldsymbol{R}^s)$ of all C^∞ divergenceless vector fields $\boldsymbol{v}$ on $\boldsymbol{R}^s$ ($s = 2$ or 3), that vanish (rapidly in all derivatives) at infinity. It is equipped with the usual Lie bracket

$$[\boldsymbol{v}_1, \boldsymbol{v}_2] = (\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_2 - (\boldsymbol{v}_2 \cdot \nabla)\boldsymbol{v}_1 \tag{6}$$

for vector fields $\boldsymbol{v}_1, \boldsymbol{v}_2 \in g$. In (6) the vector fields have the interpretation of fluid velocities, as they describe infinitesimal changes in spatial configurations. They belong to the *tangent space* to $SDiff(\boldsymbol{R}^s)$ at the identity. Of course the property of being divergenceless is preserved by the Lie bracket, so that we have a proper Lie subalgebra of the full algebra of vector fields on the spatial manifold. Each such vector field generates a one-parameter subgroup (a *flow*) in the group G.

Note that $M = \boldsymbol{R}^2$ may be endowed with the usual symplectic structure associated with the Poisson bracket (5). Then G is actually the group of *symplectic diffeomorphisms* of the plane, and g is the algebra of Hamiltonian vector fields.

Regarding G as a symmetry group, we expect its continuous unitary representations (CURs) to describe the possible kinematics of systems arising from quantization of the classical motion. Self-adjoint operators in such a representation, indexed by the vector fields, are to be recovered as generators of the corresponding one-parameter unitary subgroups. Thus we shall obtain a self-adjoint representation of the Lie algebra of divergenceless vector fields. The velocity fields are serving as test functions for a *local current algebra*, as described in [15–17]. The idea of using this algebra to study quantized fluid motion was already included in the program proposed by Rasetti and Regge [12].

Now the velocity fields $\boldsymbol{v}$, being smooth, cannot serve as the correct collective coordinates with which to describe discontinuities or singularities in the fluid flow. Yet smoothness is needed for g to close as a Lie algebra. The coadjoint orbit method for constructing group representations automatically brings in the *dual* to the Lie algebra, which we denote g'. The space g' consists of continuous, linear mappings from g to $\boldsymbol{R}$. For finite-dimensional Lie algebras such a dual space is always isomorphic, as a vector space, to the original algebra; but in the infinite-dimensional case it is not. Its elements here are *generalized* vector fields, *i.e.* vector fields whose components are generalized functions (distributions). These can have discontinuities or singularities; *e.g.* Δ-function singularites such as occur in the case of point vortices, derivatives of Δ-functions, and so forth. This is one reason why the description of quantized vortex motion by an infinite-dimensional Lie group is essentially different from descriptions based on finite-dimensional Lie groups or algebras. We interpret such generalized vector fields, dual to the velocity fields, as *momentum density* fields in the theory.

For $\boldsymbol{A} \in g'$, let $\langle \boldsymbol{A}, \boldsymbol{v} \rangle = \int \boldsymbol{A}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d^s x$ ($s = 2$ or 3) denote the value of $\boldsymbol{A}$ on $\boldsymbol{v}$. As $\nabla \cdot \boldsymbol{v} = 0$, we have $\boldsymbol{v} = \nabla \times \chi$, with χ defined up an arbitrary gradient, so that there is a sort of "gauge freedom" in choosing the stream function χ. Since $\boldsymbol{v} \to 0$ rapidly as $|\boldsymbol{x}| \to \infty$, we can choose this gauge so that $\chi \to 0$ rapidly at infinity. We write $\chi_{\boldsymbol{v}}$ with the subscript to make explicit the dependence of χ on $\boldsymbol{v}$. For a two-dimensional fluid, χ can be visualized as being always perpendicular to the plane of the fluid; *i.e.*, a scalar function multiplied by the unit normal vector. In three dimensions, it is a vector whose direction is arbitrary.

