FINITE QED AND QUANTUM GAUGE FIELD THEORY*

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We investigate QED and quantized Yang–Mills theories coupled to matter fields in the framework of causal perturbation theory which goes back to Epstein and Glaser. In this approach gauge invariance is expressed by a simple commutator relation for the S-matrix. It has been proven in all orders of perturbation theory. The corresponding gauge transformations are simple transformations of the free fields only. In spite of this simplicity gauge invariance implies the usual Ward rsp. Slavnov–Taylor identities and unitarity on the physical subspace.

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1. Introduction

The word 'finite' means that we work with *causal perturbation theory*, which goes back to Epstein and Glaser [1]. No-ultraviolet divergences appear in this approach. One works exclusively with free fields, which are mathematically well-defined and performs only justified operations with them. By considering a simple gauge transformation of the free fields, we shall obtain the known Ward rsp. Slavnov–Taylor identities which express the usual gauge invariance.

In the causal method the problems are separated: First we prove *(re)normalizability* by simple power counting. Then we prove *gauge invariance*. The latter implies *unitarity on the physical subspace*. The interaction is switched off by a function $g(x)$, $g \in \mathcal{S}(\boldsymbol{R}^4)$. *Infrared divergences* and the problem of confinement appear only in the adiabatic limit $g(x) \to 1$, which is not considered in these lectures.

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2. Short introduction to causal perturbation theory

The idea of constructing the S-matrix by means of causality goes back to Stückelberg [2] and Bogoliubov and Shirkov [3]. This program was carried out correctly by Epstein and Glaser [1]. However they considered scalar fields only. These lectures report on the extension to gauge theories.

In causal perturbation theory one makes the ansatz of a formal power series in the coupling constant e for the S-matrix

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 ... d^4x_n \, T_n(x_1, ..., x_n) g(x_1) ... g(x_n) . \tag{2.1}$$

The test-function $g \in \mathcal{S}(\boldsymbol{R}^4)$ switches the interaction and $T_n(x_1, ..., x_n) \sim e^n$ is an operator-valued distribution. The T_n's are constructed by induction on the order n. The input is $T_1(x)$ in terms of free fields. T_1 is roughly speaking given by the interaction Lagrangian density. In QED we have

$$T_1(x) = ie : \overline{\psi}(x) \gamma^\mu \psi(x) : A_\mu(x) , \tag{2.2}$$

where A_μ, ψ, $\overline{\psi} \stackrel{\text{def}}{=} \psi^+ \gamma^0$ are free field operators

$$\Box A_\mu = 0 , \qquad (i\gamma^\mu \partial_\mu - m)\psi = 0 . \tag{2.3}$$

The step from $n-1$ to n in the inductive construction of the T_n is uniquely determined by translation invariance and *causality* only. First we have to do some preparations. The formal power series (2.1) can be inverted

$$S(g)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 ... d^4x_n \, \tilde{T}_n(x_1, ..., x_n) g(x_1) ... g(x_n) \tag{2.4}$$

with

$$\tilde{T}_n(X) \stackrel{\text{def}}{=} \sum_{r=1}^{n} (-1)^r \sum_{P_r} T_{n_1}(X_1) ... T_{n_r}(X_r) . \tag{2.5}$$

Herein the second sum runs over all partitions of $X \stackrel{\text{def}}{=} \{x_1, ..., x_n\}$ into r disjoint subsets

$$X = X_1 \cup ... \cup X_r, \quad X_j \neq \emptyset, \quad |X_j| = n_j, . \tag{2.6}$$

Now we summarize the inductive step as a recipe. For the derivation of this construction from causality and translation invariance and for many details see [1, 4]. With (2.5) we can construct the operator-valued distributions

R'_n, A'_n which are a tensor product of T_k's of lower order k, $1 \leq k \leq n-1$ and, therefore, known by the induction assumption;

$$R'_n(x_1, ...; x_n) \stackrel{\text{def}}{=} \sum_{X,Y} T_{n-k}(Y, x_n)\tilde{T}_k(X) , \tag{2.7}$$

$$A'_n(x_1, ...; x_n) \stackrel{\text{def}}{=} \sum_{X,Y} \tilde{T}_k(X)T_{n-k}(Y, x_n) , \tag{2.8}$$

where $X \stackrel{\text{def}}{=} \{x_{i_1}, ..., x_{i_k}\}$, $Y \stackrel{\text{def}}{=} \{x_{i_{k+1}}, ..., x_{i_{n-1}}\}$, $X \cup Y = \{x_1, ..., x_{n-1}\}$ and the sum is over all partitions of this kind with $1 \leq k \equiv \mid X \mid \leq n-1$. One can prove that

$$D_n \stackrel{\text{def}}{=} R'_n - A'_n \tag{2.9}$$

has causal support

$$\operatorname{supp} D_n(x_1, ...; x_n) \subset (\Gamma^+_{n-1}(x_n) \cup \Gamma^-_{n-1}(x_n)) , \tag{2.10}$$

where

$$\Gamma^{\pm}_{n-1}(x_n) \stackrel{\text{def}}{=} \{(x_1, ..., x_n) \in \boldsymbol{R}^{4n} | x_j \in x_n + \bar{V}^{\pm},\ \forall j = 1, ..., n-1\} . \tag{2.11}$$

The crucial step in the inductive construction is the *correct distribution splitting* of D_n

$$D_n = R_n - A_n , \tag{2.12}$$

with

$$\operatorname{supp} R_n(x_1, ...; x_n) \subset \Gamma^+_{n-1}(x_n) \quad \text{and} \quad \operatorname{supp} A_n(x_1, ...; x_n) \subset \Gamma^-_{n-1}(x_n) . \tag{2.13}$$

For this pupose we expand the operator-valued distributions in normally ordered form, *e.g.* for QED

$$F_n(x_1, ..., x_n) = \sum f_n(x_1 - x_n, ..., x_{n-1} - x_n) : \overline{\psi}(x_j) ... \psi(x_l) ... A(x_m) ... : , \tag{2.13a}$$

where $F = R'$, A', D, R, A, T. The coefficients f_n are C-number distributions. Due to translation invariance, they depend on the relative coordinates only and, therefore, are responsible for the support properties. Consequently, the splitting must be done in these C-number distributions. Obviously, the critical point for the splitting is the UV-point

$$\Gamma^+_{n-1}(x_n) \cap \Gamma^-_{n-1}(x_n) = \{(x_1, ..., x_n) \in \boldsymbol{R}^{4n} | x_1 = x_2 = ... = x_n\} . \tag{2.14}$$

