AXIOMATIC QUANTUM THEORY*

W. LÜCKE

Arnold Sommerfeld Institute for Mathematical Physics
Technical University of Clausthal
Leibniz Str. 10, D-38678 Clausthal, Federal Republic of Germany
e-mail: aswl@ptal.pt.tu-clausthal.de

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"It is difficult and perhaps still somewhat controversial to summarize the tenets of quantum physics." [Haag, 1990]

The basic logical structure of standard quantum mechanics and relativistic quantum field theory is outlined.

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1. Introduction

The aim of the following lecture is to give some rough overview over most essential structures underlying all working quantum theoretical models as well as axiomatic and algebraic quantum field theory.

Before specializing to ordinary Hilbert space quantum theory it will be explained why common sense reasoning cannot be applied naively and the pragmatic procedure (*quantum reasoning*) of getting along with this situation is briefly described. Then the more special structure underlying ordinary quantum theory will be postulated rather than derived, since

"It is not yet possible to deduce the present form of quantum mechanics from completely plausible and natural axioms." [Mackey, 1963, p. 62]

Within this framework equivalence between the quantum logical and the algebraic formulation will be established. Finally, the Kastler–Kastler theory of local observables and Wightman's axiomatic field theory will be indicated.

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Unfortunately, due to lack of time, essential issues like canonical quantization, GNS represention, or spontaneous symmetry breaking cannot be discussed.

2. Basic logical structure

2.1. Fundamental postulates

Every known concrete quantum theory of closed systems is a *statistical theory* of the following type. It is affiliated with

1. A set $\mathcal{Q}$ of **macroscopic** prescriptions for preparing a 'state' of the system under consideration.
2. A set $\mathcal{X}$ of **macroscopic** prescriptions for performing *simple* 'tests' (called *questions* by Piron) on the system under consideration, [1] *i.e.* tests with only two possible outcomes referred to as 'yes' or 'no'.
3. A mapping [2]
$$\text{pr} : \mathcal{Q} \times \mathcal{X} \longrightarrow [0,1]$$
with the following interpretation: [3]
    pr (S,T) is the probability for the outcome 'yes' when performing a simple 'test' corresponding to T on the system in a 'state' corresponding to S.

Obviously, the 'tests' $T \in \mathcal{X}$ cannot separate elements $S_1, S_2 \in \mathcal{Q}$, which are equivalent in the following sense:

$$S_1 \sim S_2 \overset{\text{def}}{\Longleftrightarrow} \text{pr}(S_1,T) = \text{pr}(S_2,T)\ \forall\, T \in \mathcal{X}\,.$$

Similarly the 'states' $S \in \mathcal{Q}$ cannot separate 'tests' $T_1, T_2 \in \mathcal{X}$, which are equivalent in the sense that

$$T_1 \sim T_2 \overset{\text{def}}{\Longleftrightarrow} \text{pr}(S,T_1) = \text{pr}(S,T_2)\ \forall\, S \in \mathcal{Q}\,.$$

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[1] We do not require that these tests can be performed within an arbitrarily small time interval.

[2] Actually — as well known for open systems [Davies, 1976] — the probability for the outcome 'yes' or 'no' in a test performed before the 'state' state is prepared need not have any meaning. However for all known models of **closed** quantum systems the 'states' can be imagined as having been prepared as early as one likes. This is essential for standard scattering theory.

[3] We do **not** claim that S uniquely characterizes a **microscopic** state, nor do we claim that T fixes the microscopic details of a test!

Therefore the appropriate mathematical formalism deals with the equivalence classes $[S]$ (also called *states*) and $[T]$ (also called *propositions* or *questions*) together with the (consistent) assignment

$$\omega(\boldsymbol{P}) \stackrel{\text{def}}{=} \operatorname{pr}(S,T) \quad \text{for } \omega = [S]\,,\ \boldsymbol{P} = [T]$$

rather than the specific prescriptions S, T and the mapping pr.

$\mathcal{Q}$ and $\mathcal{X}$ are always (more or less implicitly) chosen such that the following three conditions are fulfilled: [4]

(I_1): For every $\boldsymbol{P} \in \mathcal{L} \stackrel{\text{def}}{=} \{[T] : T \in \mathcal{X}\}$ there is also an element $\neg\boldsymbol{P} \in \mathcal{L}$ fulfilling [5]

$$\omega(\neg\boldsymbol{P}) = 1 - \omega(\boldsymbol{P}) \quad \text{for all } \omega \in \mathcal{S} \stackrel{\text{def}}{=} \{[S] : S \in \mathcal{Q}\}\,.$$

(I_2): Let $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots \in \mathcal{L}$. Then there is an element $\boldsymbol{I} \in \mathcal{L}$ such that for all $\omega \in \mathcal{S}$

$$\omega(\boldsymbol{I}) = 1 \text{ if and only if } \omega(\boldsymbol{P}_j) = 1 \text{ for } j = 1, 2, \ldots$$

(I_3): Let $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots \in \mathcal{L}$ be such that

$$\omega(\boldsymbol{P}_j) = 1 \Longrightarrow \omega(\boldsymbol{P}_k) = 0 \quad \text{for all } \omega \in \mathcal{S} \quad \text{whenever } j < k\,.$$

Then there is an element $\boldsymbol{S} \in \mathcal{L}$ fulfilling

$$\omega(\boldsymbol{S}) = \omega(\boldsymbol{P}_1) + \omega(\boldsymbol{P}_2) + \ldots \quad \forall\,\omega \in \mathcal{S}\,.$$

Note that (I_1) defines a mapping $\neg : \mathcal{L} \longrightarrow \mathcal{L}$. Moreover, there is always a natural semi-ordering of the elements of $\mathcal{L}$ given by

$$P_1 \prec P_2 \stackrel{\text{def}}{\Longleftrightarrow} \omega(P_1) \leq \omega(P_2)\ \forall\,\omega \in \mathcal{S}\,. \tag{1}$$

Theorem 2.1 (Structure Theorem) *If $\mathcal{L} \neq \emptyset$ and $\mathcal{S}$ fulfill conditions (I_1)—(I_3), then $(\mathcal{L}, \prec, \neg)$, with $\prec$ given by (1) and $\neg$ given by (I_1), is a **logic**, i.e. a σ-complete weakly modular lattice $(\mathcal{L}, \prec)$. Moreover, under*

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[4] These conditions are designed to allow for classical reasoning as far as possible. Implicit in (I_3) and (I_1) is the following *standardization postulate*:
For every $\boldsymbol{P} \in \mathcal{L} \setminus \{\mathbf{o}\}$ there exist a state $\omega \in \mathcal{S}$ with $\omega(\boldsymbol{P}) = 1$ and a state $\omega' \in \mathcal{S}$ with $\omega'(\boldsymbol{P}) = 0$. Therefore semi-transparent windows, *e.g.*, cannot be used for simple tests.

[5] A more general framework was suggested in [Mielnik,1974].

these conditions, every $\omega \in \mathcal{S}$ is a probability measure over $(\mathcal{L}, \prec, \neg)$ fulfilling the ***Jauch-Piron condition***

$$\Big(\omega(\boldsymbol{P}_1) = 1 = \omega(\boldsymbol{P}_2) \implies \omega(\boldsymbol{P}_1 \wedge \boldsymbol{P}_2) = 1\Big) \quad \forall \boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathcal{L}. \tag{2}$$

Proof: See [Doebner and Lücke, 1991, appendix] (see also [Maczyński, 1974] for related results). ■

Motivated by the structure theorem, we rely on the following

Axiom 1. Every physical system can be modeled by some logic $(\mathcal{L}, \prec, \neg)$ in the following way:
(i) For every preparable statistical state there is a probability measure ω on $(\mathcal{L}, \prec, \neg)$ fulfilling (2) and for every performable simple test there is an element $\boldsymbol{P} \in \mathcal{L}$ such that

$$\omega(\boldsymbol{P}) = \text{probability for the result 'yes'}.$$

(ii) For all $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathcal{L}$ we have

$$\boldsymbol{P}_1 = \boldsymbol{P}_2 \iff (\omega(\boldsymbol{P}_1) = \omega(\boldsymbol{P}_2)\ \forall \omega \in \mathcal{S}),$$

where $\mathcal{S}$ denotes the set of all probability measures ω on $(\mathcal{L}, \prec, \neg)$ corresponding to preparable statistical states.
(iii) Given $\omega \in \mathcal{S}$ and $\boldsymbol{P} \in \mathcal{L}$ with $\omega(\boldsymbol{P}) \neq 0$, there is a unique $\omega_{,\boldsymbol{P}} \in \mathcal{S}$ fulfilling[6]

$$\omega_{,\boldsymbol{P}}(\boldsymbol{P}') = \omega(\boldsymbol{P}')/\omega(\boldsymbol{P}) \quad \forall \boldsymbol{P}' \prec \boldsymbol{P}.$$

Corollary 2.2 *If Axiom 1 is fulfilled, the following statements hold:*

1. *For all $\boldsymbol{P} \in \mathcal{L}$ and $\omega \in \mathcal{S}$:*

$$\omega(\boldsymbol{P}) = 1 \Longrightarrow \omega_{,\boldsymbol{P}} = \omega.$$

2. *For every atom[7] $\boldsymbol{Z}$ of $(\mathcal{L}, \prec, \neg)$ there is a unique $\omega_{\boldsymbol{Z}} \in \mathcal{S}$ with $\omega_{\boldsymbol{Z}}(\boldsymbol{Z}) = 1$.*
3. *Defining*

$$\omega_{,\boldsymbol{P}}(\boldsymbol{P}')/\omega(\boldsymbol{P}) \stackrel{\text{def}}{=} 0 \quad \textit{if}\ \omega(\boldsymbol{P}) = 0,$$

[6] Conditions (iii) and (ii) imply that for every $\boldsymbol{P} \in \mathcal{L} \setminus \{\mathbf{o}\}$ there is a *state* $\omega \in \mathcal{S}$ with $\omega(\boldsymbol{P}) = 1$ — which would also be a consequence of (I_3) and (I_1). Naively interpreted, $\omega_{,\boldsymbol{P}}(\boldsymbol{P}')$ describes the *conditional probability* in the state ω for $E_{\boldsymbol{P}'}$ — defined by (9) — being true provided $E_{\boldsymbol{P}}$ is true. In ordinary quantum theory $\boldsymbol{T}_{\omega_{,\boldsymbol{P}}}$ is given by $\boldsymbol{P}\boldsymbol{T}_\omega \boldsymbol{P}/\omega(\boldsymbol{P})$.