Now $\langle \boldsymbol{A}, \boldsymbol{v} \rangle = \int (\nabla \times \boldsymbol{A})(\boldsymbol{x}) \cdot \chi_{\boldsymbol{v}}(\boldsymbol{x})\, d^s x$, as long as the contribution of any boundary term to the integration by parts vanishes. This condition does not require that $\boldsymbol{A}(\boldsymbol{x})$ vanish at infinity, only that its growth be correctly offset by the rapid vanishing of χ. Then if we define the *vorticity density* $\boldsymbol{B} = \nabla\times\boldsymbol{A}$, we have the fact that (as an element of g') $\boldsymbol{A}$ is only defined up to an arbitrary gradient, and is uniquely specified by $\boldsymbol{B}$. Thus $\langle \boldsymbol{A}, \boldsymbol{v}\rangle = \int \boldsymbol{B}(\boldsymbol{x})\cdot\boldsymbol{\chi}(\boldsymbol{x})\, d^s x = \langle \boldsymbol{B}, \boldsymbol{\chi}\rangle$. The fact is also useful that

$$\nabla\times(\boldsymbol{v}_1\times\boldsymbol{v}_2) = [\boldsymbol{v}_1, \boldsymbol{v}_2], \tag{7}$$

so that a stream function for the Lie bracket (satisfying the above gauge condition) is given by $\boldsymbol{\chi}_{[\boldsymbol{v}_1,\boldsymbol{v}_2]} = \boldsymbol{v}_1\times\boldsymbol{v}_2$.

Next we proceed to obtain representations of $SDiff(\boldsymbol{R}^s)$ on coadjoint orbits through the Kirillov–Kostant–Souriau method of geometric quantization [19].

First we write the adjoint and coadjoint representations of G. Orbits in the coadjoint representation are interpreted as reduced phase spaces for the classical theory, and include point vortices of the sort described above. Then we look at some specific coadjoint orbits to ascertain whether or not a *polarization* exists. Finding a polarization amounts to choosing a set of phase-space coordinates (half of them) that are simultaneously observable in the quantum theory. Absence of a polarization means, in our context, that there is a physical obstacle to quantization–namely, the Heisenberg uncertainty principle cannot be satisfied on the coadjoint orbit in question. The requirement that a polarization exist thus leads to mathematically rigorous conclusions about the types of quantum configurations that are possible; see also [18] for some further discussion.

The adjoint representation acts on g, and is given (for $\boldsymbol{v}_1, \boldsymbol{v}_2 \in g$) by $(Ad\,\boldsymbol{v}_1)\boldsymbol{v}_2 = [\boldsymbol{v}_1, \boldsymbol{v}_2]$. For $\phi \in G$) we have $\boldsymbol{v}' := (Ad\,\phi)\boldsymbol{v} = [J_{\phi^{-1}}\boldsymbol{v}]\circ\phi$; where J_ϕ is the derivative of the diffeomorphism given by the matrix $[J_\phi]^j_k = \partial_k\phi^j$. Equivalently, $\boldsymbol{\chi}' = (Ad\,\phi)\boldsymbol{\chi} = [J_\phi]^\dagger[\boldsymbol{\chi}\circ\phi]$, where $\dagger$ denotes the matrix transpose. The coadjoint representation acts on g', and is defined (for $\boldsymbol{A} \in g'$) by taking $\boldsymbol{A}' := (Coad\phi)\boldsymbol{A}$ to satisfy $\langle \boldsymbol{A}', \boldsymbol{v}\rangle = \langle \boldsymbol{A}, (Ad\,\phi^{-1})\boldsymbol{v}\rangle$. Then $\boldsymbol{A}' = [J_\phi]^\dagger[\boldsymbol{A}\circ\phi]$, or equivalently, $\boldsymbol{B}' = ([J_{\phi^{-1}}]\boldsymbol{B})\circ\phi$.

Suppose that a specific orbit Δ in the coadjoint representation has been selected. For a point $\boldsymbol{A}_0 \in \Delta$, let K be the stability subgroup of G; *i.e.*, the group of all diffeomorphisms ϕ leaving $\boldsymbol{A}_0$ fixed. As usual, Δ may be identified with the quotient space G/K, and the dimension of Δ is the codimension of K in G. A polarization in the orbit Δ requires a group H, with $K \subset H \subset G$, where the codimension of H in G is half the codimension of K in G. Letting h be the Lie algebra of H, the condition that must hold is: $\langle \boldsymbol{A}_0, [\boldsymbol{v}_1, \boldsymbol{v}_2]\rangle = 0$ for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in h$. Then $\boldsymbol{A}_0$ will define a *character* (a one-dimensional unitary representation) of H, which can induce the desired representation of G.