In order to measure the behavior of the C-number distribution f in the vicinity of this point, one defines an index $\omega(f)$, which is called the *singular order of f at $x = 0$* [1, 4]. We will need the following example: Let D^a, $a \stackrel{\text{def}}{=} (a_1, ..., a_m)$, be a partial differential operator. Then

$$\omega(D^a \delta^{(m)}(x_1, ..., x_m)) = |a| \stackrel{\text{def}}{=} a_1 + ... + a_m \,. \tag{2.15}$$

If $\omega(d_n) < 0$, the splitting of d_n is trivial and uniquely given by multiplication with a step fuction [1, 4]

$$r_n(x_1 - x_n, ...) = \Theta(\sum_j v_j (x_j - x_n)) d_n(x_1 - x_n, ...) \,,$$
$$\text{with } v_j \in V^+ \text{ arbitrary}, \quad \forall j \,. \tag{2.16}$$

If $\omega(d_n) \geq 0$, one must do the splitting more carefully [1, 4]. Moreover it is not unique. One has an undetermined polynomial which is of degree $\omega(d_n)$ (the degree cannot be higher since renormalizability requires $\omega(r_n) = \omega(d_n)$),

$$r_n(x_1 - x_n, ..., x_{n-1} - x_n) = r_n^0(...)$$
$$+ \sum_{|a|=0}^{\omega(d_n)} C_a D^a \delta^{(4(n-1))}(x_1 - x_n, ..., x_{n-1} - x_n) \,, \tag{2.17}$$

where r_n^0 is a special splitting solution and C_a are the undetermined normalization constants. If one does the splitting also in this case by multiplying with a Θ-function, one obtains the usual, UV-divergent Feynman rules. But this procedure is mathematically inconsistent. From R_n one constructs

$$T_n' \stackrel{\text{def}}{=} R_n - R_n' \tag{2.18}$$

and finally T_n is obtained by symmetrization of T_n'

$$T_n(x_1, ..., x_n) = \sum_{\pi \in \mathcal{S}_n} \frac{1}{n!} T_n'(x_{\pi 1}, ..., x_{\pi n}) \,. \tag{2.19}$$

One can prove that this is the correct n-point distribution of $S(g)$ (2.1), fulfilling the requirements of causality and translation invariance. Note

$$\omega \stackrel{\text{def}}{=} \omega(t_n) = \omega(r_n) = \omega(d_n) \,. \tag{2.20}$$

If one does the distribution splitting (2.12) correctly no UV-divergences appear. The undetermined normalization polynomial goes over from r_n to t_n

$$t_n(x_1 - x_n, ..., x_{n-1} - x_n) = t_n^0(...) + \sum_{|a|=0}^{\omega} C_a D^a \delta^{(4(n-1))}(x_1 - x_n, ..., x_{n-1} - x_n) . \quad (2.21)$$

In QED one can prove by means of scaling properties [4]

$$\omega = 4 - b - \tfrac{3}{2} f , \quad (2.22)$$

where b is the number of external photons and f is the number of external pairs $(\overline{\psi}, \psi)$. The fact that ω is bounded in the order n of the perturbation series, is the *(re)normalizability* of QED.

The normalization constants C_a are restricted by the following requirements:

— Lorentz covariance, SU(N)-covariance in the case of Yang–Mills theories,

— pseudo-unitarity $S(g)^K = S(g)^{-1}$, where K is a conjugation related to the ajoint (see Section 5(b)),

— invariance with respect to permutations of certain vertices, for example the inner vertices of a diagram,

— invariance with respect to the parity transformation (P), time reversal (T) and charge conjugation (C),

— the existence of the adiabatic limit $g(x) \to 1$ and

— gauge invariance.

3. Finite QED

(a) Distribution splitting

In x-space the unknown $r_n(x_1 - x_n, ...)$ has its support in $\Gamma_{n-1}^+(x_n)$ (2.13). Performing the Fourier transformation in the relative coordinates this support property goes over into an analyticity statement [1]: $\hat{r}_n(p_1, ..., p_{n-1})$ is the boundary value of an analytic function — analytic in $(\boldsymbol{R}^{4(n-1)} + i\Gamma^+)$. For $p = (p_1, ..., p_{n-1})$, with $p_j \in V^+$, $\forall j = 1, ..., n-1$, a special splitting solution, the so-called *'central solution'*, can be obtained from $\hat{d}_n$ by a dispersion integral [1, 4, 5]

$$\hat{r}_n^0(p) = \frac{i}{2\pi} \int dt \, \frac{\hat{d}_n(tp)}{(t - i0)^{\omega+1}(1 - t + i0)} , \quad (3.1)$$

where $\omega = \omega(d_n)$. In the case of trivial splitting $\omega(d_n) < 0$, (3.1) is essentally obtained by Fourier transformation of $r_n(x) = \Theta(vx)d_n(x)$ and choosing $v := p$ (2.16). The values of $\hat{r}_n^0(p)$ in other domains of p can be obtained by analytic continuation. The central solution fulfils

$$(D^a \hat{r}_n^0)(0) = 0\,, \qquad \forall |a| \leq \omega\,. \tag{3.2}$$

Since this property fixes *all* normalization constants C_a, $|a| \leq \omega$ in (2.17), the central solution is characterized by (3.2) *uniquely*.

The central solution $\hat{r}_n^0(p)$ has nearly all symmetries listed at the end of Section 2. Especially, it is covariant [1] and gauge invariant [5]. But in QED the existence of the adiabatic limit [4,6] requires the normalization

$$\Sigma(p, -p, 0, ...0)|_{\gamma^\nu p_\nu = m} = 0 \tag{3.3}$$

for the self-energy $\Sigma(p_1, ..., p_{n-1})$ of the electron in contrast to (3.2).

The dispersion integral (3.1) does not exist in massless theories (*e.g.* in the massless Yang–Mills theories considered below) because of an infrared divergence. Then, instead of $p = 0$ in (3.2), one can work with a totally space-like subtraction point q, but the splitting solution with subtraction point $q \neq 0$ is generally not covariant [7]. However, the central solution exists in every pure massive theory [1] and most probably in QED. (The latter is not yet proven completely, but by computing a typically dangerous diagram we found that the nonvanishing mass of the fermions seems to be just sufficient to ensure the existence of (3.1).)