[7] See Appendix C for the definition of an ***atom Z***. Obviously, $\omega_{\boldsymbol{Z}}$ is a *pure state*.

*we have for all **compatible*** $\boldsymbol{P}$, $\boldsymbol{P}' \in (\mathcal{L}, \prec, \neg)$

$$\omega(\boldsymbol{P}') = \omega(\boldsymbol{P})\, \omega_{,\boldsymbol{P}}(\boldsymbol{P}') + \omega(\neg \boldsymbol{P})\, \omega_{,\neg\boldsymbol{P}}(\boldsymbol{P}') \quad \forall \omega \in \mathcal{S} \tag{3}$$

and

$$\omega_{,\boldsymbol{P}_1,\boldsymbol{P}_2} = \omega_{,\boldsymbol{P}_1 \wedge \boldsymbol{P}_2} \quad \forall \omega \in \mathcal{S}\,. \tag{4}$$

Just for simplicity we add the following assumption, fulfilled in ordinary quantum theory: [8]

$$\boxed{\mathcal{S} = \text{ set of \textbf{all} probability measures on } (\mathcal{L}, \prec, \neg)\,.} \tag{5}$$

2.2. Terminology

A semi-ordered set $(\mathcal{L}, \prec)$ (*poset*) is called a ***lattice***, if [9] both

$$\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \stackrel{\text{def}}{=} \inf_{\mathcal{L}} \{\boldsymbol{P}_1, \boldsymbol{P}_2\} \quad \text{and} \quad \boldsymbol{P}_1 \vee \boldsymbol{P}_2 \stackrel{\text{def}}{=} \sup_{\mathcal{L}} \{\boldsymbol{P}_1, \boldsymbol{P}_2\}$$

exist for arbitrary $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathcal{L}$. If

$$\mathbf{0} \stackrel{\text{def}}{=} \inf \mathcal{L}$$

exists, we say $(\mathcal{L}, \prec)$ has a ***universal lower bound*** $\mathbf{0}$. If

$$\mathbf{1} \stackrel{\text{def}}{=} \sup \mathcal{L}$$

exists, we say $(\mathcal{L}, \prec)$ has a ***universal upper bound*** $\mathbf{1}$. A bijection $\neg : \mathcal{L} \longrightarrow \mathcal{L}$ is called an ***orthocomplementation*** of $(\mathcal{L}, \prec)$ if the latter has universal upper and lower bounds and the following requirements are fulfilled for arbitrary $\boldsymbol{P}, \boldsymbol{P}' \in \mathcal{L}$:

(O_1): $\boldsymbol{P} \wedge \neg \boldsymbol{P} = \mathbf{0}$,
(O_2): $\boldsymbol{P} \vee \neg \boldsymbol{P} = \mathbf{1}$,
(O_3): $\neg(\neg \boldsymbol{P}) = \boldsymbol{P}$,
(O_4): $\boldsymbol{P} \prec \boldsymbol{P}' \Longrightarrow \neg \boldsymbol{P}' \prec \neg \boldsymbol{P}$.

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[8] In orthodox quantum theory (5) is a consequence of Footnote 3 and σ-convexity.

[9] For quantum logic: $E_{\boldsymbol{P}_1 \wedge \boldsymbol{P}_2}$ is certain if and only if both $E_{\boldsymbol{P}_1}$ **and** $E_{\boldsymbol{P}_2}$ are certain.

An orthocomplemented lattice $(\mathcal{L}, \prec, \neg)$ is called ***distributive***, if [10]

$$\boldsymbol{P}_1 \wedge (\boldsymbol{P}_2 \vee \boldsymbol{P}_3) = (\boldsymbol{P}_1 \wedge \boldsymbol{P}_2) \vee (\boldsymbol{P}_1 \wedge \boldsymbol{P}_3) \tag{6}$$

holds for all $\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3 \in \mathcal{L}$.

An **ordered** pair$(\boldsymbol{P}_1, \boldsymbol{P}_2) \in \mathcal{L} \times \mathcal{L}$ is called ***compatible*** if [11]

$$\boldsymbol{P}_1 = (\boldsymbol{P}_1 \wedge \boldsymbol{P}_2) \vee (\boldsymbol{P}_1 \wedge \neg \boldsymbol{P}_2) \ . \tag{7}$$

An **ordered** pair$(\boldsymbol{P}_1, \boldsymbol{P}_2) \in \mathcal{L} \times \mathcal{L}$ is called ***modular*** if (6) holds for all $\boldsymbol{P}_3 \prec \boldsymbol{P}_1$, *i.e.* if

$$\mathcal{L} \ni \boldsymbol{P}_3 \prec \boldsymbol{P}_1 \Longrightarrow \boldsymbol{P}_1 \wedge (\boldsymbol{P}_2 \vee \boldsymbol{P}_3) = (\boldsymbol{P}_1 \wedge \boldsymbol{P}_2) \vee \boldsymbol{P}_3 \,. \tag{8}$$

An orthocomplemented lattice $(\mathcal{L}, \prec, \neg)$ is called ***weakly modular*** [12] if

$$\boldsymbol{P}_1 \perp \boldsymbol{P}_2 \Longrightarrow (\boldsymbol{P}_1, \boldsymbol{P}_2) \ \text{modular}\,,$$

holds for all $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathcal{L}$, where

$$\boldsymbol{P}_1 \perp \boldsymbol{P}_2 \overset{\text{def}}{\Longleftrightarrow} (\boldsymbol{P}_1 \prec \neg \boldsymbol{P}_2) \ .$$

It is easy to prove, that $(\mathcal{L}, \prec, \neg)$ is weakly modular if and only if

$$\boldsymbol{P}_2 \prec \boldsymbol{P}_3 \Longrightarrow (\boldsymbol{P}_2\,, \boldsymbol{P}_3) \ \text{compatible}$$

holds for all $\boldsymbol{P}_2\,, \boldsymbol{P}_3 \in \mathcal{L}$.

$(\mathcal{L}, \prec, \neg)$ is called an ***orthomodular lattice*** if it is a weakly modular orthocomplemented lattice.

> **Remark:** An orthocomplemented lattice $(\mathcal{L}, \prec, \neg)$ is weakly modular if and only if
>
> $$(\boldsymbol{P}_2\,, \boldsymbol{P}_3) \ \text{compatible} \Longleftrightarrow (\boldsymbol{P}_3\,, \boldsymbol{P}_2) \ \text{compatible}$$

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[10] Note that, thanks to orthocomplementation, also

$$\boldsymbol{P}_1 \vee (\boldsymbol{P}_2 \wedge \boldsymbol{P}_3) = (\boldsymbol{P}_1 \vee \boldsymbol{P}_2) \wedge (\boldsymbol{P}_1 \vee \boldsymbol{P}_3)$$

holds for all $\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3 \in \mathcal{L}$, if (6) does.

[11] In a logic $(\mathcal{L}, \prec, \neg)$ the ordered pair $(\boldsymbol{P}_1, \boldsymbol{P}_2)$ is compatible if and only the sublogic generated by $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ is classical (see [Piron 1976, Definition (2.14) and Theorem (2.19)]). Also this shows that $(\boldsymbol{P}_1, \boldsymbol{P}_2)$ is compatible if and only $(\boldsymbol{P}_2, \boldsymbol{P}_1)$ is compatible.

[12] It is called ***modular***, if all pairs $(\boldsymbol{P}_1, \boldsymbol{P}_2) \in \mathcal{L} \times \mathcal{L}$ are modular.

holds for all $\boldsymbol{P}_2$, $\boldsymbol{P}_3 \in \mathcal{L}$. $(\mathcal{L}, \prec, \neg)$ [Birkhoff, 1967, Theorem 21, p. 53].

A lattice $(\mathcal{L}, \prec)$ is called (σ-)***complete*** [13] if both $\inf_{\mathcal{L}} M$ and $\inf_{\mathcal{L}} M$ exist for every (countable) $M \subset \mathcal{L}$.

A ***logic*** is a σ-complete orthomodular lattice. [14]

A logic $(\mathcal{L}, \prec, \neg)$ is called ***classical*** (or *Boolean algebra*) if it is distributive. [15] Otherwise it is called a ***quantum logic***.

A ***probability measure*** on the logic $(\mathcal{L}, \prec, \neg)$ is a mapping $\omega : \mathcal{L} \longrightarrow [0, 1]$ fulfilling the following two conditions: [16]

(S_1): $\omega(\mathbf{1}) = 1$,

(S_1): For every countable subset $\{\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots\} \subset \mathcal{L}$ we have

$$\omega\left(\sup_{\mathcal{L}}\{\boldsymbol{P}_j : j = 1, 2, \ldots\}\right) = \sum_{j=1}^{\infty} \omega\left(\boldsymbol{P}_j\right) \text{ if } \boldsymbol{P}_j \perp \boldsymbol{P}_k \text{ for } j \neq k .$$

2.3. Some warnings

In 'orthodox quantum mechanics' [Primas, 1981] (without superselection rules [17]) the logic $(\mathcal{L}, \prec, \neg)$ described in Section 2.1 is realized as follows (*standard quantum logic*):

- $\mathcal{L}$ is given as the set of all projection operators [18] in some separable complex Hilbert space $\mathcal{H}$ of dimension ≥ 2.

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[13] As shown by Mac Neille, every poset can be embedded into a complete lattice in such a way that semi-ordering is preserved as well as greatest lower bounds and lowest upper bounds existing in the poset [Birkhoff, 1967, Theorem V,22].

[14] This is the definition is equivalent to that given in [Varadarajan, 1968, p. 05]. There are also others adding conditions on the set of states (see, *e.g.*, [Pulmannová, 1985]).

[15] We do not require classical logics to be atomic (see Appendix C).

[16] Varadarajan calls such ω just *measures* on $(\mathcal{L}, \prec, \neg)$ [Varadarajan, 1968, p. 113], unless $(\mathcal{L}, \prec, \neg)$ is classical.