Consider, then, some particular coadjoint orbits.

The orbit containing one pure point vortex, as in Sec. 2, is written by letting $\boldsymbol{B} = \boldsymbol{\Omega}_{\boldsymbol{y}} = \kappa\,\delta^{(2)}(\boldsymbol{x} - \boldsymbol{y})$, describing a vortex centered at $\boldsymbol{y}$. We have $(Coad\,\phi)\,\boldsymbol{\Omega}_{\boldsymbol{y}} = \boldsymbol{\Omega}_{\phi^{-1}(\boldsymbol{y})}$, so that the orbit is parameterized by the values of $\boldsymbol{y} \in \boldsymbol{R}^2$. The stability group $K_{\boldsymbol{y}_0}$ associated with fixed $\boldsymbol{y}_0$ is just the subgroup $\{\phi \in G\,|\,\phi^{-1}(\boldsymbol{y}_0) = \boldsymbol{y}_0\}$. But $K_{\boldsymbol{y}_0}$ is a maximal subgroup of G, which means that the desired group H intermediate between K and G cannot exist. Likewise for N pure point vortices, the vorticity distribution is given by $\boldsymbol{B} = \boldsymbol{\Omega}_{\{(\kappa_1,\boldsymbol{y}_1),\ldots,(\kappa_N,\boldsymbol{y}_N)\}} = \kappa_1\,\boldsymbol{\Omega}_{\boldsymbol{y}_1} + \ldots + \kappa_N\,\boldsymbol{\Omega}_{\boldsymbol{y}_N}$, and

$$(Coad\,\phi)\,\boldsymbol{\Omega}_{\{(\kappa_1,\boldsymbol{y}_1),\ldots,(\kappa_N,\boldsymbol{y}_N)\}} = \kappa_1\,\boldsymbol{\Omega}_{\phi^{-1}(\boldsymbol{y}_1)} + \ldots + \kappa_N\,\boldsymbol{\Omega}_{\phi^{-1}(\boldsymbol{y}_N)}\,. \quad (8)$$

Diffeomorphisms in the stability group K leave the individual points $\boldsymbol{y}_j$ fixed, or else permute those for which the κ_j are equal. As in [4] and [5], the components of this stability group map naturally to braids. Since each $\boldsymbol{y}_j$ ranges over $\boldsymbol{R}^2$ the coadjoint orbit is $2N$-dimensional, but there is still no polarization. Groups H intermediate between K and G exist, and can be obtained by relaxing the constraints on some of the $\boldsymbol{y}_j$; but these do not obey the polarization condition that $\langle \boldsymbol{A}_0, [\boldsymbol{v}_1, \boldsymbol{v}_2]\rangle = 0$ for arbitrary $\boldsymbol{v}_1, \boldsymbol{v}_2 \in h$. It is natural to consider the possibility of spreading out the distribution of vorticity. If Δ is the coadjoint orbit containing a single uniform vortex patch (*i.e.*, a disk of fixed radius with uniform, nonvanishing vorticity in its interior), it is not difficult to show that the polarization we seek is still non-existent.

The absence of polarizations in these cases cannot be rectified by complexification of the coadjoint orbit. We are forced to conclude that, in the plane, pure point vortices are actually incompatible with the assumptions of the model, as are uniform patches of vorticity. Such results might lead one to be skeptical of our geometric quantization scheme, were it not for the fact that nontrivial vortex configurations in two dimensions do exist.

The most elementary of these are *point vortex dipoles*. Let $\boldsymbol{\Omega}_{(\boldsymbol{y},\lambda)}$ be defined by

$$\langle \boldsymbol{\Omega}, \chi\rangle = -(\lambda^j \partial_j \chi)(\boldsymbol{y})\,, \quad (9)$$

