QUANTIZATION OF IDEAL FLUIDS: PERTURBATIVE OR NON–PERTURBATIVE APPROACH*,**

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Perturbative and non-perturbative types of approaches to quantization of ideal fluid flows are considered and compared. The results on stability of particular vortex structures obtained in the framework of the standard energy-Casimir method are reminded for the purpose of checking connection between stability and quantizability. Results on geometric quantizability derived by Goldin, Menikoff and Sharp for these structures are also reminded. The discrepancy between results of these two approaches being an evidence for non-perturbative character of quantization of ideal fluids is stressed. New non-perturbative approach exploiting ideas from Ashtekar programme of quantization of gravity is formulated. Some applications of the new approach in description of superfluid helium are briefly shown.

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1. Introduction

Quantization of physical theories needs good knowledge of their phase spaces. As a matter of fact the reasonable procedure is well understood if we have the phase space which is a finite-dimensional linear space equipped with a non-degenerate symplectic form ω. The Darboux theorem enables introduction of canonical coordinates on the phase space (q^i, p_j) (coordinates

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and momenta). The quantization works then if we assign to q^i and p_i some operators $\hat{q}^i$ and $\hat{p}_i$ acting in a Hilbert space H, in such a way that usual Poisson brackets $\{q^i, p_j\} = \delta^i_j$, $\{q^i, q^j\} = 0$, $\{p_i, p_j\} = 0$ become canonical commutation relations $[\hat{q}^i, \hat{p}_j] = i\hbar\delta^i{}_j$, $[\hat{q}^i, \hat{q}^j] = 0$, $[\hat{p}_i, \hat{p}_j] = 0$ for the operators. However, there exist many physical theories which have phase spaces that are nonlinear and/or of infinite dimensions (field theories). Additionally, not always the symplectic form is degenerate. We do not have a general procedure for quantizing such theories. Of course one canalways introduce for any reasonable theory a symplectic form that is non-degenerate, with a cost of possible complication of the so obtained reduced phase space, which becomes a space of higher nonlinearity. Now, one should learn how to deal with theories with nonlinear phase spaces.

The possibility one should take firstly into account is linearization of the phase space of the theory *i.e.* consideration of the tangent space at a solution of the equations of the theory. Such procedure would be justified if the Hamiltonian of the theory also reasonably linearizes, it means its second order expansion form is not divergent and further expansion terms do not lead to another divergence. This way only theories which are nonlinearly stable can be quantized by using these methods. Since further terms lead to some small corrections to the theory, the approach can be called perturbative. Nevertheless, one should take into account also other theories, like, for example, gravity, which are not perturbatively quantizable. In this paper we show that also ideal fluids do not underlie such quantization. For this purpose we review results on stability of particular ideal fluid flows, especially of some vortex structures. We compare these results with the results on a non-perturbative quantization within the geometric quantization approach found by Goldin, Menikoff and Sharp [1, 2]. The latter approach did not lead to establishing the quantum theory but it enabled stating which particular vortex structures are quantizable within this approach.

The discrepancy between results of these two approaches proves lack of a perturbative approach to quantization of ideal fluids. We propose another method of non-perturbative quantization of ideal fluids, which applies extensively some elements of the recently proposed and presently being developed the Ashtekar program of quantization of gravity. We believe the interplay between the Ashtekar-like approach and the geometric quantization should lead us to a reliable quantized theory of ideal fluids. The plan of the paper is as follows: firstly we report the results on stability of ideal fluids flows, in particular of some vortex structures, this way investigating perturbative approach. Secondly, we remind results on geometric quantization of the vortex structures. Thirdly, I propose a new Ashtekar-like approach and show an application in description of superfluid helium.

2. Perturbative approach

Presently well developed method of analysis of stability of fluids called the energy — Casimir method has its origin in the work of Arnold [3]. I will not discuss it here extensively, because of the lack of space. Let me mention only that instead of examining the Hamiltonian one considers a sum of the Hamiltonian and of Casimir functions and all constants of motion. The method gives the following results for stability of particular vortex structures: $2D$ point vortices and $2D$ vortex patches are stable, $3D$ vortex filaments are unstable, $3D$ vortex tubes seem to be also unstable.