[17] A system modeled by $(\mathcal{L}, \prec, \neg)$ is said to possess ***superselection rules*** if the ***center***

$$\mathcal{C}(\mathcal{L}, \prec, \neg) \stackrel{\text{def}}{=} \{\boldsymbol{P} \in \mathcal{L} : (\boldsymbol{P}, \boldsymbol{P}') \text{ compatible } \forall\, \boldsymbol{P}' \in \mathcal{L}\}$$

of $(\mathcal{L}, \prec, \neg)$ is nontrivial ($\mathcal{L} \neq \mathcal{C} \neq \{\mathbf{0}, \mathbf{1}\}$). $(\mathcal{L}, \prec, \neg)$ is called ***irreducible*** if $\mathcal{C} = \{\mathbf{0}, \mathbf{1}\}$.

[18] Their specific physical identification depends on the *dynamics*, as discussed in [Mielnik, 1974] and [Lücke, 1995].

- For arbitrary $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathcal{L}$ we have

$$\boldsymbol{P}_1 \prec \boldsymbol{P}_2 \stackrel{\text{def}}{\Longleftrightarrow} \boldsymbol{P}_1 \leq \boldsymbol{P}_2$$
$$\Longleftrightarrow (\langle \Psi \mid \boldsymbol{P}_1 \Psi \rangle \leq \langle \Psi \mid \boldsymbol{P}_1 \Psi \rangle \ \forall \Psi \in \mathcal{H}) \ .$$

- For every $\boldsymbol{P} \in \mathcal{H}$ we have

$$\neg \boldsymbol{P} \stackrel{\text{def}}{=} \mathbf{1} - \boldsymbol{P} .$$

If dim $(\mathcal{H}) \geq \ni$, by Gleason's theorem [Gleason, 1957] for every $\omega \in \mathcal{S}$ there is a unique positive trace class operators [19] $\boldsymbol{T}_\omega \in \mathcal{B}(\mathcal{H})$ fulfilling

$$\omega(\boldsymbol{P}) = \text{trace}\ (\boldsymbol{T}_\omega \boldsymbol{P}) \quad \forall\, \boldsymbol{P} \in \mathcal{L} .$$

From this it is easily seen that **none** of the $\omega \in \mathcal{S}$ can be ***dispersion free***, *i.e.* fulfill the requirement

$$\omega(\boldsymbol{P}) \in \{0, 1\} \quad \forall\, \boldsymbol{P} \in \mathcal{L}$$

(not even approximately). For a very long time this was taken as evidence for nonexistence of *hidden variables* — even though D. Bohm constructed a consistent (nonlocal) hidden variable theory in the beginning of the fifties [Bohm, 1952]. Actually, in order to avoid this conclusion one has to abandon the seemingly natural assumption that **every microscopic state** — not only those given by $\mathcal{S}$ — induces a probability measure on $(\mathcal{L}, \prec, \neg)$.

However, also from a hidden variables point of view, this assumption has to be questioned:

Lemma 2.3 (D. Pfeil) *For every set $\hat{\mathcal{L}}$ there is a* **classical** *logic $(\mathcal{B}, \prec_B, \neg_B)$ and a mapping $M : \hat{\mathcal{L}} \longrightarrow \mathcal{B}$ for which the following holds:*

> *For every mapping $\omega : \hat{\mathcal{L}} \longrightarrow [0, 1]$ there is a probability measure μ on $(\mathcal{B}, \prec_B, \neg_B)$ fulfilling*

$$\omega(\boldsymbol{P}) = \mu\Big(M(\boldsymbol{P})\Big) \quad \forall\, \boldsymbol{P} \in \hat{\mathcal{L}} .$$

Proof: In order to avoid purely technical complications [20] we consider only the case of finite

$$\hat{\mathcal{L}} = \{\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n\} .$$

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[19] Conversely, every trace class operator of trace 1 induces a probability measure on standard quantum logic.

[20] The general proof is by straightforward adaption of a construction given by Kochen and Specker [Kochen and Specker, 1967, Section I].

Then we may take

$$B \stackrel{\text{def}}{=} \{0,1\}^n \,, \quad \prec_B \stackrel{\text{def}}{=} \subset \,,$$
$$\neg_B M' \stackrel{\text{def}}{=} B \setminus M' \quad \text{for } M' \subset B \,,$$
$$M(\boldsymbol{P}_\nu) \stackrel{\text{def}}{=} \{b = (b_1, \ldots, b_n) \in B : b_\nu = 1\} \quad \text{for } \nu \in \{1, \ldots, n\} \,,$$

and

$$\mu_\omega(M') \stackrel{\text{def}}{=} \sum_{(b_1,\ldots,b_n)\in M'} \prod_{\nu=1}^{n} \omega_\nu(b_\nu) \quad \text{for } \omega \in \mathcal{S} \,,\ M' \subset B \,,$$

where

$$\omega_\nu(1) \stackrel{\text{def}}{=} 1 - \omega_\nu(0) \stackrel{\text{def}}{=} \omega(\boldsymbol{P}_\nu) \quad \text{for } \nu \in \{1, \ldots, n\} \,. \qquad \blacksquare$$

It seems natural to assign 'actual' properties $E_{\boldsymbol{P}}$ to the elements of $\mathcal{L}$ in the sense that: [21]

A system in the state $\omega \in \mathcal{S}$ has property $E_{\boldsymbol{P}}$ **with certainty** if and only if $\omega(P) = 1$. (9)

We are used giving names to these properties like 'spin up', 'positive energy' and so on. However, from the proof of Lemma 2.3 it should be clear that there is no evidence [22] for the assumption that under all circumstances — independent of any test — the system has either property $E_{\boldsymbol{P}}$ or property $E_{\neg\boldsymbol{P}}$ — even though [23]

$$\omega(\neg\boldsymbol{P}) = 1 - \omega(\boldsymbol{P}) \quad \forall\, \omega \in \mathcal{S} \,,\ \boldsymbol{P} \in \mathcal{L}$$

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[21] Assumption (iii) of Axiom 1 then says that for every $\boldsymbol{P} \in \mathcal{L}$ there is a state $\omega \in \mathcal{S}$ in which $E_{\boldsymbol{P}}$ is certain.

[22] From a 'hidden variables' point of view the 'test' enforces a transition, if necessary, of the system to a state in which either $E_{\boldsymbol{P}}$ or $E_{\neg\boldsymbol{P}}$ is certain. Typically, for micro-systems, the number of cases in which the criteria for this alternative are not specified by T cannot be neglected, causing apparent indeterminism with respect to incompatible properties $E_{\boldsymbol{P}_1}, E_{\boldsymbol{P}_2}$. According to the Kopenhagen interpretation of quantum mechanics, indeterminism is a direct consequence of the hypotheses, never accepted by Einstein, that quantum theory presents a **complete** description of physical reality.

[23] We doubt that more detailed specification of the measurement context might be of any help, here.

and even though tests corresponding to $\boldsymbol{P}$ and $\neg\boldsymbol{P}$ can typically be performed jointly. Therefore, it is no surprise that we encounter quantum peculiarities such as [24]

$$\omega(\boldsymbol{P}) = 1 \not\Longrightarrow \left(\omega(\boldsymbol{P} \wedge \boldsymbol{P}') = \omega(\boldsymbol{P}') \;\forall\, \boldsymbol{P}' \in \mathcal{L}\right) \tag{10}$$

or [25]

$$\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 = 0 \not\Longrightarrow \boldsymbol{P}_1 \prec \neg\boldsymbol{P}_2 \,. \tag{11}$$

Let us call a state $\omega \in \mathcal{S}$ ***classical***, if (3) holds for all pairs $\boldsymbol{P}$, $\boldsymbol{P}' \in (\mathcal{L}, \prec, \neg)$ — whether compatible or not.

Lemma 2.4 *A σ-complete orthocomplemented lattice $(\mathcal{L}, \prec, \neg)$ fulfilling conditions (i)–(iii) of Axiom 1 for (5) is a classical logic, if and only if all $\omega \in \mathcal{S}$ are classical.*

Proof: See Appendix B. ■

2.4. Quantum reasoning

In spite of all warnings, ***simple quantum reasoning*** according to the following rules is consistent:

- Choose a **classical** sublogic $(\mathcal{L}_c, \prec, \neg)$ of $(\mathcal{L}, \prec, \neg)$ and forget about all the other elements of $\mathcal{L}$.
- Then imagine that every **individual** — in whatever situation — has either property $E_{\boldsymbol{P}}$ or $E_{\neg\boldsymbol{P}}$ **if** $\boldsymbol{P} \in \mathcal{L}_c$.
- For $\omega \in \mathcal{S}$, imagine that $\omega(\boldsymbol{P})$ is the relative number of individuals having property $E_{\boldsymbol{P}}$ in an ensemble corresponding to ω **if** $\boldsymbol{P} \in \mathcal{L}_c$.
- Imagine that $\prec$ corresponds to common sense logical implication and that $\neg$ corresponds to common sense logical negation.

This way all quantum peculiarities are avoided. For instance, in spite of (10), we may conclude

$$\left.\begin{array}{l}\omega(P_1) = 1\,,\\ P_1 \text{ compatible with } P_2\end{array}\right\} \Longrightarrow \omega(P_1 \wedge P_2) = \omega(P_2)\;\forall\,\omega$$

or even

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[24] The set of 'states' determines the (quantum logical) relations between (equivalence classes of) tests, which must not be interpreted too naively.

[25] By (2), $\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 = 0$ means that there is no preparable property guaranteeing both $E_{\boldsymbol{P}_1}$ and $E_{\boldsymbol{P}_2}$.

P_1, P_2 compatible [26]

$$\Longrightarrow \omega(P_1 \vee P_2) = \omega(P_1 \wedge \neg P_2) + \omega(\neg P_1 \wedge P_2) + \omega(P_1 \wedge P_2) \; \forall \omega .$$

Simple quantum reasoning naturally leads to the notion of *observable*:[27]

Definition 2.5 An ***observable*** A of a physical system modeled by the logic $(\mathcal{L}, \prec, \neg)$ is a σ-morphism $\boldsymbol{E}_A$ of the Borel ring on the real line into $(\mathcal{L}, \prec, \neg)$ which is unitary, *i.e.* $\boldsymbol{E}_A(\mathbb{R}) = \mathbf{1}$.

The physical interpretation of $\boldsymbol{E}_A$ in the sense of quantum reasoning is as follows:

Given $\omega \in \mathcal{S}$ and a Borel subset Δ of $\mathbb{R}^1$ then $\omega\big(\boldsymbol{E}_A(\Delta)\big)$ can be imagined as the relative number of individuals for which $A \in \Delta$ in an ensemble corresponding to ω.