for $\boldsymbol{y} \in \boldsymbol{R}^2$ and λ a tangent vector to $\boldsymbol{R}^2$ at $\boldsymbol{y}$. Physically λ is a dipole moment of vorticity. Writing $\boldsymbol{\Omega}' = (Coad\,\phi)\,\boldsymbol{\Omega} = \boldsymbol{\Omega}_{(\boldsymbol{y}',\lambda')}$ in the coadjoint representation, we have $\boldsymbol{y}' = \phi^{-1}(\boldsymbol{y})$ and $(\lambda')^k = \partial_j(\phi^{-1})^k(\boldsymbol{y})\lambda^j$ (here we sum over the repeated index j). The coadjoint orbit is parameterized by $\lambda \neq 0$ and $\boldsymbol{y}$, and is thus four-dimensional. The stability subgroup $K_{(\boldsymbol{y}_0,\lambda_0)}$ consists of diffeomorphisms for which $\phi^{-1}(\boldsymbol{y}_0) = \boldsymbol{y}_0$, *and* for which the derivative at $\boldsymbol{y}_0$ is an upper triangular matrix (in the appropriate basis) with diagonal entries equal to one. Giving up the constraints on the derivative matrix yields a polarization group $H = \{\phi\,|\,\phi^{-1}(\boldsymbol{y}_0) = \boldsymbol{y}_0\}$. For any $\boldsymbol{v}$,

$\langle \boldsymbol{\Omega}_{(\boldsymbol{y}_0,\lambda_0)}, \boldsymbol{\chi}_{\boldsymbol{v}}\rangle = \hat{\mathbf{k}}\cdot[\lambda_0\times\boldsymbol{v}(\boldsymbol{y}_0)]$ where $\hat{\mathbf{k}}$ is the unit normal to the plane; thus $\langle \boldsymbol{\Omega}_{(\boldsymbol{y}_0,\lambda_0)}, \boldsymbol{\chi}_{\boldsymbol{v}}\rangle = 0$ if $\boldsymbol{v}\in h$. We have trivially that $\langle \boldsymbol{A}_0, [\boldsymbol{v}_1, \boldsymbol{v}_2]\rangle = 0$ for $\boldsymbol{v}_1, \boldsymbol{v}_2 \in h$, and the character defined by $\boldsymbol{A}_0$ is identically one. The result resembles a single-particle representation of the local current algebra. Now a parameterized family of interesting coadjoint orbits with polarizations can be obtained. Let

$$\boldsymbol{\Omega}^{\kappa}_{(\boldsymbol{y},\lambda)} = \kappa\,\delta^{(2)}(\boldsymbol{x}-\boldsymbol{y}) + \lambda\cdot\nabla_{\boldsymbol{x}}\,\delta^{(2)}(\boldsymbol{x}-\boldsymbol{y})\,. \tag{10}$$

In the coadjoint representation κ is invariant, while $\boldsymbol{y}$ and λ transform as in the point vortex dipole case above. In effect we have "glued" the vortex dipole to the pure point vortex, resulting in a spinning vortex dipole. The subgroups $K_{(\boldsymbol{y}_0,\lambda_0)}$ and H are just as before. For any $\boldsymbol{v}$, $\langle \boldsymbol{\Omega}^{\kappa}_{(\boldsymbol{y}_0,\lambda_0)}, \boldsymbol{\chi}_{\boldsymbol{v}}\rangle = \kappa\,\boldsymbol{\chi}_{\boldsymbol{v}}(\boldsymbol{y}_0) + \hat{\mathbf{k}}\cdot[\lambda_0\times\boldsymbol{v}(\boldsymbol{y}_0)]$. This becomes $\kappa\,\boldsymbol{\chi}_{\boldsymbol{v}}(\boldsymbol{y}_0)$ in the special case that $\boldsymbol{v}\in h$. Now, for $\boldsymbol{v}_1,\boldsymbol{v}_2\in h$, $\langle \boldsymbol{A}_0, [\boldsymbol{v}_1,\boldsymbol{v}_2]\rangle = \kappa\,\boldsymbol{\chi}_{[\boldsymbol{v}_1,\boldsymbol{v}_2]}$, which (as desired) is zero by Eq. (7). Thus A_0 determines a nontrivial character on H, and induces a representation of G. The result is a single-particle representation of the local current algebra with a nontrivial cocycle.