(b) Gauge invariance in QED

In view of non-abelian gauge theory we will formulate gauge invariance in a somewhat strange way, but we will obtain the usual Ward identities. We define the gauge charge Q by

$$Q \stackrel{\text{def}}{=} \int_{t=\text{const.}} d^3x\, (\partial_\nu A^\nu) \overleftrightarrow{\partial}_0 u \tag{3.4}$$

where $u(x)$ is an external C-number field fulfilling $\square u(x) = 0$. This definition makes sense because $(\partial_\nu A^\nu)\overleftrightarrow{\partial}_\mu u$ is a conserved current. One easily obtains

$$[Q, A^\mu] = i\partial^\mu u\,, \qquad [Q, \psi] = 0\,, \qquad [Q, \overline{\psi}] = 0\,. \tag{3.5}$$

By means of current conservation $\partial_\nu : \overline{\psi}\gamma^\nu\psi := 0$, we may write $[Q, T_1]$ as a divergence

$$[Q, T_1] = i\partial_\nu T^\nu_{1/1}\,, \qquad \text{with} \quad T^\nu_{1/1} \stackrel{\text{def}}{=} ie : \overline{\psi}\gamma^\nu\psi : u\,. \tag{3.6}$$

We call this fact gauge invariance in first order and $T^{\nu}_{1/1}$ is called a 'Q-vertex'.

We turn to second order. With $A'_2(x_1;x_2) = -T_1(x_1)T_1(x_2)$ (2.8), we get by inserting (3.6)

$$\begin{aligned}[Q, A'_2(x_1;x_2)] &= -[Q,T_1(x_1)]T_1(x_2) - T_1(x_1)[Q,T_1(x_2)] \\ &= i\partial^{x_1}_{\nu}\{-T^{\nu}_{1/1}(x_1)T_1(x_2)\} + i\partial^{x_2}_{\nu}\{-T_1(x_1)T^{\nu}_{1/1}(x_2)\}\,.\end{aligned} \tag{3.7}$$

According to the inductive construction of Epstein and Glaser, we have

$$\begin{aligned}-T^{\nu}_{1/1}(x_1)T_1(x_2) &= A'^{\nu}_{2/1}(x_1;x_2)\,, \\ -T_1(x_1)T^{\nu}_{1/1}(x_2) &= A'^{\nu}_{2/2}(x_1;x_2)\end{aligned} \tag{3.8}$$

which are the A'_2-distributions with a Q-vertex at x_1 (rsp. x_2) and an ordinary QED-vertex at x_2 (rsp. x_1). Proceeding analogously for R'_2 we obtain with $D_{2(/j)} = R'_{2(/j)} - A'_{2(/j)}$ $j = 1,2$ (2.9)

$$[Q, D_2(x_1;x_2)] = i\partial^{x_1}_{\nu}D^{\nu}_{2/1}(x_1;x_2) + i\partial^{x_2}_{\nu}D^{\nu}_{2/2}(x_1;x_2)\,. \tag{3.9}$$

Assuming that this equation can be maintained in the distribution splitting $D_{2(/j)} \to R_{2(/j)}$, $j = 1,2$ (2.12), we obtain with $T'_{2(/j)} = R_{2(/j)} - R'_{2(/j)}$ (2.18) and after symmetrization (2.19)

$$[Q, T_2(x_1,x_2)] = i\partial^{x_1}_{\nu}T^{\nu}_{2/1}(x_1,x_2) + i\partial^{x_2}_{\nu}T^{\nu}_{2/2}(x_1,x_2)\,, \tag{3.10}$$

which we call gauge invariance in second order.

In order to define gauge invariance in an arbitrary order n we define a *larger* theory by giving its first order

$$S_1(g_0, g_{1\nu}) \overset{\text{def}}{=} \int d^4x\{T_1(x)g_0(x) + T^{\nu}_{1/1}(x)g_{1\nu}(x)\}\,, \tag{3.11}$$

where T_1 is the ordinary QED-vertex (2.2) and $T_{1/1}$is the Q-vertex (3.6). The higher orders (of the larger theory) are determined by the usual inductive construction. The physically relevant S-matrix is obtained in the limit $g_0 \to 1$, $g_1 \to 0$. We define *perturbative gauge invariance* by

$$[Q, T_n(x_1,...,x_n)] = i\sum_{l=1}^{n} \partial^{x_l}_{\nu}T^{\nu}_{n/l}(x_1,...,x_n), \quad \forall n \in \mathbf{N}\,, \tag{3.12}$$

where $T^{\nu}_{n/l}(x_1,...,x_n)$ is the n-point distribution of the big theory with one Q-vertex at x_l and all other $(n-1)$ vertices are ordinary QED-vertices.

Adopting the argument in (3.7)–(3.10) to the general inductive step from $n-1$ to n (in the construction of the $T_n, T_{n/l}$), we see that gauge invariance can be violated in the distribution splitting only. Due to

$$R_n(x_1,...;x_n) = D_n(x_1,...;x_n) \quad \text{on} \quad \Gamma^+_{n-1}(x_n) \setminus \{(x_n,...,x_n)\}\,,$$

$$R_n(x_1,...;x_n) = 0 \quad \text{on} \quad \boldsymbol{R}^{4n} \setminus \Gamma^+_{n-1}(x_n)\,, \tag{3.13}$$

the possible anomaly $\mathcal{A}_n \stackrel{\text{def}}{=} [Q, T_n] - i\sum_l \partial^l T_{n/l}$ is a *local* term

$$\operatorname{supp} \mathcal{A}_n(x_1,...,x_n) \subset \{(x_n,...,x_n)\} \tag{3.14}$$

We have seen in (2.17) that $T_n, T_{n/l}$ have undetermined normalization polynomials which have their support in this point $\{(x_n,...,x_n)\}$, too. Therefore, in constructing a gauge invariant S-matrix (2.1) we proceed in the following way: We choose the normalization constants C_a in (2.17) in such a way that gauge invariance is preserved in the distribution splitting. Generally, it is highly non-trivial that such a choice is possible. However, in QED the situation is quite simple: *Choosing the mass normalization (3.3) and taking the central solution r_n^0 (3.1) in all other cases, gauge invariance (3.12) holds true* [4, 5].

In order to establish the connection of our gauge invariance (3.12) with the usual one, we have to eliminate the unphysical Q-vertex in (3.12). For this purpose we consider all terms in $T_n(x_1,...,x_n)$ containing $A_\mu(x_l)$ (l fixed)

$$T_n(x_1,...,x_n) =: t_l^\mu(x_1,...,x_n) A_\mu(x_l) : + (\text{terms without } A_\mu(x_l))\,, \tag{3.15}$$

where $t_l^\mu(x_1,...,x_n)$ contains operators $\overline{\psi}$, ψ and $A_\mu(x_k)$ for $k \neq l$. Then one easily proves [7] that (3.12) is equivalent to

$$\partial_\mu^{x_l} t_l^\mu(x_1,...,x_n) = 0, \quad \forall l = 1,...,n, \quad \forall n\,, \tag{3.16}$$

which is a condition on distributions of the physical theory only. Choosing for example $l = 3$ and collecting all terms with operators $:\overline{\psi}(x_1)...\psi(x_2):$ in (3.16), one obtains the usual Ward identity connecting the vertex Λ with the electron self-energy Σ [4, 5]

$$\partial_\mu^{x_3} \Lambda^\mu(x_1 - x_n, ...) + ie[\delta(x_1 - x_3) - \delta(x_2 - x_3)] \\ \times \Sigma(x_1 - x_n, x_2 - x_n, x_4 - x_n, ...) = 0\,. \tag{3.17}$$

The set of all Ward identities implies the operator gauge invariance (3.12).