Consequently, the expectation value [28] for A in an ensemble corresponding to ω is given by the Stieltjes integral

$$\overline{A}(\omega) = \int \lambda \, \mathrm{d}\omega \left(\boldsymbol{E}_A \left((-\infty, \lambda] \right) \right) . \tag{12}$$

In orthodox quantum theory $\boldsymbol{E}_A$ is a projection valued measure [29] and (12) can also be written as

$$\overline{A}(\omega) = \text{trace } (\boldsymbol{T}_\omega \boldsymbol{A}) , \tag{13}$$

where

$$\boldsymbol{A} \stackrel{\text{def}}{=} \int \lambda \, \mathrm{d}\boldsymbol{E}_A \left((-\infty, \lambda] \right) \tag{14}$$

is a self-adjoint operator (*spectral representation*).

Simple quantum reasoning can be applied to a whole family observables $A_1, A_2, \ldots$ if and only if all the pairs

$$\Big(\boldsymbol{E}_{A_j}(\varDelta_j), \boldsymbol{E}_{A_k}(\varDelta_k) \Big) \; , \quad \varDelta_j \, , \varDelta_k \in \mathbb{R}$$

---

[26]

$$\begin{aligned} P_1 \vee P_2 = (P_1 \vee P_2) \wedge 1 &= (P_1 \wedge \neg P_2) \vee P_2 \\ &= (P_1 \wedge \neg P_2) \vee (\neg P_1 \wedge P_2) \vee (P_1 \wedge P_2) \end{aligned}$$

[27] The Borel ring on $\mathbb{R}^1$ could be replaced by an arbitrary classical logic.

[28] Of course, the expectation value may be infinite!

[29] Of course, in general, tests corresponding to the $\boldsymbol{E}_A(\Delta)$ can never be exactly realized. Therefore many people prefer to use just positive operator valued measures.

are compatible. For ***bounded*** A_j, *i.e.* if $E_{A_j}(\Delta_j) = 1$ for suitable compact $\Delta_j \in \mathbb{R}$, this is equivalent to pairwise commutativity of the corresponding (bounded) self-adjoint operators $\boldsymbol{A}_j$.

In order to make predictions for multiple tests one has to know how states change as a result of a simple test. Here we assume [30]

> ***Lüders' Postulate.*** For every $\boldsymbol{P} \in \mathcal{L}$ there is a corresponding *ideal test* causing a transition[31] $\omega \mapsto \omega_{,\boldsymbol{P}}$ whenever the result is 'yes'.

Remark: The Lüders postulate ensures that an **ideal** test corresponding to $\boldsymbol{P}$ destroys none of the properties $E_{\boldsymbol{P}'}$ with $\boldsymbol{P}'$, $\boldsymbol{P}$ compatible.
Proof: Let $\boldsymbol{P}'$, $\boldsymbol{P}$ be compatible and

$$\omega(\boldsymbol{P}') = 1\,, \quad \omega(\boldsymbol{P}) \neq 0\,.$$

Then

$$1 = \omega(\boldsymbol{P}') \underset{(3)}{=} \omega(\boldsymbol{P})\omega_{\boldsymbol{P}}(\boldsymbol{P}') + (1 - \omega(\boldsymbol{P}))\,\omega_{\neg\boldsymbol{P}}(\boldsymbol{P}')$$

implies $\omega_{\boldsymbol{P}}(\boldsymbol{P}') = 1$. ■

By Lüders' postulate, [32] given the initial state $\omega \in \mathcal{S}$, the probability for the *homogeneous history* $(\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n)$ — *i.e.* for getting the answer 'yes' for all subsequent ideal tests of a series corresponding to $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n \in \mathcal{L}$ – should be [33]

$$\omega(\boldsymbol{P}_1)\omega_{,\boldsymbol{P}_1}(\boldsymbol{P}_2)\cdots\omega_{,\boldsymbol{P}_1,\ldots,\boldsymbol{P}_{n-1}}(\boldsymbol{P}_n)\,.$$

Consistent quantum reasoning with respect to histories leads to the modern notion of *decoherent histories.*

Given a history $(\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n)$ not corresponding to a simple test, we can no longer be sure that there is an initial state for which $(\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n)$ is

---

30 Usually, a test causes a much more drastic change of the state or even ends by absorbing the corresponding individual. An ideal test, typically, would be approximately realized by a highly efficient filter.

31 Remember the second statement of Corollary 2.2, however. If $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ are atoms of $(\mathcal{L}, \prec, \neg)$, $\omega_{\boldsymbol{Z}_1}(\boldsymbol{Z}_2)$ is called the *transition probability* for the transition $\omega_{\boldsymbol{Z}_1} \longrightarrow \omega_{\boldsymbol{Z}_2}$.

32 In the relativistic theory Lüders' postulate causes interesting problems [Schlieder, 1971] (see also [Mittelstaed, 1983], [Mittelstaed and Stachow, 1983]).

33 Naively interpreted, $\omega(\boldsymbol{P}_1)\omega_{,\boldsymbol{P}_1}(\boldsymbol{P}_2)\cdots\omega_{,\boldsymbol{P}_1,\ldots,\boldsymbol{P}_{n-1}}(\boldsymbol{P}_n)$ is the probability for joint validity of the properties $E_{\boldsymbol{P}_1}, \ldots, E_{\boldsymbol{P}_1}$ in the state ω. Usually (see, *e.g.*, [Omnès, 1994], [Griffith, 1995]), unfortunately, this is formulated in the Schrödinger picture, thus imposing unnecessary restrictions.

certain, *i.e.*, for which $\omega(\boldsymbol{P}_1)\omega_{,\boldsymbol{P}_1}(\boldsymbol{P}_2)\cdots\omega_{,\boldsymbol{P}_1,\ldots,\boldsymbol{P}_{n-1}}(\boldsymbol{P}_n)=1$. Therefore the 'logic' of histories is weaker than that for simple tests and may provide a useful basis for generalizing quantum theory [Isham, 1995].

2.5. Dynamics

Definition 2.6 A ***symmetry*** of a physical system modeled [34] by the logic $(\mathcal{L},\prec,\neg)$ is an automorphism of $(\mathcal{L},\prec,\neg)$, *i.e.* a bijection of $\mathcal{L}$ onto itself preserving the least upper bound and the orthocomplementation.

In orthodox quantum theory, by Wigner's theorem [Piron,1976, §3–2] for every automorphism α of $(\mathcal{L},\prec,\neg)$ there is an operator $\boldsymbol{V}\in\mathcal{B}(\mathcal{H})$ which is either unitary or anti-unitary and fulfills

$$\alpha(\boldsymbol{P})=\boldsymbol{V}\boldsymbol{P}\boldsymbol{V}^*\quad\forall\boldsymbol{P}\in\mathcal{L}\subset\mathcal{B}(\mathcal{H})\,. \tag{15}$$

For simplicity we consider only those systems for which time translation is a symmetry: [35]

> **Axiom 2.** For every time t there is a symmetry α_t with the following physical interpretation:
>
> Let T be a macroscopic prescription for performing a simple test corresponding to $\boldsymbol{P}\in\mathcal{L}$. Then the prescription T_t to do everything prescribed by T just with time delay t characterizes a test corresponding to $\alpha_t(\boldsymbol{P})$.
>
> $t\longmapsto\alpha_t$ is ***weakly continuous***, *i.e.*, for fixed $\boldsymbol{P}\in\mathcal{L}$ and $\omega\in\mathcal{S}$ the probability $\omega(\alpha_t(\boldsymbol{P}))$ is a **continuous** function of t.

The family $\{\alpha_t\}_{t\in\mathbb{R}}$ determines the ***dynamics*** of the system. According to its definition it has to be a weakly continuous 1-parameter group of transformations:

$$\alpha_0=\mathrm{id}\,,\qquad\alpha_{t_1}\circ\alpha_{t_2}=\alpha_{t_1+t_2}\quad\forall t_1\,,\,t_2\in\mathbb{R}\,. \tag{16}$$

---

[34] If (5) does not hold one should also require $\alpha^*(\mathcal{S})=\mathcal{S}$ and then the inverse of a symmetry need not be a symmetry (A. Bohm's point of view).

[35] Note that the dual of a symmetry has always an inverse in the set of all probability measures. In this sense evolution can always be extrapolated backwards in time!

For orthodox quantum mechanics this implies that there is a self-adjoint operator, the ***Hamiltonian***, $\boldsymbol{H}$ on $\mathcal{H}$ fulfilling

$$\alpha_t(\boldsymbol{P}) = \mathrm{e}^{it\boldsymbol{H}}\boldsymbol{P}\mathrm{e}^{-it\boldsymbol{H}} \quad \forall t \in \mathbb{R}\,,\ \boldsymbol{P} \in \mathcal{L}, \tag{17}$$

but:

> "... we omit its surprisingly difficult proof, which involves some theorems about group cocycles." [Davies, 1976, p. 26]

(see Section 3.3 and Appendix A for a sketchy proof).

The Hamiltonian is unique only up to an additive multiple of the identity operator. But, in any case, it has to be bounded from below. This has been exploited in a nice, easy to prove, theorem by Hegerfeldt which, unfortunately, caused a lot of irritation [36] in connection with some misleading application to Fermi's two-atoms problem [Hegerfeldt, 1994]:

Theorem 2.7 (Hegerfeldt) *Let $\boldsymbol{H}$ be a self-adjoint operator on $\mathcal{H}$ which is semibounded from below. Moreover let $\boldsymbol{P}$ be a nonnegative bounded operator on $\mathcal{H}$ and $\Psi \in \mathcal{H}$. Then either*

$$\left\langle \Psi \mid \mathrm{e}^{i\boldsymbol{H}t}\boldsymbol{P}\mathrm{e}^{-i\boldsymbol{H}t}\Psi \right\rangle = 0 \quad \forall t \in \mathbb{R}$$

or

$$t_1 \neq t_2 \Longrightarrow \int_{t_1}^{t_2} \left\langle \Psi \mid \mathrm{e}^{i\boldsymbol{H}t}\boldsymbol{P}\mathrm{e}^{-i\boldsymbol{H}t}\Psi \right\rangle > 0\,\mathrm{d}t \quad \forall t_1\ t_2 \in \mathbb{R}\,.$$

Remark: Even if $(\mathcal{L}, \prec, \neg)$ is isomorphic to the standard quantum logic one may use a nonlinear realization of $(\mathcal{L}, \prec, \neg)$ in Hilbert space such that time evolution has to be described by a group of nonlinear transformations [Lücke, 1995] — in full agreement with Mielnik's program for handling nonlinear Schrödinger equations [Mielnik, 1974].