Coadjoint orbits for finitely many point vortices, point vortex dipoles, or higher vortex multipoles in the plane, are finite-dimensional. Quantum theories modeled on such orbits, when they exist, represent infinitely many independent self-adjoint operators in Hilbert spaces of wave functions on finite-dimensional configuration-spaces. We next turn to some infinite-dimensional coadjoint orbits where polarizations exist. These can describe extended quantum configurations, with an actual infinity of degrees of freedom.

Consider first the case of *vortex filaments* in the plane. For $0\le\alpha\le 2\pi$, let $\boldsymbol{C}(\alpha)$ be a parameterized curve in $\boldsymbol{R}^2$, *i.e.* an arc or a loop. If $\boldsymbol{C}$ is sufficiently smooth it is equivalent to the following pair: an *unparameterized* curve $\Gamma = \{\boldsymbol{C}(\alpha)\}$, parameterized by its own arc length s, together with the function $\gamma = d\alpha/ds$. This pair permits us to think of $\boldsymbol{C}$ entirely in terms of the objects (Γ,γ) in its target space $\boldsymbol{R}^{\,2}$, the physical space of the theory. Define the momentum density $\boldsymbol{A}_{\boldsymbol{C}}$ as a distribution by

$$\langle \boldsymbol{A}_{\boldsymbol{C}}, \boldsymbol{v}\rangle = \int_{\Gamma} ds\,\gamma(s)\,\hat{\mathbf{k}}\cdot\boldsymbol{\chi}_{\boldsymbol{v}}(\Gamma(s))\,. \tag{11}$$

Then the corresponding vorticity density is $\boldsymbol{B} = \gamma\,\delta_{\Gamma}\,\hat{\mathbf{k}}$, where δ_{Γ} is a delta-function concentrated on the filament Γ, and $\gamma\hat{\mathbf{k}}$ is the vorticity density function on Γ (normal to the plane). *Coad*(ϕ) does two things: it moves Γ to $\phi^{-1}\Gamma$, and it changes the vorticity density distribution γ. Hence the stability subgroup K consists of diffeomorphisms that leave Γ fixed (as a set), *and* leave γ invariant. A polarization group H is obtained by relaxing the latter condition. This permits a consistent quantum theory in which the configuration space is the set of *unparameterized* arcs or loops. The densities γ are the conjugate variables, about which information is lost when "positional" measurements are made in the plane. Representations of G are induced on such coadjoint orbits by nontrivial cocycles on H. It is also possible to demonstrate that coadjoint orbits of $SDiff(\boldsymbol{R}^2)$ containing *nonuniform* vortex patches in the plane have polarizations.

Next consider the case $G = SDiff(\boldsymbol{R}^3)$, and a coadjoint orbit containing a one-dimensional vortex filament. In g' such a filament is specified by the unparameterized curve $\Gamma \subset \boldsymbol{R}^3$, and a (singular) vorticity density vector with support on Γ. The direction of this vorticity must be tangential to Γ, and its magnitude γ must be constant along the length of Γ. In this sense lines of vorticity in $\boldsymbol{R}^3$ are analogous to lines of magnetic flux. Then $\langle \boldsymbol{\Omega}_\Gamma, \boldsymbol{\chi}_{\boldsymbol{v}} \rangle = \gamma \int_\Gamma \boldsymbol{\chi}_{\boldsymbol{v}} \cdot d\boldsymbol{s}$, where $d\boldsymbol{s}$ is an infinitesimal length vector. The stability subgroup K_Γ is $\{\phi \in G \,|\, \phi^{-1}\Gamma = \Gamma\}$. It is apparent that a polarization for such an orbit cannot exist. Indeed, suppose a diffeomorphism in H moves one point off of Γ; then its composition with diffeomorphisms in K_Γ (which are unrestricted away from Γ) leads to unrestricted diffeomorphisms. It is also instructive to understand the absence of a polarization at the level of the Lie algebra. The algebra k_Γ of the stability subgroup consists of divergenceless vector fields which on Γ are tangent to Γ. If a polarization existed, the Lie algebra h_Γ would have to be a proper subalgebra of g containing k_Γ and satisfying $(\Omega_\Gamma, \chi_{[\boldsymbol{v}_1, \boldsymbol{v}_2]}) = 0$ for $\boldsymbol{v}_1, \boldsymbol{v}_2 \in h_\Gamma$. Then $\int_\Gamma d\boldsymbol{s} \cdot (\boldsymbol{v}_1 \times \boldsymbol{v}_2) = 0$. These conditions can only be fulfilled by restricting h_Γ to vector fields whose normal components at each point in Γ lie in a fixed direction. The cross product of two such vector fields is then normal to Γ. Unfortunately for the desired polarization, the class of such vector fields does not close as a Lie algebra. Thus, in three-space, the classically-permitted one-dimensional vortex filaments are quantum-mechanically forbidden. As in the pure point vortex case, complexification of the coadjoint orbit cannot overcome this obstacle.