Which are the *gauge transformations* belonging to our gauge invariance (3.12)? Let us consider the transformations

$$A^\mu_\lambda \stackrel{\text{def}}{=} e^{-i\lambda Q} A^\mu e^{i\lambda Q} = A^\mu - i\lambda[Q, A^\mu] = A^\mu + \lambda\partial^\mu u\,,$$
$$\psi_\lambda \stackrel{\text{def}}{=} e^{-i\lambda Q}\psi e^{i\lambda Q} = \psi\,, \qquad \overline{\psi}_\lambda \stackrel{\text{def}}{=} e^{-i\lambda Q}\overline{\psi} e^{i\lambda Q} = \overline{\psi}\,, \tag{3.18}$$

where we have used (3.5). Note that the terms of higher order in λ vanish, since they are higher commutators $[Q, ...[Q, A]]$ and since $[Q, A] = i\partial u$ is a C-number field. We emphasize that the transformed fields A^μ_λ, ψ_λ, $\overline{\psi}_\lambda$ fulfil the wave rsp. Dirac equation (2.3), they are *free* fields, too. By means of our gauge invariance (3.12) we obtain

$$\begin{aligned}\lim_{g\to 1}[Q, S_n(g)] &= \lim_{g\to 1}\frac{1}{n!}\int d^4x_1 ...[Q, T_n(x_1, ...)]g(x_1)...\\
&= \lim_{g\to 1}\frac{i}{n!}\sum_l \int d^4x_1 ...\partial_\nu^{x_l} T^\nu_{n/l}(x_1, ...)g(x_1)...\\
&= -\lim_{g\to 1}\frac{i}{n!}\sum_l \int d^4x_1 ...d^4x_l ...T^\nu_{n/l}(x_1, ...)g(x_1)...\partial_\nu g(x_l)... = 0\,.\end{aligned} \tag{3.19}$$

However, the adiabatic limit $g \to 1$ of $S_n(g)$ is infrared divergent. But these divergences are known to be logarithmic. Moreover, $T^\nu_{n/l}$ has the same infrared behavior as T_n [7]. Therefore, the logarithmic divergence coming from $T^\nu_{n/l}$ is overcompensated by $\partial_\nu g(x_l)$, which goes linearly to zero [6]. Then we conclude

$$\lim_{g\to 1}[e^{-i\lambda Q}S_n(g)e^{i\lambda Q} - S_n(g)] = 0, \quad \forall\lambda \in \boldsymbol{R}\,. \tag{3.20}$$

Note that $e^{-i\lambda Q}S_n(g)e^{i\lambda Q}$ is the transformed S-matrix of n-th order which is obtained from $S_n(g)$ by replacing all external legs A, ψ, $\overline{\psi}$ by A_λ, ψ_λ, $\overline{\psi}_\lambda$. But the adiabatic limits of the individual terms in (3.20) do not exist.

4. Yang–Mills theories with matter fields

The theory is constructed inductively from the following first order

$$T_1(x) = T_1^A(x) + T_1^\psi(x) + ...\,, \tag{4.1}$$

with

$$T_1^A(x) \stackrel{\text{def}}{=} \frac{ig}{2} f_{abc} : A_{\mu a}(x)A_{\nu b}(x)F_c^{\nu\mu}(x) :\,, \tag{4.2}$$

$$T_1^\psi(x) \stackrel{\text{def}}{=} ig' j_{\mu a}(x)A_a^\mu(x) \tag{4.3}$$

is a QED-like coupling, where the matter current $j_{\mu a}$ is defined by

$$j_{\mu a}(x) \stackrel{\text{def}}{=} \frac{1}{2} : \overline{\psi}_\alpha(x)\gamma_\mu(\lambda_a)_{\alpha\beta}\psi_\beta(x) : \tag{4.4}$$

Herein, g, g' are coupling constants, which are independent for the time being, f_{abc} are the structure constants of the group SU(N) and $\frac{-i}{2}\lambda_a$, $a = 1, ..., N^2 - 1$ denote the generators of the fundamental representation of SU(N). The gauge potentials A_a^μ, $F_a^{\mu\nu} \stackrel{\text{def}}{=} \partial^\mu A_a^\nu - \partial^\nu A_a^\mu$ are massless and fulfil the wave equation. The matter fields ψ_α and $\overline{\psi}_\alpha \stackrel{\text{def}}{=} \psi_\alpha^+\gamma^0$ satisfy the free Dirac equation (2.3) with a colour independent mass $m \geq 0$ [8]. Therefore, the matter current is conserved

$$\partial^\mu j_{\mu a}(x) = 0\,. \tag{4.5}$$

The dots in (4.1) signify a further coupling which will be introduced later.

(a) Definition of gauge invariance

We adopt the procedure for QED given in section 3(b). Analogously to (3.4) we define the gauge charge Q by [9]

$$Q \stackrel{\text{def}}{=} \int\limits_{t=\text{const.}} d^3x \sum_a (\partial_\nu A_a^\nu) \overleftrightarrow{\partial}_0 u_a \tag{4.6}$$

where $u_a(x)$ is an external C-number field fulfilling $\Box u_a(x) = 0$ for a moment. One easily obtains

$$[Q, A_a^\mu] = i\partial^\mu u_a\,, \qquad [Q, F_a^{\mu\nu}] = 0, \qquad [Q, \psi_\alpha] = 0, \qquad [Q, \overline{\psi}_\alpha] = 0 \tag{4.7}$$

and by means of (4.5)

$$[Q, T_1^\psi] = i\partial_\nu T_{1/1}^{\psi\nu}, \qquad \text{with the Q} - \text{vertex} \quad T_{1/1}^{\psi\nu} \stackrel{\text{def}}{=} ig' j_a^\nu u_a\,. \tag{4.8}$$