3. General quantum theory

Usually one is only concerned with suitable sublogics of standard quantum logic. Therefore we assume the following: [37]

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[36] See, *e.g.*, B. Schroer: *Reminiscences about Many Pitfalls and Some Successes of QFT Within the Last Three Decades*, hep-th/9410085, pp. 7–8; to appear in Reviews in Mathematical Physics.

[37] The C^*-algebraic approach may be considered as a preliminary step: One has to find a suitable representation (superselection structure) and take the weak closure in this representation to get the von Neumann algebra.

Axiom 3. There is a **separable**[38] Hilbert space $\mathcal{H}$ and a **von Neumann algebra** $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ by which $(\mathcal{L}, \prec, \neg)$ is realized in the following way:

- $\mathcal{L} = \mathcal{L}_{\mathcal{M}} \stackrel{\text{def}}{=} \{\boldsymbol{P} \in \mathcal{M} : \boldsymbol{P}^* = \boldsymbol{P} = \boldsymbol{P}^2\}$,
- $\boldsymbol{P}_1 \prec \boldsymbol{P}_2 \stackrel{\text{def}}{\Longleftrightarrow} \boldsymbol{P}_1 \leq \boldsymbol{P}_2 \quad \forall \boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathcal{L}$,
- $\neg\boldsymbol{P} \stackrel{\text{def}}{=} \mathbf{1} - \boldsymbol{P} \quad \forall \boldsymbol{P} \in \mathcal{L}$.

3.1. Von Neumann algebras

A ***von Neumann algebra*** is a subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ some Hilbert space, that is given by the ***commutant***

$$\mathcal{N}' \stackrel{\text{def}}{=} \{\boldsymbol{A} \in \mathcal{B}(\mathcal{H}) : [\boldsymbol{A}, \boldsymbol{B}]_- = \boldsymbol{0}\ \forall \boldsymbol{B} \in \mathcal{N}\}$$

of some $*$-invariant subset $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$:

$$\mathcal{M} = (\mathcal{N} \cup \mathcal{N})' \qquad \left(\text{and hence } \mathcal{M} = \mathcal{M}''\right) .$$

The projection operators of a von Neumann algebra always form a sublogic $(\mathcal{L}_{\mathcal{M}}, \prec, \neg)$ of the corresponding standard quantum logic. [39] This does not hold for arbitrary C^*-algebras. [40]

A von Neumann algebra $\mathcal{M}$ is called a ***factor*** if its ***center***

$$\mathcal{Z}(\mathcal{M}) \stackrel{\text{def}}{=} \mathcal{M} \cap \mathcal{M}''$$

is trivial, *i.e.*, if $\mathcal{Z}(\mathcal{M}) = \mathbb{C}\mathbf{1}$. This equivalent to $(\mathcal{L}_{\mathcal{M}}, \prec, \neg)$ being irreducible, [41] *i.e.* to the absence of ***superselection rules***.

---

[38] For an interesting application of nonseparable Hilbert space see, *e.g.*, [Buchholz, 1982].

[39] Here, $\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 = \text{s-}\lim_{n\to\infty}(\boldsymbol{P}_1\boldsymbol{P}_2)^n$. Necessary and sufficient conditions for a quantum logic to be isomorphic to a sublogic am standard quantum logic are given in [Gudder, 1979].

[40] For example, the C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ generated by $\mathbf{1}$ and $\mathcal{C}(\mathcal{H})$ contains exactly those projection operators $\boldsymbol{P}$ for which either $\boldsymbol{P}$ itself or $\mathbf{1} - \boldsymbol{P}$ has finite rank. This cannot be consistent with σ-completeness, required for a logic.

[41] Recall footnote 17.

A factor of ***type*** I_n, $n \in \{2, \ldots, \infty\}$, is the von Neumann algebra $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ of dimension n, *i.e.*, the corresponding logic is a standard quantum logic.

3.2. State functionals

A ***state functional*** on a von Neumann algebra $\mathcal{M}$ is a mapping $\boldsymbol{A} \to \omega(\boldsymbol{A})$ of $\mathcal{M}$ into the **complex** numbers fulfilling the following three conditions:

$$(S_1) : \omega(\boldsymbol{A} + \alpha\boldsymbol{B}) = \omega(\boldsymbol{A}) + \alpha\omega(\boldsymbol{B}) \qquad (\textit{linearity})$$

$$(S_2) : \omega(\mathbb{1}) = 1 \qquad (\textit{normalization})$$

$$(S_3) : \omega(\boldsymbol{A}^*\boldsymbol{A}) \geq 0 \qquad (\textit{positivity}^{42})$$

One can easily show that the following three conditions conditions are fulfilled for every state ω on $\mathcal{M}$:

(i) $|\omega(\boldsymbol{A}^*\boldsymbol{B})|^2 \leq \omega(\boldsymbol{A}^*\boldsymbol{A})\omega(\boldsymbol{B}^*\boldsymbol{B})$ ***(Cauchy Schwarz inequality)***
(ii) $\omega(\boldsymbol{A}^*) = \overline{\omega(\boldsymbol{A})}$ ***(hermiticity)***
(iii) $\omega(\boldsymbol{A}_n) \to \omega(\boldsymbol{A})$ if $\|\boldsymbol{A} - \boldsymbol{A}_n\| \to 0$ ***(continuity)***

Lemma 3.1 *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and let $\{\boldsymbol{A}_i\}_{i \in I} \subset \mathcal{M}$ be an increasing net of positive operators fulfilling $\sup_{i\in I} \|\boldsymbol{A}_i\| < \infty$. Then $\sup_{i \in I} \boldsymbol{A}_i$ exists with respect to $\mathcal{B}(\mathcal{H})$ and is an element of the algebra $\mathcal{M}$.*

Proof: See [Bratteli and Robinson, 1979, Lemma 2.4.19].

A state functional of the von Neumann algebra ω on $\mathcal{M}$ is called ***normal***, [43] if

$$\omega(\sup_{i\in I} \boldsymbol{A}_i) = \sup_{i\in I} \omega(\boldsymbol{A}_i)$$

---

[42] It would be quite tedious to show in general for C^*-algebras that $\boldsymbol{A}^*\boldsymbol{A} = -\boldsymbol{B}^*\boldsymbol{B} \Longrightarrow \boldsymbol{A}^*\boldsymbol{A} = 0$.

[43] In general, a state ω on a von Neumann algebra is called ***singular***, if for every nonzero projection operator $\boldsymbol{P}$ there is another nonzero projection operator $\boldsymbol{P}'$ for which $\omega(\boldsymbol{P}') = 0$ and $\boldsymbol{P}' \prec \boldsymbol{P}$. An example for such a state is

$$\omega(\boldsymbol{A}) \stackrel{\text{def}}{=} \lim_{N\to\infty} \sum_{\nu=1}^{N} \text{trace } (\boldsymbol{P}_\nu \boldsymbol{A}) ,$$

where the $\boldsymbol{P}_\nu$ denote rank-1 projection operators corresponding to a complete orthonormal basis of $\mathcal{H}$.

holds for every net fulfilling the requirements of Lemma 3.1. The following two theorems shows that the normal state functionals are the algebraic equivalent to the probability measures.

Theorem 3.2 (generalized Gleason theorem) *Let $\mathcal{M}$ be a von Neumann algebra with no type I_2 summand.* [44] *Every finitely additive probability measure ω on $\mathcal{L}_{\mathcal{M}}$ can be extended to a* ***state functional*** *on $\mathcal{L}_{\mathcal{M}}$. This state functional is normal if and only if the corresponding probability measure is completely additive.*

Proof: See [Maeda, 1989]. ■

Theorem 3.3 *A state ω on the von Neumann algebra $\mathcal{M}$ is normal if and only if there is a positive trace class operator* [45] $\boldsymbol{T}_\omega \in \mathcal{B}(\mathcal{H}) \supset \mathcal{M}$ *with*

$$\omega(\boldsymbol{A}) = \text{trace } (\boldsymbol{T}_\omega \boldsymbol{A}) \quad \forall\, \boldsymbol{A} \in \mathcal{M}\,.$$

Proof: [Bratteli and Robinson, 1979, Theorem 2.4.21]. ■

From now on we identify the states with their corresponding normal state functionals. Then the bounded [46] observables A can be identified with the self-adjoint elements $\boldsymbol{A} \in \mathcal{M}$ in the sense of (13).

3.3. Symmetry groups

Thanks to the generalized Gleason theorem one can show the following:

Corollary 3.4 *Let $\mathcal{M}$ be a von Neumann algebra and $\{\alpha_t\}_{t\in\mathbb{R}}$ a weakly continuous 1-parameter group of symmetries of $(\mathcal{L}_{\mathcal{M}}, \prec, \neg)$. Then $\{\alpha_t\}_{t\in\mathbb{R}}$ is the restriction to $\mathcal{L}_{\mathcal{M}}$ of a weakly* continuous* [47] *1-parameter group of C^*-automorphisms* [48] *of $\mathcal{M}$.*

---

[44] Of course, the theorem cannot be true if $\mathcal{M}$ is a type I_2 factor since then

$$\omega(\mathbb{1} - \boldsymbol{P}) = 1 - \omega(\boldsymbol{P}) \geq 0 \quad \forall\, \boldsymbol{P} \in \mathcal{L}_{\mathcal{M}}$$

is the only requirement for a probability measure ω.

[45] Note, however, that $\boldsymbol{T}_\omega$ is no longer unique, in general.

[46] The self-adjoint operators corresponding to unbounded observables of $(\mathcal{L}_{\mathcal{M}}, \prec, \neg)$ are said to be ***affiliated*** to $\mathcal{M}$. An operator $\boldsymbol{A}$ on $\mathcal{H}$ is affiliated to $\mathcal{M}$ if and only if $\boldsymbol{U}\boldsymbol{A}\boldsymbol{U}' = \boldsymbol{A}$ holds for all unitary operators $\boldsymbol{U} \in \mathcal{M}'$ [Dixmier, 1969, I, 1, exerc. 10].

[47] $\{\alpha_t\}_{t\in\mathbb{R}}$ is ***weakly* continuous*** iff $\omega(\alpha_t(\boldsymbol{A}))$ is continuous in t for all normal states ω and all $\boldsymbol{A} \in \mathcal{M}$ [Bratteli and Robinson, 1979, Proposition 2.4.3].