However, coadjoint orbits of $SDiff(\boldsymbol{R}^3)$ containing two-dimensional *vortex surfaces* have polarizations. Consider, for example, an element of g' specified by an infinite ribbon-shaped surface $\Sigma \subset \boldsymbol{R}^3$, together with a (singular) vorticity density $\gamma(\boldsymbol{s})$, $\boldsymbol{s} \in \Sigma$. One must take $\gamma(\boldsymbol{s})$ to be tangent to Σ, and furthermore to be tangential at the edges of the ribbon. Thus we shall have $\langle \boldsymbol{B}, \chi \rangle = \int_\Sigma \gamma(\boldsymbol{s}) \cdot \chi(\boldsymbol{s})\, d^2 s$. Furthermore, if Γ is a smooth curve crossing the ribbon transversely (from one bounding edge to the other), the total vorticity of the ribbon across Γ is independent of Γ — *i.e.*, it takes a constant value all along the ribbon. A diffeomorphism ϕ in the coadjoint representation acts on $\boldsymbol{B} = (\Sigma, \gamma)$ to transform both the ribbon's surface and the vorticity density. The stability subgroup K contains diffeomorphisms which preserve Σ (as a set), and also preserve the lines of vorticity within it. A polarization H is obtained when the latter constraint is relaxed. The configuration space that results is the space of unparamaterized ribbons with fixed total vorticity, but *without* information regarding the distribution of that vorticity within the ribbon. Such ribbons (or related structures discussed below) should be taken as the fundamental, extended quantum vortex configurations in $\boldsymbol{R}^3$, rather than one-dimensional strings.

Similar polarizations exist for other coadjoint orbits, whose elements can be various (diffeomorphism-invariant) modifications of the ribbon example. For example the edges of the infinite ribbon may be identified to give an infinite tube of vorticity. The ribbon may close on itself to form a ring, giving a configuration in a bounded spatial region. With both of these modifications, we have a coadjoint orbit whose elements are tori. The ribbon may have one or more twists, or it may separate and rejoin, leaving holes. The vortex tube may be knotted. Several ribbons and tubes may be intertwined, and so forth. Solid "ropes" of vorticity, with nonuniform vorticity density, are also permitted. Of course two-dimensional surfaces of vorticity in $\boldsymbol{R}^3$ have degrees of freedom beyond those of one-dimensional filaments. These include the extra continuous degrees of freedom, as well as discrete topological parameters associated with twists, knots, and holes. It is therefore to be expected that the model presented will have observable consequences — *e.g.*, for the specific heat of matter whose excitations in the quantum description include such configurations. To derive such consequences is one long-range goal of our research.

I shall conclude this section with some comments on the overall status of the geometric quantization program in the present context.

The coadjoint orbit serves as a reduced classical phase-space — that is, the values of all conserved quantities are preserved on the orbit. This observation does not address the question of stability — whether or not the classical motion is such that small deviations from the orbit remain small. In treating a fluid in various models, we would expect the stability (or lack of it) in quantum configurations to be of particular interest and to require investigation.

For quantization to be possible on a coadjoint orbit Δ, it must have several good characteristics. The one on which we have focused is the existence of a polarization, which expresses the compatibility of the resulting quantum theory with the uncertainty principle. Once one has found a polarization group H, the quantum configuration space Δ^c for the orbit is the space of leaves in the foliation defined by H. It may be necessary (see below) to construct the configuration-space from a union of such coadjoint orbits–in general, an uncountable union.