But $[Q, T_1^A]$ is not a divergence. *Gauge invariance requires additional couplings and fields* [9]: Let u_a, $\tilde{u}_a$ be two free, scalar Fermi fields ('ghost fields')

$$\begin{aligned} &\Box u_a = 0, \quad \Box \tilde{u}_a = 0, \quad \{u_a, u_b\} = 0, \quad \{\tilde{u}_a, \tilde{u}_b\} = 0\,, \\ &\{u_a, \tilde{u}_b\} = -i\delta_{ab}D_0(x-y)\,, \end{aligned} \tag{4.8a}$$

where D_0 is the mass zero Pauli–Jordan distribution. We replace in the definition (4.6) of Q the C-number field u_a by this operator-valued ghost field u_a and obtain the anticommutators

$$\{Q, u_a\} = 0, \qquad \{Q, \tilde{u}_a\} = -i\partial_\nu A_a^\nu\,. \tag{4.9}$$

Moreover, we introduce the ghost coupling

$$T_1^u(x) \stackrel{\text{def}}{=} -igf_{abc} : A_{\mu a}(x)u_b(x)\partial^\mu \tilde{u}_c(x) : . \tag{4.10}$$

Then, together with (4.8), we have gauge invariance in first order [9]

$$[Q, T_1^A(x) + T_1^u(x)] = i\partial_\nu (T_{1/1}^{A\nu}(x) + T_{1/1}^{u\nu}(x)) , \tag{4.11}$$

with the Q-vertices

$$T_{1/1}^{A\nu}(x) \stackrel{\text{def}}{=} igf_{abc} : A_{\mu a}(x)u_b(x)F_c^{\nu\mu}(x) : , \tag{4.12}$$

$$T_{1/1}^{u\nu}(x) \stackrel{\text{def}}{=} -\frac{ig}{2}f_{abc} : u_a(x)u_b(x)\partial^\nu \tilde{u}_c(x) : . \tag{4.13}$$

Gauge invariance in arbitrary order is defined in exactly the same way as in QED: Inserting $T_1 \stackrel{\text{def}}{=} T_1^A + T_1^u + T_1^\psi$ and $T_{1/1}^\nu \stackrel{\text{def}}{=} T_{1/1}^{A\nu} + T_{1/1}^{u\nu} + T_{1/1}^{\psi\nu}$ in (3.11), the definition is given by (3.12).

To probe uniqueness of the construction, we have tried to establish gauge invariance by working with *bosonic* ghosts [9] in contrast to the fermionic ones introduced above. This works fine in first order, but for second order tree diagrams gauge invariance is violated.

Since the gauge charge Q is a Fermi operator, we have

$$Q^2 = 0 , \tag{4.14}$$

which will be important in the proof of unitarity on the physical subspace (sect.5(b,c)).

Let us turn to the *gauge transformations* which are transformations of *free* fields only. Up to a sign (which provides a closer connection to the BRS-transformations [11]), they are defined completely analogous to (3.18). Their infinitesimal versions are given by

$$\delta\phi(x) = \partial_\lambda|_{\lambda=0}[e^{-i\lambda Q'}\phi(x)e^{i\lambda Q'}], \qquad \phi = A_a^\mu, F_a^{\mu\nu}, u_a, \tilde{u}_a, \psi, \overline{\psi}, \tag{4.15}$$

where

$$Q' \stackrel{\text{def}}{=} (-1)^{Q_g} Q, \quad Q_g \stackrel{\text{def}}{=} i \int\limits_{t=\text{const.}} d^3x : \tilde{u}_a(x)\overleftrightarrow{\partial}_0 u_a(x) : . \tag{4.16}$$

Q_g is the ghost charge operator [10]: $[Q_g, u_a] = -u_a$, $[Q_g, \tilde{u}_a] = \tilde{u}_a$. Due to $[Q', T_n(X)] = (-1)^{Q_g}[Q, T_n(X)]$, our gauge invariance (3.12) implies the invariance (up to a divergence) of $T_n(X)$ with respect to these transformations (4.15): $e^{-i\lambda Q'}T_n(X)e^{i\lambda Q'} - T_n(X)$ =divergence. However, (3.19-20) cannot be adopted from QED, since the infrared behavior is worse in non-abelian gauge theories. The explicit expressions for the transformations (4.15) are

$$\delta A_a^\mu = (-1)^{Q_g}\partial^\mu u_a, \qquad \delta\tilde{u}_a = -(-1)^{Q_g}\partial_\nu A_a^\nu \tag{4.17}$$

by means of (4.7), (4.9), and the infinitesimal transformations of the other fields F, u, ψ, $\overline{\psi}$ vanish. These transformations are the *free field version of the famous BRS-transformations* [11], *i.e.* the terms $\sim g^0$ in $\delta^{BRS}\phi$ agree with (4.17). However, we emphasize that the BRS-transformations are transformations of *interacting* fields, whereas we transform *free* fields only. Moreover, the compensation of gauge variations is completely different: The pure Yang–Mills Lagrangian is BRS-invariant alone:

$$\delta^{BRS}(-\tfrac{1}{4}F^a_{\mu\nu\,\mathrm{int}}F^{a\,\mu\nu}_{\mathrm{int}}) = 0\,;$$

whereas we need the ghost coupling T_1^u (4.10) to cancel the gauge variation of the 3-gluon coupling T_1^A (4.2) (see (4.11)). Note that we work always in a fixed gauge, namely the Feynman gauge.

Analogously to (2.22) one can prove for the singular order the following result [7]

$$\omega = 4 - b - g - d - \tfrac{3}{2}f\,, \tag{4.18}$$

where b is the number of external gauge bosons (A, F), g the number of external ghosts (u, $\partial\tilde{u}$), d the number of derivatives on external legs (F, $\partial\tilde{u}$) and f is the number of external pairs ($\overline{\psi}$, ψ). This proves the *(re)normalizability* of this model.