[48] A C^*-automorphism α of $\mathcal{M}$ is a linear automorphism α of $\mathcal{M}$ fulfilling the

Proof: See Appendix A. ■

Therefore, the weakly continuous 1-parameter groups of ***symmetries*** of a system modeled by $(\mathcal{L}_{\mathcal{M}}, \prec, \neg)$ can be identified with the weakly* continuous 1-parameter groups of C^*-automorphisms of $\mathcal{M}$. The most important 1-parameter group of symmetries is the time translation symmetries. If $\mathcal{M}$ is of type I these C^*-automorphisms are generated in the standard way by some (in general unbounded) Hamiltonian $\boldsymbol{H}$:

Theorem 3.5 *Let $\{\alpha_t\}_{t\in\mathbb{R}^1}$ be a weakly 0∗ continuous one parameter group of C^*-automorphisms of $L(H)$. Then there is a unique self-adjoint operator $\boldsymbol{H}$ fulfilling*

$$\alpha_t(\boldsymbol{A}) = e^{\frac{i}{\hbar}\boldsymbol{H}t}\boldsymbol{A}e^{-\frac{i}{\hbar}\boldsymbol{H}t} \quad \forall \boldsymbol{A} \in \mathcal{L}\mathcal{H}),\ t \in \mathbb{R}^1 .$$

Proof: See [Bratteli and Robinson, Example 3.2.35] and Stone's theorem. ■

4. Relativistic quantum theory

For a relativistic quantum theory Axiom 2 has to be enhanced:

> **Axiom 2'.** There is a weakly* continuous representation of $\mathcal{P}_+^\uparrow$ by C^*-automorphism $\alpha_{\Lambda,x}$ of $\mathcal{M}$, with obvious interpretation generalizing that of the dynamics
>
> $$\alpha_t \stackrel{\text{def}}{=} \alpha_{\mathbb{1},(t,0,0,0)} \quad \text{for } t \in \mathbb{R} .$$

4.1. Algebras of local observables

Let $\mathcal{M}(\mathcal{O})$ denote the subalgebra of $\mathcal{M}$ generated by all those $\boldsymbol{P} \in \mathcal{L}_{\mathcal{M}}$ corresponding to tests that can be performed within the space-time region

$$\mathcal{O} \in \mathcal{K} \stackrel{\text{def}}{=} \{\text{open double cones}\} .$$

---

conditions

$$\alpha(\boldsymbol{AB}) = \alpha(\boldsymbol{A})\alpha(\boldsymbol{B}) \quad \forall\, \boldsymbol{A}, \boldsymbol{B} \in \mathcal{M},$$

and

$$\alpha(\boldsymbol{A}^*) = (\alpha(\boldsymbol{A}))^* \quad \forall\, \boldsymbol{A} \in \mathcal{M} .$$

This identification implies [49]

- $$\mathcal{O}_1 \subset \mathcal{O}_1 \Longrightarrow \mathcal{M}(\mathcal{O}_1) \subset \mathcal{M}(\mathcal{O}_2) \qquad \forall\, \mathcal{O}_1\,,\, \mathcal{O}_2 \in \mathcal{K}$$

and

$$\alpha_{\Lambda,x}\left(\mathcal{M}(\mathcal{O})\right) = \mathcal{M}\left(\Lambda\mathcal{O} + x\right) \quad \forall\,(\Lambda, x) \in \mathcal{P}_+^\uparrow\,,\; \mathcal{O} \in \mathcal{K}\,.$$

According to [Haag and Kastler, 1964] one also assumes

$$\mathcal{M} = \left(\bigcup_{\mathcal{O}\in\mathcal{K}} \mathcal{M}(\mathcal{O})\right)''\,.$$

Then the following Problem arises:

> $\mathcal{M}$ depends on the representation of $\bigcup_{\mathcal{O}\in\mathcal{K}} \mathcal{M}(\mathcal{O})$, but only tests corresponding to $\boldsymbol{P} \in \mathcal{M}(\mathcal{O})$ are realistic, *i.e.*, only $[\mathcal{M}(\mathcal{O})]$, $\mathcal{O} \in \mathcal{K}$, can be experimentally determined (in principle).

The solution of this problem is based on the following Haag-Kastler assumptions:

There is a separable Hilbert space $\mathcal{H}$, a continuous unitary representation [50] $\boldsymbol{U}(A, x)$ of iSL(2,$\mathbb{C}$) in $\mathcal{H}$, and a faithful irreducible representation π_0 of the

$$C^*\text{-completion of } \bigcup_{\mathcal{O}\in\mathcal{K}} \mathcal{M}(\mathcal{O})$$

---

49 Usually **all** projection operators in $\mathcal{M}(\mathcal{O})$ are considered as corresponding to tests performable within $\mathcal{O}$. However, this is not as evident as tacitly assumed in standard presentations like [Haag, 1992] since it implies that the interpretation of $\boldsymbol{A}$ and ω depends on the selected space-time region $\mathcal{O}$: The unit operator considered as an element of $\mathcal{M}(\mathcal{O})$ has to be identified with the maximal equivalence class $\mathbf{1}_{\mathcal{O}}$ of simple tests performable within $\mathcal{O}$ and $\omega \in \mathcal{S}$ has to be **locally** interpreted via

$$\omega(\boldsymbol{A}) = \frac{\text{counting rate for } \boldsymbol{A}}{\text{counting rate for } \mathbf{1}_{\mathcal{O}}}\,.$$

50 Physical interpretation:

$$\boldsymbol{U}(A, x)\pi_0(\boldsymbol{B})\boldsymbol{U}(A, x)^{-1} = \pi_0\left(\alpha_{\Lambda_A,x}(\boldsymbol{A})\right) \text{ for } \boldsymbol{B} \in \mathcal{M}(\mathcal{O})\,.$$

in $\mathcal{H}$ such that the net of ***local algebras***

$$\mathcal{A}(\mathcal{O}) \stackrel{\text{def}}{=} \pi_0\left(\mathcal{M}(\mathcal{O})\right) \,,\; \mathcal{O} \in \mathcal{K},$$

fulfills the following requirements:

isotony

$$: \mathcal{O}_1 \subset \mathcal{O}_2 \Longrightarrow \mathcal{A}\left(\mathcal{O}_1\right) \subset \mathcal{A}\left(\mathcal{O}_2\right) ,$$

locality:

$$\mathcal{O}_1 \,\text{spacelike}\, \mathcal{O}_2 \;\Longrightarrow\; \left[\mathcal{A}\left(\mathcal{O}_1\right), \mathcal{A}\left(\mathcal{O}_2\right)\right]_- = 0 ,$$

covariance:

$$U(A,x)\mathcal{A}(\mathcal{O})U(A,x)^{-1} = \mathcal{A}\left(\Lambda_A(\mathcal{O}) + x\right) ,$$

spectrum condition: [51]

$$\text{supp}\tilde{\varphi} \cap \overline{V_+} = 0 \Longrightarrow \int U(\mathbb{1},x)\varphi(x)\,\mathrm{d}x = 0 ,$$

cyclic vacuum:

$$\overline{\bigcup_{\mathcal{O}\in\mathcal{K}} \mathcal{A}(\mathcal{O})\,\Omega} = \mathcal{H} , \qquad U(A,x)\Omega = \Omega .$$

Now $\mathcal{M}$ can be constructed following the Haag–Doplicher–Roberts approach: [52]

1. Determine the physically relevant irreducible representations π_μ (***superselection sectors***) of

$$\mathcal{A} \stackrel{\text{def}}{=} C^*\text{-completion of } \bigcup_{\mathcal{O}\in\mathcal{K}} \mathcal{A}(\mathcal{O})$$

   via *localized endomorphisms* [53] of $\mathcal{A}$.
2. Take

$$\mathcal{M}(\mathcal{O}) = \bigoplus_{\mu} \pi_\mu\left(\mathcal{A}(\mathcal{O})\right) .$$

---

[51] ***Fourier transform.*** $\tilde{\varphi}(p) \stackrel{\text{def}}{=} (2\pi)^{-2} \int \varphi(x) e^{ipx}\,\mathrm{d}x$.

[52] See [Haag, 1992] and references given there.

[53] An endomorphism ρ of $\mathcal{A}$ is called ***localized*** (in $\mathcal{O}$) if

$$\rho(A) = A \quad \forall\, A \in \mathcal{O}' \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^4 : x\,\text{spacelike}\,\mathcal{O}\right\} .$$

The guiding principle of this approach is:

All physical information — especially the $\boldsymbol{S}$-matrix [Lücke, 1983] — is already encoded in the local net structure!
Now the question arises:

How to construct a concrete local net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{K}}$ fulfilling the Haag–Kastler assumptions?
In the simplest case $\mathcal{A}(\mathcal{O})$ is generated by all

$$\exp\left(i\int \boldsymbol{A}(x)\varphi(x)\,\mathrm{d}x\right), \quad \varphi=\varphi^*\in\mathcal{S}(\mathcal{O}),$$

where $\boldsymbol{A}(x)$ a given (observable) scalar Wightman field.

4.2. *Quantum fields*

The observable $\boldsymbol{A}(x)$ (in the distributional sense) of a scalar field is usually characterized by the following Wightman axioms [Streater and Wightman, 1989]:

(W0): Poincaré symmety implemented by continuous unitary representation $\boldsymbol{U}(\Lambda,x)$ of $\mathcal{P}_+^\uparrow$ on separable Hilbert space $\mathcal{H}$.

(W2): $\tilde{\varphi}\cap\overline{V_+}=0 \Longrightarrow \int \boldsymbol{U}(\mathbb{1},x)\varphi(x)\,\mathrm{d}x=0$.

(W3): There is a normed vector $\Omega\in\mathcal{H}$, unique up to a phase factor, fulfilling
$$\boldsymbol{U}(\mathbb{1},x)\Omega=\Omega \quad \forall\, x\in\mathbb{R}^4.$$

(W4): $\boldsymbol{A}(x)$ is an operator valued generalized function over a space $\mathcal{T}$ of *test functions* on $\mathbb{R}^4$, with invariant dense domain $D\subset\mathcal{H}$; $\mathcal{T}=\mathcal{S}(\mathbb{R}^4)$ (Schwartz space of tempered functions).