Besides having a polarization, the orbit must satisfy a condition of *integrality*. This permits imposition of a periodic boundary condition specifying appropriate domains of wave functions for the operators in Hilbert space; physically it is the quantum condition that prevents destructive "self-interference" of the wave functions. In the examples above integrality holds trivially, because we defined the group to include only diffeomorphisms which tend toward the identity at infinity. Rotating a vortex configuration by 2π (via a one-parameter family of diffeomorphisms) can only be accomplished by a final diffeomorphism that retains a "twist" in its action on the space; the requirement of integrality therefore imposes no new constraint. If however the Lie algebra g is taken to include the generators of global rotations, integrality becomes non-trivial. The corresponding constraint is equivalent to the well-known Feynmann–Onsager condition, and results in quantization of the total vorticity.

Finally, for the actual construction of the space that carries the representation as a Hilbert space, one needs a *measure* on the configuration space having the property of quasi-invariance under the diffeomorphism group action. This means that the class of zero-measure sets is preserved under diffeomorphisms; it permits the group representation to be unitary. Physically, it makes possible meaningful calculations of expectation values of observables with respect to probability distributions. When the configuration space is infinite-dimensional (as it is for path-spaces, loop-spaces, ribbon-spaces, and so forth) the existence of such quasi-invariant measures is a nontrivial problem. In that case the measure is typically expected by be carried by a configuration space built from the union of uncountably many coadjoint orbits, in which the smoothness class of the configurations is less than C^{∞}.

4. Creation and annihilation of vortices

The above analysis leads to conclusions about what kinds of "elementary" vortex configurations are possible in the quantized theory of vortices in two and three space dimesnions. To discuss multi-vortex quantum systems and to explore their properties as energy is put in or removed and the temperature changes, a framework for describing creation and annihilation of vortices is needed. The purpose of this section is to elaborate on an approach I proposed with Sharp [4, 5], modeled on our method for introducing "anyon" creation and annihilation fields through their intertwining relations with a hierarchy of diffeomorphism group representations.

Anyons are particles or excitations in two-dimensional space whose statistics can interpolate those of bosons and fermions. This possibility was first conjectured by Leinaas and Myrheim from the topology of the configuration space [20]. Independently, it was demonstrated rigorously from the interpretation of the continuous unitary representations of $Diff(\boldsymbol{R}^{\,2})$. The diffeomorphism group approach led also to a number of the important physical and mathematical properties of anyons [21], and to the role of braid group representations [22–24]. Anyons find application to surface phenomena such as the fractional quantum Hall effect [25–27]. A review of the diffeomorphism group approach to anyons may be found in [28].

To construct creation and annihilation fields for anyons, Sharp and I were able to begin with the hierarchy of N-anyon representations introduced in [21]. Let $U_N(f)$ be an N-anyon representation of the additive group of real-valued scalar functions on $\boldsymbol{R}^{\,2}$, and $V_N(\phi)$ an N-anyon representation of the group of diffeomorphisms of $\boldsymbol{R}^{\,2}$, in the Hilbert space $\mathcal{H}_N$ for N identical anyons ($N = 0, 1, 2, \ldots$). For each N these unitary operators satisfy the semidirect product group law

$$U(f_1)V(\phi_1)U(f_2)V(\phi_2) = U(f_1 + f_2 \circ \phi_1)V(\phi_1\phi_2)\,, \tag{12}$$

where $\phi_1\phi_2$ denotes composition of diffeomorphisms.

Now let $h_1 \in \mathcal{H}_1$ be a one-anyon state (*i.e.*, a wave function), and let $\psi^*(h_1)$, $\psi(h_1)$ be intertwining operators (creation and annihilation fields) labeled by h_1; that is, $\psi^*(h_1) : \mathcal{H}_N \to \mathcal{H}_{N+1}$ and $\psi(h_1) : \mathcal{H}_{N+1} \to \mathcal{H}_N$. As usual $\psi(h_1)$ annihilates the (unique) vacuum state $h_0 \in \mathcal{H}_0$. Thus the Hilbert space $\mathcal{H}_1$ determines the kind of configuration that ψ^* creates and ψ annihilates, and the elements h_1 of $\mathcal{H}_1$ serve as test functions of creation and annihilation fields. The intertwining conditions

$$\begin{aligned} U_{N+1}(f)\psi^*(h_1) &= \psi^*(U_{N=1}(f)h_1)U_N(f)\,, \\ V_{N+1}(\phi)\psi^*(h_1) &= \psi^*(V_{N=1}(\phi)h_1)V_N(\phi)\,, \end{aligned} \tag{13}$$

were proposed to characterize the fact that the representations U_N and V_N belong to a hierarchy, and in that case serve as defining equations for the field ψ^*. The adjoint of these equations describes the field ψ.