(b) Four-gluon coupling and universality of charge

Gauge invariance for second order tree diagrams yields powerful restrictions [8, 9]. Due to (3.7)–(3.10) gauge invariance can be violated only in the distribution splitting and only by local terms. Therefore, we consider *local* terms only ($\sim D^a\delta(x_1 - x_2)$) in

$$[Q, T_2(x_1, x_2)] - i\sum_{l=1}^{2}\partial_\mu^{x_l}T^\mu_{2/l}(x_1, x_2)\,. \tag{4.19}$$

Let us consider the three tree terms

$$T_2(x_1,x_2) = \frac{-ig^2}{4} : A_{\mu a}(x_1)A_{\nu b}(x_1)A_{\rho d}(x_2)A_{\lambda e}(x_2)$$
$$: f_{abc}f_{dec}\{[g^{\mu\lambda}(\partial^\nu\partial^\rho D^F(x_1-x_2)$$
$$+C_a g^{\nu\rho}\delta(x_1-x_2)) - g^{\nu\lambda}(\partial^\mu\partial^\rho D^F(x_1-x_2)$$
$$+ C_a g^{\mu\rho}\delta(x_1-x_2))] - [\lambda \longleftrightarrow \rho]\} + \ldots, \quad (4.20)$$
$$T^\nu_{2/1}(x_1,x_2) = \frac{-ig^2}{2} : A_{\mu a}(x_1)u_b(x_1)A_{\rho d}(x_2)A_{\lambda e}(x_2)$$
$$: f_{abc}f_{dec}\{[g^{\mu\lambda}(\partial^\nu\partial^\rho D^F(x_1-x_2)$$
$$+C_b g^{\nu\rho}\delta(x_1-x_2)) - g^{\nu\lambda}(\partial^\mu\partial^\rho D^F(x_1-x_2)$$
$$+ C_b g^{\mu\rho}\delta(x_1-x_2))] - [\lambda \longleftrightarrow \rho]\} + \ldots \quad (4.21)$$

and the term of $T_{2/2}$ obtained from (4.21) by exchanging x_1 and x_2. (Note $T_{2/2}(x_1,x_2) = T_{2/1}(x_2,x_1)$.) These tree diagrams have singular order $\omega = 0$ and, therefore, a *free normalization term* $\sim C_{a,b}\delta(x_1-x_2)$ has been added. Now we collect all local terms in (4.19) with external legs $: A_{\mu a}(x_1)\partial_\nu u_b(x_1)A_{\rho d}(x_2)A_{\lambda e}(x_2) :$. Due to $[Q, A_{\mu a}(x)] = i\partial_\mu u_a(x)$, we have two equal terms from Q commuted with (4.20) which contribute. There is also a contribution from $i\partial_\nu^{x_1}T^\nu_{2/1}(x_1,x_2)$, generated by the divergence $\partial_\nu^{x_1}$ acting on $u_b(x_1)$ in (4.21). The considered terms vanish (*i.e.* gauge invariance is fulfilled in this case) iff

$$C_a = C_b\,. \quad (4.22)$$

Next we collect all local terms in (4.19) with operators

$$: A_{\mu a}(x_1)u_b(x_1)A_{\rho d}(x_2)A_{\lambda e}(x_2) : .$$

There is only a contribution from the divergence $\partial_\nu^{x_1}$ acting on the numerical distribution in the term (4.21) of $T^\nu_{2/1}(x_1,x_2)$. By using $\Box D^F = \delta$ we conclude that gauge invariance means

$$C_b = -\tfrac{1}{2} \quad (4.23)$$

in this sector. Hence, gauge invariance fixes the values of C_a, C_b uniquely. The C_a-normalization term (in (4.20)) is the *4-gluon interaction*. It propagates to higher orders in the inductive construction of the T_n's (see Section 4(b) of Ref. [12]).

In order to derive the *universality of charge*

$$g' = g\,, \quad (4.24)$$

where g (rsp. g') is the coupling constant in $(T_1^A + T_1^u)$ (4.2), (4.10) (rsp. in T_1^ψ (4.3)), we collect all local terms in (4.19) with external legs $:u_a(x_1)A_{\mu b}(x_1)\overline{\psi}(x_2)...\psi(x_2):$. $\partial_\nu^{x_1}$ acting on the numerical distribution in

$$T^\nu_{2/1}(x_1,x_2) = \frac{igg'}{2} :u_a(x_1)A_{\mu b}(x_1)\overline{\psi}(x_2)\gamma_\rho\lambda_c\psi(x_2): \\ \cdot f_{abc}[g^{\mu\rho}\partial^\nu D^F(x_1-x_2) - g^{\nu\rho}\partial^\mu D^F(x_1-x_2)] + ... \tag{4.25}$$

produces a non-local and a local term. The latter is

$$\partial_\nu^{x_1}T^\nu_{2/1}(x_1,x_2) = \frac{igg'}{2} :u_a(x_1)A_{\mu b}(x_1)\overline{\psi}(x_2)\gamma^\mu\lambda_c\psi(x_2) \\ : f_{abc}\delta(x_1-x_2) + \dots . \tag{4.26}$$

Let us consider the two C-conjugated Compton diagrams

$$T^\nu_{2/1}(x_1,x_2) = -\frac{ig'^2}{4}[:u_a(x_1)A_{\mu b}(x_2)\overline{\psi}(x_2)\gamma^\mu\lambda_b S^F(x_2-x_1)\lambda_a\gamma^\nu\psi(x_1): \\ + :u_a(x_1)A_{\mu b}(x_2)\overline{\psi}(x_1)\gamma^\nu\lambda_a S^F(x_1-x_2)\lambda_b\gamma^\mu\psi(x_2):] + \dots . \tag{4.27}$$

Using

$$\partial_\nu^{x_1}(\overline{\psi}(x_1)\gamma^\nu S^F(x_1-x_2)...) = -i\overline{\psi}(x_2)\delta(x_1-x_2)... , \\ \partial_\nu^{x_1}(...S^F(x_2-x_1)\gamma^\nu\psi(x_1)) = ...i\delta(x_2-x_1)\psi(x_2) , \tag{4.28}$$

we obtain two further local contributions in $\partial_\nu^{x_1}T^\nu_{2/1}(x_1,x_2)$. These three local terms cancel by means of

$$2if_{abc}\lambda_c + \lambda_b\lambda_a - \lambda_a\lambda_b = 0 \tag{4.29}$$

if and only if $g' = g$ (4.24). In QED the u-field couples to the matter field only. Therefore, the term (4.25)–(4.26) is absent. The sum (4.27) of the two C-conjugated Compton diagrams is already gauge invariant there.

(c) Outline of the proof of gauge invariance

Since the distribution splitting is done in terms of the C-number distributions (2.13a), we have to express the operator gauge invariance (3.12) by the so-called 'Cg-identities', the C-number identities for gauge invariance. They correspond to the Ward identities in QED and are obtained by collecting all terms in the operator decomposition (2.13a) of (3.12) which belong to a particular combination $:\mathcal{O}:$ of external field operators [7, 8, 13] (as we did in Subsection 4(b) before). The set of all Cg-identities implies the operator gauge invariance (3.12). But this statement cannot be reversed, because the operator decomposition of (3.12) is not unique if $\delta^{(4)}$-distributions are present [7, 8]. Therefore, we proceed in another way: *Instead of proving the operator gauge invariance (3.12), we prove the corresponding Cg-identities (by induction on n), which are a stronger statement.* In this framework the Cg-identities for R'_n, A'_n, $D_n \stackrel{\text{def}}{=} R'_n - A'_n$ can be proven by means of the Cg-identities for T_k, $\tilde{T}_k$ in lower orders $1 \leq k \leq n-1$ [8].