(W5): $\left.\begin{array}{l}\boldsymbol{U}(\Lambda,a)D=D,\\ \boldsymbol{U}(\Lambda,a)\boldsymbol{A}(x)\boldsymbol{U}(\Lambda,a)^{-1}=\boldsymbol{A}(\Lambda x+a)\end{array}\right\}\ \forall\,\boldsymbol{U}(\Lambda,a)$.

(W6): x spacelike $y\Longrightarrow[\boldsymbol{A}(x),\boldsymbol{A}(y)]_-=0$

(W7): Ω cyclic with respect to the *smeared* field operators
$$\boldsymbol{A}(\varphi)=\int \boldsymbol{A}(x)\varphi(x)\,\mathrm{d}x\,,\ \varphi\in\mathcal{T}.$$

For $\boldsymbol{A}(x)$ to be 'observable' one has to add the requirement [54]

$$\varphi=\varphi^* \Longrightarrow \boldsymbol{A}(\varphi)\text{ essentially self-adjoint}^{55}\text{ on }D.$$

[54] Naive interpretation: $\boldsymbol{A}(\varphi)$ is the observable for $\int A(x)\varphi(x)\,\mathrm{d}x$, with $A(x)$ a measurable field.

Of course, for constructing realistic dynamics, like quantum electrodynamics, one also needs unobservable fields like Dirac spinors $\Psi(x)$ or electromagnetic potentials $A^\mu(x)$. One even has to use an indefinite metric if one wants $A^\mu(x)$ to be covariant and local. The situation becomes even worse in nonlinear gauge theories. Nevertheless, the Wightman frame with obvious generalization of (W5) should be adequate for all **observable** fields (with tempered high energy behaviour). Thus quantum electrodynamics should finally be given by an observable Wightman tensor field $F^{\mu\nu}(x)$ for the electromagnetic field strength and a Wightman vector field $j^\mu(x)$ for the charge-current density. The more regrettable is the following fact:

> No Wightman field with nontrivial S-matrix is known [56] (on 4-dimensional space-time)!

Maybe the main obstruction comes from the purely technical assumption $\mathcal{T} = \mathcal{S}(\mathbb{R}^4)$ implying ad hoc high energy restrictions [Wightman, 1981]. Therefore one should use a more general framework within which the main results of axiomatic field theory are still valid ([Lücke,1984], [Lücke, 1986]). As a first step one could try to construct theories of Efimov's type (see, *e.g.*, [Efimov, 1974]).

Appendix A

Symmetries of von Neumann logics

As a simple consequence of the generalized Gleason theorem we have the following:

Corollary A.1 *Let $\mathcal{M}$ be a W^* algebra on the* **separable** *complex Hilbert space $\mathcal{H}$ and let α be an automorphism of the corresponding logic $(\mathcal{L}_\mathcal{M}, \prec, \neg)$. Then there is a bijection φ of $\mathcal{M}_s \stackrel{\text{def}}{=} \{A = A^* \in \mathcal{M}\}$ onto itself and a bijection φ_* of*

$$N_\mathcal{M} \stackrel{\text{def}}{=} \{\textbf{normal states on } \mathcal{M}\}$$

onto itself fulfilling the following two conditions:

$$\alpha(P) = \varphi(P) \quad \forall\, P \in \mathcal{L}_\mathcal{M}\,,$$

$$(\varphi_*\omega)\,(\varphi(A)) = \omega(A) \forall\, \omega \in N_\mathcal{M}\,,\ A \in \mathcal{M}_s\,.$$

---

[55] See [Borchers and Zimmermann, 1964] for a useful criterion.

[56] Even apparently nontrivial models turned out belong to the Borchers class of generalized free fields [Rehren, 1995].

Theorem A.2 *Let $\mathcal{M}$ be a von Neumann algebra, let φ be a bijection of $\mathcal{M}_s \stackrel{\text{def}}{=} \{A = A^* \in \mathcal{M}\}$ onto itself, and let φ_* be a bijection of*

$$N_{\mathcal{M}} \stackrel{\text{def}}{=} \{\textit{normal states on } \mathcal{M}\}$$

onto itself fulfilling

$$(\varphi_*\omega)(\varphi(A)) = \omega(A) \qquad \textit{for all } \omega \in N_{\mathcal{M}} \textit{ and } A = A^* \in \mathcal{M}\,.$$

Then φ has a unique continuation to a **Jordan automorphism** *of $\mathcal{M}$, i.e. to a linear bijection φ of $\mathcal{M}$ onto itself fulfilling* [57]

$$\varphi(A^*) = (\varphi(A))^*\,,\ \varphi(\{A, B\}) = \{\varphi(A), \varphi(B)\} \qquad \textit{for all } A, B \in \mathcal{M}\,.$$

Sketch of Proof: φ_* is easily seen to be *affine*, *i.e.* to fulfill the condition

$$\varphi_*\left(\lambda\omega_1 + (1-\lambda)\omega_2\right) = \lambda(\varphi_*\omega_1) + (1-\lambda)(\varphi_*\omega_2)$$

for all $\lambda \in [0,1]$ and all $\omega_1, \omega_2 \in N_{\mathcal{M}}$:

$$\begin{aligned}[\varphi_*\left(\lambda\omega_1 + (1-\lambda)\omega_2\right)](\varphi\varphi^{-1}A) &= \left(\lambda\omega_1 + (1-\lambda)\omega_2\right)(\varphi^{-1}A) \\ &= \lambda\omega_1(\varphi^{-1}A) + (1-\lambda)\omega_2(\varphi^{-1}A) \\ &= \lambda(\varphi_*\omega_1)(\varphi\varphi^{-1}A) + (1-\lambda)(\varphi_*\omega_2)(\varphi\varphi^{-1}A) \\ &= [\lambda(\varphi_*\omega_1) + (1-\lambda)(\varphi_*\omega_2)](A) \quad \forall A \in \mathcal{M}_s\,.\end{aligned}$$

According to Bratelli and Robinson [Bratteli and Robinson, 1979, Theorem 3.2.8] this implies the statement of Theorem A.2. ■

One easily proves the following:

Lemma A.3 *Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra and let φ be a Jordan automorphism of $\mathcal{M}$. Then the following statements hold:*

1. $0 \neq A \in \mathcal{M} \Longrightarrow 0 < \varphi(A^*A)$.
2. *The restriction of φ to $\mathcal{L}_{\mathcal{M}}$ is an automorphism of $(\mathcal{L}_{\mathcal{M}}, \prec, \neg)$.*
3. $\omega \in N_{\mathcal{M}} \Longrightarrow \varphi_*\omega \in N_{\mathcal{M}}$, *where* $(\varphi_*\omega)(A) \stackrel{\text{def}}{=} \omega\left(\varphi^{-1}(A)\right)\ \forall A \in \mathcal{M}$.

Now we see that symmetries of a von Neumann logic $(\mathcal{L}_{\mathcal{M}}, \prec, \neg)$ are in 1-1-correspondence to the Jordan automorphisms of $\mathcal{M}$. At least for factors $\mathcal{M}$ the latter have to be either C^*-automorphisms or C^*-antiautomorphisms:

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[57] $A \circ B \stackrel{\text{def}}{=} \frac{1}{2}\{A, B\}$ is the so-called *Jordan product.*

Theorem A.4 *Let φ be a Jordan automorphism of the von Neumann algebra $\mathcal{M}$. Then there is a $\boldsymbol{P} \in \mathcal{L}_{\mathcal{M}} \cap \mathcal{M}'$ for which*

$$\alpha(\boldsymbol{A}) \stackrel{\text{def}}{=} \varphi(\boldsymbol{A})\boldsymbol{P}$$

is a ***C^*-automorphism*** *of $\mathcal{M}$, i.e. a Jordan automorphism fulfilling*

$$\alpha(\boldsymbol{AB}) = \alpha(\boldsymbol{A})\alpha(\boldsymbol{B}) \quad \forall \boldsymbol{A}, \boldsymbol{B} \in \mathcal{M},$$

and

$$\widehat{\alpha}(\boldsymbol{A}) \stackrel{\text{def}}{=} \varphi(\boldsymbol{A})(\mathbb{1} - \boldsymbol{P})$$

a ***C^*-antiautomorphism*** *of $\mathcal{M}$, i.e. a Jordan automorphism fulfilling*

$$\widehat{\alpha}(\boldsymbol{AB}) = \widehat{\alpha}(\boldsymbol{B})\widehat{\alpha}(\boldsymbol{A}) \quad \forall \boldsymbol{A}, \boldsymbol{B} \in \mathcal{M},$$

Proof: See [Bratteli and Robinson, 1979, Proposition 3.2.2].■

Since the Jordan automorphisms for time translation may be written $\alpha_t = \alpha_{t/2} \circ \alpha_{t/2}$ they are also C^*-**auto**morphisms, [58] by Theorem A.4. Let us close this section with the following

> **Warning:** The algebraic formalism requires the observables to have no physical dimensions, otherwise addition of observables would make no sense in general. Hence a system of physical units should be specified to which the numbers characterized by the observables refer.

Appendix B

Classical states

Lemma B.1 *Let $(\mathcal{L}, \prec, \neg)$ be a σ-complete orthocomplemented lattice fulfilling conditions (i)–(iii) of Axiom 1 for (5). Then the following conditions are equivalent:* [59]

(i) $(\mathcal{L}, \prec, \neg)$ *is a classical logic.*

(ii) $(\boldsymbol{P}_1 \wedge \boldsymbol{P}_2) \vee (\boldsymbol{P}_1 \wedge \neg\boldsymbol{P}_2) \vee (\neg\boldsymbol{P}_1 \wedge \boldsymbol{P}_2) \vee (\neg\boldsymbol{P}_1 \wedge \neg\boldsymbol{P}_2) = \mathbb{1} \quad \forall \boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathcal{L}$.

(iii) All probability measures ω on $(\mathcal{L}, \prec, \neg)$ are classical.

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[58] For generalizations see [Roos, 1985].

[59] To have equivalence of (i) and (ii) it is sufficient to postulate weak modularity instead of condition (ii) of Axiom 1 [Piron, 1976, Theorem (2.15)].