The meaning of (13) is that ψ^* creates a one-anyon configuration in $\boldsymbol{R}^{\,2}$, while h_1 averages over such configurations. Both U and ψ^* act locally. The result of creating a configuration and then transforming by a diffeomorphism is the same as that of transforming first, and then creating the transformed configuration; where the rule for transforming individual configurations is given by $V_{N=1}(\phi)$. In the case of anyons, we found that the intertwining fields obey q-commutation relations. The essential point is that the q-commutator bracket is a *consequence* of the diffeomorphism group representations for anyons, together with the general intertwining property (13) of the fields. It is not put in "by hand" as a starting assumption [29].

We expect that fields can be defined as intertwining operators of diffeomorphism group representations not only for point-like particles like anyons, but for extended quantum configurations like vortex filaments, ribbons, and tubes. Then we shall focus on the second equation of (13), with V_N representing the group of area- or volume-preserving diffeomorphisms. The "test function" h_1 must be a one-vortex Hilbert space vector. The fields ψ^* and ψ, before averaging by h_1, do not depend directly on points in space but on one-vortex configurations (the argument of h_1). Integration takes place over the one-vortex configuration-space with respect to the quasi-invariant measure discussed above (here assumed to exist).

Consider as an example how this could apply to the case of quantized vortex filaments in two dimensions. The one-vortex configuration can be an arbitrary, unparameterized, non-self-intersecting loop (of some smoothness class), with fixed total vorticity and unit area. Call the space of all such loops Δ_1^c. Elements of $SDiff(\boldsymbol{R}^2)$ act on Δ_1^c, deforming the loops as values of the argument of the vectors h_1 in $\mathcal{H}_1$. The field ψ^* (before averaging with h_1) creates a new loop, bringing us into a two-vortex configuration space Δ_2^c. But we now have many different generic possibilities: (a) The two loops can be non-intersecting. (b) The loops can intersect twice, with overlapping area β, $0 < \beta < 1$. Note that the case of loops tangent at one point is excluded as non-generic; it should enter only as a measure zero set. (c) The loops can intersect $2n$ times ($n > 2$), with distinct "generalized knot classes" describing the patterns of intersection, and additional β-parameters describing the appropriate areas. These situations correspond to distinct, diffeomorphism-invariant subspaces of Δ_2^c, and hence to different subspaces of the two-vortex Hilbert space $\mathcal{H}_2$, invariant under the diffeomorphism group but connected by way of the creation and annihilation fields. Likewise $\mathcal{H}_N$ decomposes into subspaces parameterized by knot classes and area parameters β.

Though we do not know explicitly how to construct quasi-invariant measures μ_N for the N-vortex or even the 1-vortex configuration space, it may be possible to use the knot classes and area parameters β to describe the *relative weights* given by μ_N to each diffeomorphism-invariant subspace of Δ_N. Analogously, one may be able to use the knot classes in $\boldsymbol{R}^3$ to parameterize measures on configuration spaces of vortex tubes. This is one direction of our continuing research.

5. Conclusion

As we have seen, geometric quantization based on diffeomorphism groups offers the possibility of describing mathematically the quantum theory of extended vortex configurations, and tells us what types of configurations are possible. In addition, we have the possibility of defining vortex creation and annihilation operators intertwining multiple-vortex representations to obtain a field theory, where the arguments of the resulting fields are not points in the physical space but single-vortex configurations. Finally we hope to use the knot classification of intertwined vortices as diffeomorphism-invariant coordinates for the partial description of quasi-invariant measures.

I would like to thank the organizers of the Second International German-Polish Symposium on Particles and Fields in Zakopane, Poland, for the privilege of presenting this work. Los Alamos National Laboratory provided hospitality during the course of the research. The major results reviewed in this talk were obtained in collaboration with Ralph Menikoff and David Sharp. I would also like to thank Robert Owczarek for interesting discussions.

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