Again the crucial step is the distribution splitting [7, 8, 14]: The problem is much harder than in QED, because the central solution (3.1) does not exist. Similarly to (3.13)–(3.14), the Cg-identities can be violated in the splitting only and solely by *local* terms. Therefore, the possible anomaly a (*i.e.* the possible violation of the Cg-identity) has the following form

$$[:\mathcal{O}:]: \qquad \sum_{l=1}^{n} \partial_\alpha^{x_l} t_l^\alpha + t_0 \stackrel{\text{def}}{=} a = \sum_c K_c D^c \delta, \qquad K_c = \text{const.}, \quad (4.30)$$

where we have written the general form of a non-trivial Cg-identity [13] (belonging to the operator combination $:\mathcal{O}:$) on the l.h.s.. One easily finds by means of (4.18) that the terms on the l.h.s. are singular of order $|\mathcal{O}|+1$, with

$$|\mathcal{O}| \stackrel{\text{def}}{=} 4 - b - g - d - \frac{3}{2} f\,, \quad (4.31)$$

where b is the number of gluons, g the number of ghosts, d the number of derivatives in $\mathcal{O}$ and f the number of pairs $(\overline{\psi}, \psi)$ in $\mathcal{O}$. Consequently, due to (2.15), only terms with

$$|c| \leq |\mathcal{O}| + 1 \quad (4.32)$$

appear on the r.h.s. of (4.30). The terms on the l.h.s. in (4.30) have certain symmetries. For example, they are covariant, SU(N)-invariant, C-invariant, invariant with respect to permutations of certain vertices etc.. With these symmetries we restrict the constants K_c in the ansatz for the anomaly a on the r.h.s. of (4.30). Then we perform finite renormalizations (2.17)

$$t_l \to t'_l \stackrel{\text{def}}{=} t_l + n_l, \qquad t_0 \to t'_0 \stackrel{\text{def}}{=} t_0 + n_0 \quad (4.33)$$

which remove the possible anomalies a in *all* Cg-identities

$$\sum_{l=1}^{n} \partial_\alpha^{x_l} t_l'^\alpha + t'_0 = 0\,. \quad (4.34)$$

If a certain distribution t_l, t_0 appears in several Cg-identities, the different normalizations of t_l, t_0 must be compatible. It is highly non-trivial that one has enough freedom of normalization to remove *all* possible anomalies. The problem is to find enough symmetry properties for the restriction of the K_c's in (4.30). With the usual symmetries we do not succeed in all cases. For certain t-distributions we need a weak additional assumption about the infrared behavior [14]. However, this assumption seems to be true. Note that we have to consider Cg-identities with

$$|\mathcal{O}| + 1 \geq 0 \tag{4.35}$$

only, because of (4.32). There are only 12 non-trivial Cg-identities of this kind.

(d) Compatibility of the normalization conditions

At the end of Section 2 we have listed various normalization conditions. Now we sketch the proof of their compatibility [8] (apart from the existence of the adiabatic limit $g \to 1$). Similar to gauge invariance, all these symmetry properties can get lost in the distribution splitting only. We start with a covariant splitting solution r rsp. $t = r - r'$. (The existence of a covariant splitting solution is not a triviality in massless theories. It is proven by means of cohomological arguments in [7].) Then we symmetrize r rsp. t with respect to permutations of certain vertices, with respect to pseudo-unitarity (see Section 5(a)) and to the discrete transformations P, T, C, and obtain r^s rsp. $t^s = r^s - r'$. This is a symmetrization over a *finite* group G of symmetry transformations and covariance is maintained. Since $d = r' - a'$ possesses all these symmetries, r^s is still a splitting solution of d. Then we start our proof of the Cg-identities with these symmetrized distributions, *i.e.* we insert t_l^s, t_0^s for t_l, t_0 in (4.30). Then the corresponding anomaly $a^s \stackrel{\text{def}}{=} \sum_l \partial^l t_l^s + t_0^s$ must be invariant with respect to G, too. Let n_l, n_0 remove this anomaly: $\sum_l \partial^l n_l + n_0 = -a^s$. We symmetrize n_l, n_0 with respect to G and call them n_l^s, n_0^s. By means of the symmetry of a^s, we conclude $\sum_l \partial^l n_l^s + n_0^s = -a^s$. Then $t_l'^s \stackrel{\text{def}}{=} t_l^s + n_l^s$ and $t_0'^s \stackrel{\text{def}}{=} t_0^s + n_0^s$ are covariant, invariant with respect to G and fulfil the Cg-identities.

(e) Slavnov–Taylor identities

The Cg-identities contain distributions with one Q-vertex. Moreover, their coordinates refer to external and *inner* vertices. By inserting the Cg-identities in each other and integrating the inner vertices with $g(x) \equiv 1$, the Q-vertices can be eliminated and we obtain [18] the famous Slavnov–Taylor identities [15, 16, 17], which express the usual gauge invariance. The latter identities merely contain distributions of the physical theory (without

Q-vertex) and their coordinates are external vertices rsp. momenta only. However there is an exception: For the Cg-identities with an external pair $(\overline{\psi}, \psi)$, the Q-vertex cannot be eliminated completely. But in this case also *Taylor [16] was forced to introduce the Q-vertex* $T^{\psi}_{1/1}$ *(4.3)* to formulate his identities.

Note that the integration of the inner vertices with $g(x) \equiv 1$ is infrared dangerous. However, it seems [18] that no infrared divergences appear, if *all external momenta are off-shell.* (If this would be wrong, the distributions in the (usual) Slavnov–Taylor identities would be infrared divergent.)

5. Unitarity on the physical subspace

One important application of gauge invariance (3.12) is unitarity on the physical subspace (chapter 5 of [14]). The following proof holds true for the Yang–Mills theory of Section 4 and for QED (Section 3). However, in the case of QED, we have to modify the definition of gauge invariance slightly, because we will need $Q^2 = 0$ (4.14). The latter is wrong for Q defined by (3.4), if u is an external C-number field. Therefore, similarly to the non-abelian case, we let u be a free, scalar Fermi field. Due to the spin and statistics theorem of Pauli [19], this requires the presence of a second free, scalar Fermi field $\tilde{u}$, and u, $\tilde{u}$ fulfil (4.8a). However, $\tilde{u}$ does not interact at all and u is only coupled to the matter current by $T^{\nu}_{1/1}$. In this way we get a formulation of gauge invariance (3.12) in QED with $Q^2 = 0$.