Proof for $(i) \Longleftarrow (ii)$: [60]
Since

$$(P_1 \wedge P_2) \vee (P_1 \wedge \neg P_2) \prec P_1\,, \quad (\neg P_1 \wedge P_2) \vee (\neg P_1 \wedge \neg P_2) \prec \neg P_1\,,$$

we have

$$\omega\left((P_1 \wedge P_2) \vee (P_1 \wedge \neg P_2)\right) \leq \omega(P_1)\,,$$
$$\omega\left((\neg P_1 \wedge P_2) \vee (\neg P_1 \wedge \neg P_2)\right) \leq \omega(\neg P_1)\,.$$

for all $\omega \in \mathcal{S}$. Since, on the other hand,

$$\omega\left((P_1 \wedge P_2) \vee (P_1 \wedge \neg P_2)\right) + \omega\left((\neg P_1 \wedge P_2) \vee (\neg P_1 \wedge \neg P_2)\right) \underset{(ii)}{=} 1$$
$$= \omega(P_1) + \omega(\neg P_1)$$

for all $\omega \in \mathcal{S}$, this implies

$$\omega\left((P_1 \wedge P_2) \vee (P_1 \wedge \neg P_2)\right) = \omega(P_1)$$

for all $\omega \in \mathcal{S}$ and hence

$$(P_1 \wedge P_2) \vee (P_1 \wedge \neg P_2) = P_1\,,$$

by condition (ii) of Axiom 1. This means that all pairs (P_1, P_2) are compatible, hence (i), by Footnote 10.

Proof for $(ii) \Longleftarrow (iii)$:
Applying (4) to $\omega = \omega_{,P_1}$ we get

$$\omega_{,P_1} \geq \omega_{,P_1}(P_2)\omega_{,P_1,P_2}\,.$$

Since, by definition, $\omega_{,P_1}(\neg P_1) = 0$, therefore

$$\omega_{,P_1}(P_2) = 0 \quad \text{or} \quad \omega_{,P_1,P_2}(P_1) = 1\,.$$

Since, by definition, also $\omega_{,P_1,P_2}(P_2) = 1$, the **Jauch-Piron condition** (2) gives

$$\omega_{,P_1}(P_2) = 0 \quad \text{or} \quad \omega_{,P_1,P_2}(P_1 \wedge P_2) = 1\,.$$

Since P_1 , P_2 are arbitrary, we also have:

$$\left(1 - \omega_{,P_1}(P_2)\right) = 0 \text{ or } \omega_{,P_1,\neg P_2}(P_1 \wedge \neg P_2) = 1\,,$$
$$\omega_{,\neg P_1}(P_2) = 0 \text{ or } \omega_{,\neg P_1,P_2}(\neg P_1 \wedge P_2) = 1\,,$$
$$(1 - \omega_{,\neg P_1}(P_2)) = 0 \text{ or } \omega_{,\neg P_1,\neg P_2}(\neg P_1 \wedge \neg P_2) = 1\,.$$

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[60] See also [Piron, 1976, [Theorem (2.15)].

By double application of (2):

$$\begin{aligned}\omega &= \omega(\boldsymbol{P}_1)\omega_{,\boldsymbol{P}_1} + (1-\omega(\boldsymbol{P}_1))\,\omega_{,\neg\boldsymbol{P}_1}\\ &= \omega(\boldsymbol{P}_1)\left(\omega_{,\boldsymbol{P}_1}(\boldsymbol{P}_2)\omega_{,\boldsymbol{P}_1,\boldsymbol{P}_2} + (1-\omega_{,\boldsymbol{P}_1}(\boldsymbol{P}_2))\,\omega_{,\boldsymbol{P}_1,\neg\boldsymbol{P}_2}\right)\\ &\quad + (1-\omega(\boldsymbol{P}_1))\left(\omega_{,\neg\boldsymbol{P}_1}(\boldsymbol{P}_2)\omega_{,\neg\boldsymbol{P}_1,\boldsymbol{P}_2} + (1-\omega_{,\neg\boldsymbol{P}_1}(\boldsymbol{P}_2))\,\omega_{,\neg\boldsymbol{P}_1,\neg\boldsymbol{P}_2}\right).\end{aligned}$$

Therefore

$$\begin{aligned}\boldsymbol{P} \perp (\boldsymbol{P}_1 \wedge \boldsymbol{P}_2) \vee (\boldsymbol{P}_1 \wedge \neg\boldsymbol{P}_2) \vee (\neg\boldsymbol{P}_1 \wedge \boldsymbol{P}_2) \vee (\neg\boldsymbol{P}_1 \wedge \neg\boldsymbol{P}_2)&\\ \Longrightarrow \omega(\boldsymbol{P}) = 0 \;\forall\, \omega&\end{aligned}$$

and hence

$$\boldsymbol{P} \perp (\boldsymbol{P}_1 \wedge \boldsymbol{P}_2) \vee (\boldsymbol{P}_1 \wedge \neg\boldsymbol{P}_2) \vee (\neg\boldsymbol{P}_1 \wedge \boldsymbol{P}_2) \vee (\neg\boldsymbol{P}_1 \wedge \neg\boldsymbol{P}_2) \Longrightarrow \boldsymbol{P} = 0\,.$$

This implies (ii). **Proof for** $(iii) \Longleftarrow (i)$:
(i) implies

$$\omega_{,\boldsymbol{P}}(\boldsymbol{P}') = \omega(\boldsymbol{P} \wedge \boldsymbol{P}')/\omega(\boldsymbol{P}) \quad \forall\, \boldsymbol{P}\,,\, \boldsymbol{P}' \in \mathcal{L}\,,\, \omega \in \mathcal{S}$$

and hence (iii). ■

Appendix C

Covering law

Let $(\mathcal{L}, \prec)$ be a lattice with universal lower bound $\mathbf{o}$. Given $\boldsymbol{P}_1 \prec \boldsymbol{P}_2 \neq \boldsymbol{P}_1$, one says that $\boldsymbol{P}_2$ ***covers*** $\boldsymbol{P}_1$ if

$$[\boldsymbol{P}_1, \boldsymbol{P}_2]_{\mathcal{L}} \stackrel{\text{def}}{=} \{\boldsymbol{P} \in \mathcal{L} : \boldsymbol{P}_1 \prec \boldsymbol{P} \prec \boldsymbol{P}_2\} = \{\boldsymbol{P}_1, \boldsymbol{P}_2\}\,.$$

$\boldsymbol{P} \in \mathcal{L}$ is called an ***atom*** of $(\mathcal{L}, \prec)$ if it covers $\mathbf{o}$. $(\mathcal{L}, \prec)$ is called ***atomic*** if every interval $[\mathbf{o}, \boldsymbol{P}]_{\mathcal{L}}$ with $\boldsymbol{P} \neq \mathbf{o}$ contains at least one atom.

A logic $(\mathcal{L}, \prec, \neg)$ is said to fulfill the ***covering law*** if

1. $(\mathcal{L}, \prec)$ is atomic.
2. For all $\boldsymbol{Z}$, $\boldsymbol{P} \in \mathcal{L}$:

$$\left.\begin{matrix}\boldsymbol{Z} \text{ atom}\,, \\ \boldsymbol{Z} \wedge \boldsymbol{P} = 0\end{matrix}\right\} \Longrightarrow \boldsymbol{Z} \vee \boldsymbol{P} \text{ covers } \boldsymbol{P}\,.$$

Assuming irreducibility and the covering law in addition to Axiom 1 one may prove that — apart from some exceptional cases — $(\mathcal{L}, \prec, \neg)$ is isomorphic to the logic of all projection operators on some *generalized Hilbert space* [Piron, 1976, Section 3–1].

Appendix D

Quantum logic via constraints

A simple example, given in [Doebner and Lücke, 1991], shows that already the standardization postulate (Footnote 4) may lead to quantum logic:

Let $M = \{a_1, a_2, b_1, b_2\}$ be a set of 4 elements, $(\mathbf{B}, \subset, M \setminus .)$ the corresponding **classical** logic of all subsets of M, and W the set of all probability measures on $(\mathbf{B}, \subset, M \setminus .)$. Restrict now W by assuming that only those $\omega = \omega_\lambda$ correspond to experimentally realizable statistical ensembles, for which there is a quadruple $\lambda = (\lambda_1, \ldots, \lambda_4)$ of nonnegative numbers that fulfills the following five conditions:

$$\lambda_1 + \ldots + \lambda_4 = \tfrac{1}{2} \tag{18}$$

$$\omega_\lambda(\{a_1\}) = \lambda_1 + \lambda_4\,,\ \omega_\lambda(\{a_2\}) = \lambda_2 + \lambda_3\,,$$

$$\omega_\lambda(\{b_1\}) = \lambda_1 + \lambda_2\,,\ \omega_\lambda(\{b_2\}) = \lambda_3 + \lambda_4\,,$$

i.e.:

$$\mathcal{X} = \{\omega_\lambda\}\,.$$

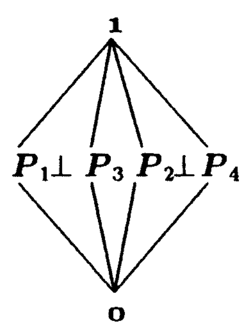
The standardization postulate allows properties $E \in \mathcal{B} = 2^M$ to be 'measurable' only if

$$\omega_\lambda(E) = 1 \text{ and } \omega_{\lambda'}(E) = 0 \text{ for suitable } \lambda, \lambda' \text{ depending on } E\,.$$

Then $\mathcal{Q}$ consists, apart from the empty set $\emptyset$ and the total set M, of the following four subsets of M :

$$T_1 \stackrel{\text{def}}{=} \{a_1, b_1\}\,,\ T_2 \stackrel{\text{def}}{=} \{a_2, b_1\}\,,\ T_3 \stackrel{\text{def}}{=} \{a_2, b_2\}\,,\ T_4 \stackrel{\text{def}}{=} \{a_1, b_2\}\,.$$

Conditions (I_1)—(I_3) are fulfilled for $\mathcal{Q}$, $\mathcal{X}$, but $(\mathcal{L}, \prec, \neg)$ is no longer classical. Indeed, $(\mathcal{L}, \prec, \neg)$ corresponds to the Hasse diagramm



where:

$$\mathbf{o} \stackrel{\text{def}}{=} [\emptyset] = \{\emptyset\} \;, \quad \mathbf{1} \stackrel{\text{def}}{=} [M] = \{M\} \;, \quad \boldsymbol{P}_j \stackrel{\text{def}}{=} [T_j] = \{T_j\} \;.$$

Of course, in order to model such typical quantum effects like nonexistence of dispersion free states or violation of Bell's inequality one has to introduce equivalence classes in a nontrivial way.

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