3. Non-perturbative approach: geometric quantization

Goldin, Menikoff and Sharp [1, 2] developed another method of quantization, based on geometric quantization. The crucial role in the method plays the symmetry group G of the system. One should establish then its stability subgroup H. Existence of an appropriate Hilbert space needs satisfaction of some integrality conditions, which in the case of the ideal fluid are expressible as the condition of quantization of the vortex flux, well known from the theory of superfluid helium. Further one should proceed towards reduction of the Hilbert space by introducing so called polarization. In classical theories polarization could mean independence of the sections of the Hilbert space, representing wave functions of the system, on the momentum variables. In the general situation one looks for a subgroup K of the symmetry group G, which satisfies the conditions: $H \subset K \subset G$ and $\dim(G/K) = \dim(K/H)$. Quantizability means then existence of such a polarization. In this sense the following vortex structures were found to be quantizable by Goldin, Menikoff and Sharp [1, 2]: $2D$ point vortices, $2D$ vortex patches and $3D$ vortex filaments are not quantizable but $3D$ vortex tubes are.

The lack of correspondence between quantizability and stability shows nonexistence of a perturbative approach to quantization. Unfortunately, the authors could not complete the whole program of geometric quantization of ideal fluids. The reason is there are difficulties with construction of measures invariant under action of the volume preserving diffeomorphism group, which is the symetry group. Such measures are necessary for constructing the unitary representations of the group whose space becomes the seeked Hilbert space of the quantum theory. Further we propose another non-perturbative approach, which could be treated as an alternative to geometric quantization or at least as a tool for solving some of the geometric quantization problems.

4. Non-perturbative approach: Ashtekar-like program

In the recently formulated Ashtekar program of quantization of gravity [4, 5] the gravitational field is described by an $SL(2,\mathbb{C})$ connection one-form A. One starts from a $3+1$ splitting of the space-time and considers the connection one-form on the Cauchy surface Σ. The space of such one-forms is the configuration space of the theory. Together with conjugate momenta they build up the phase space of the theory. The system is invariant with respect to diffeomorphisms of Σ, the evolution in time is governed by the constraints connected with the diffeomorphisms of the fourth, *i.e.* time, dimension.

It occurred reasonable to introduce nonlocal coordinates in the phase space. For this purpose one introduces holonomies of the connections along a family of loops γ_i, $i = 1, ..., n$ in Σ. Important role play traces of the exponents of holonomies with respect to some representations R_i of the gauge group $SL(2,\mathbb{C})$: $\mathrm{Tr}_{R_i} \exp\left(\oint_{\gamma_i} A\right)$.

These could be interpreted as some physical observables of the what it would be a quantum theory. Then one can construct their expectation values as quantities:

$$\int DAe^{iS(A)}\Pi_i \mathrm{Tr}_{R_i} \exp\left(\oint_{\gamma_i} A\right), \qquad (1)$$

where $S(A) = \frac{k}{4\pi}\mathrm{Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$, k — an integer, DA — a measure on the space of connections.

The quantities are diffeomorphism and gauge-invariant and are also knot invariants for links $\bigcup_i \gamma_i$.

Quantization of ideal fluid can be also considered along similar lines. With the usual Euler velocity v_i one associates a one-form $A = v_i dx^i$ on the Cauchy surface Σ, which becomes a $U(1)$ gauge field. Then one constructs physical observables:

$$\mathrm{Tr}_{R_i} \exp\left(\oint_{\gamma_i} A\right), \qquad (2)$$

where R_i are representations of U(1).

Their "expectation values" are then given by the formula (1). In my recent papers [7, 6, 8] I argued the field theory operating only with a $U(1)$ connection form describes critical superfluid helium. Therefore the field theoretic distribution function is given by:

$$\int DA \exp iS(A). \qquad (3)$$

It was also argued in my paper [8] that the statistical sum of the critical superfluid helium is equal to the product of statistical sums of $2D$ Ising models. This way was proved logarithmic divergence in heat capacity temperature dependence in the neighborhood of the critical temperature.

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