(a) Pseudo-unitarity on the whole Fock space

We work in a positive definite Fock-Hilbert space $\mathcal{F}$. We quantize the gauge field A^{μ}_a as four independent scalar fields [4] for each (fixed) colour a. The spatial components are hermitian: $A^{j\,+} = A^j$, $j = 1, 2, 3$. In order to have a covariant commutator $[A^{\mu}, A^{\nu}] \sim g^{\mu\nu}$, A^0 must be skew-hermitian

$$A^{0\,+} = -A^0 \,. \tag{5.1}$$

We define another conjugation 'K', by giving its action on the free field operators

$$\begin{aligned} &A^{0\,K} \stackrel{\text{def}}{=} A^0, \quad A^{j\,K} \stackrel{\text{def}}{=} A^{j\,+} = A^j, \quad u^K \stackrel{\text{def}}{=} u\,, \\ &\tilde{u}^K \stackrel{\text{def}}{=} -\tilde{u}, \quad \psi^K \stackrel{\text{def}}{=} \psi^+, \quad \overline{\psi}^K \stackrel{\text{def}}{=} \overline{\psi}^+ \,. \end{aligned} \tag{5.2}$$

By inserting the explicit expressions (2.2) rsp. (4.2), (4.3),(4.10) and remembering (2.5) one easily verifies

$$T_1(x)^K = -T_1(x) = \tilde{T}_1(x)\,. \tag{5.3}$$

This is pseudo-unitarity $S(g)^K = S(g)^{-1}$ in first order. In the inductive construction of the T_n's, this property can get lost in the distribution splitting only. By a symmetrization of an arbitrary splitting solution one obtains a pseudo-unitary splitting solution [1, 4]. In other words: If one restricts the normalization constants C_a in (2.17) in a suitable way, pseudo-unitary goes over from (5.3) to higher orders

$$T_n(x)^K = \tilde{T}_n(x)\,. \tag{5.3a}$$

(b) The physical subspace $\mathcal{F}_{\text{phys}}$

On the physical fields which are ψ, $\overline{\psi}$ and the transversal gauge bosons $A_a^{\perp}$, the conjugation 'K' (5.2) agrees with the adjoint. The unphysical fields are: u_a, $\tilde{u}_a$, A_a^0 and the longitudinal gauge bosons $A_a^{\parallel}$. Let N be the particle number operator of the unphysical particles. Then we define the *physical subspace* to be the kernel of N, *i.e.* the space without unphysical particles

$$\mathcal{F}_{\text{phys}} \stackrel{\text{def}}{=} \ker N, \tag{5.4}$$

and we denote the orthogonal projector on $\mathcal{F}_{\text{phys}}$ (with respect to the positive definite scalar product) by P_{phys}, *i.e.* $\mathcal{F}_{\text{phys}} = P_{\text{phys}}\mathcal{F}$.

Now our gauge charge Q (3.4), (4.6) enters the game. The orthogonal projector on the kernel of Q is called P

$$P\mathcal{F} \stackrel{\text{def}}{=} \ker Q\,. \tag{5.5}$$

Due to $Q^2 = 0$ (4.14) we know $\text{ran}\, Q \subset \ker Q$, where $\text{ran}\, Q$ is the range of Q. More precisely the situation is the following [14]

$$\ker Q = \mathcal{F}_{\text{phys}} \oplus \text{ran}\, Q, \qquad \mathcal{F}_{\text{phys}} \perp \text{ran}\, Q\,, \tag{5.6}$$

where the orthogonality is always with respect to the positive definite scalar product.

(c) Unitarity on the physical subspace

By the expression 'unitarity on the physical subspace' we mean the heuristical equation

$$\lim_{g\to 1} P_{\text{phys}} S(g)^+ P_{\text{phys}} S(g) P_{\text{phys}} = P_{\text{phys}}\,. \tag{5.7}$$

We will sketch the proof [14] of the following perturbative version of (5.7)

$$\tilde{T}_n^P(X) = P_{\text{phys}} T_n(X)^+ P_{\text{phys}} + \text{div}\,, \tag{5.8}$$

where 'div' is a sum of divergences (similar to the r.h.s. of the definition (3.12) of gauge invariance) and $\tilde{T}_n^P$ is the n-point distribution of the S-matrix inverted on $\mathcal{F}_{\text{phys}}$

$$(P_{\text{phys}} S(g) P_{\text{phys}})^{-1} = P_{\text{phys}} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 ... d^4x_n\, \tilde{T}_n^P(x_1, ..., x_n) g(x_1) ... g(x_n)\,. \tag{5.9}$$

In order to prove (5.8) we start with

$$\tilde{T}_n^P(X) = \sum_{r=1}^{n} (-1)^r \times \sum_{P_r} P_{\text{phys}} T_{n_1}(X_1) P_{\text{phys}} T_{n_2}(X_2) P_{\text{phys}} ... P_{\text{phys}} T_{n_r}(X_r) P_{\text{phys}}\,, \tag{5.10}$$

which is the usual inversion of a power series (see (2.5)–(2.6)). We want to get rid of the inner projectors P_{phys}, *i.e.* the ones which are sandwiched between two $T_{n_l}(X_l)$. By means of *gauge invariance* $[Q, T_n] = \text{div}$, one can prove

$$P_{\text{phys}} T_n(X) P_{\text{phys}} = P_{\text{phys}} T_n(X) P + \text{div}_1 \tag{5.11}$$

and

$$P T_n(X) P = T_n(X) P + \text{div}_2\,. \tag{5.12}$$

With (5.11) we first replace in (5.10) the inner P_{phys} by P's (5.5), and then eliminate these inner P's by using (5.12). In both steps we get divergence terms. It results

$$\begin{aligned} \tilde{T}_n^P(X) &= \sum_{r=1}^{n} (-1)^r \sum_{P_r} P_{\text{phys}} T_{n_1}(X_1) T_{n_2}(X_2) ... T_{n_r}(X_r) P_{\text{phys}} + \text{div}_3 \\ &= P_{\text{phys}} \tilde{T}_n(X) P_{\text{phys}} + \text{div}_3 = P_{\text{phys}} T_n(X)^K P_{\text{phys}} + \text{div}_3 \\ &= P_{\text{phys}} T_n(X)^+ P_{\text{phys}} + \text{div}_3\,, \end{aligned} \tag{5.13}$$

where we have inserted (2.5), *pseudo-unitarity* (5.3a) and used the fact that the conjugation 'K' agrees with the adjoint on $\mathcal{F}_{\text{phys}}$.

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