RENORMALIZATION OF GAUGE FIELD THEORIES*

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There is, at the present time, a renewal of interest (M. Pawlowski, R. Raczka, Warsaw preprint SINS-IP/VIII/1995) in using massive non-abelian vector boson theories without the unobserved Higgs particle. Such theories violate renormalizability by power counting. They are nevertheless thought to be renormalizable. A review of this renormalizability problem is made here in the light of gauge invariance.

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1. Power counting

Let us briefly summarize the power counting technique. In a Feynman graph integral the superficial degree of divergence is defined as the number of momenta in the numerator minus the number of momenta in the denominator. If the vector boson propagator is of the form

$$D_{\mu\nu}(k) = -\frac{i}{k^2 - m^2 + i\epsilon}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right],$$

the superficial degree of divergence is given by

$$\omega = 4 - b_E - \frac{1}{2}f_E - \sum_i n_i[g_{(i)}] + 2\,b_{I,V},$$

where b_E, f_E are respectively the number of external bosonic and fermionic lines, $b_{I,V}$ is the number of internal vector boson lines and $[g_{(i)}]$ is the mass dimension of the various coupling constants occuring n_i times in the graph.

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In the case of dimensionless coupling constants, which will be the case considered here, ω increases with the number of vector boson internal lines and the theory is not renormalizable by power counting. This does not mean that the theory is not renormalizable at all. Indeed,the Dyson theorem [2] establishing the convergence of a graph when this graph and all its subgraphs are superficially convergent involves the superficial convergence only as a sufficient condition. It may happen that the unwanted divergences in a graph cancel out.

2. Gauge theories

Gauge theories offer the possibility to change the propagator owing to the freedom of the gauge choice. One must however be careful with the gauge choice because this notion has not the same meaning for a mathematician or a physicist. Let us therefore review the various notions of gauge fixing used by physicists.

2.1. The gauge choice in a simple example

In order to be as clear as possible, let us consider the simplest gauge theory in the world [3]. It is given by the Lagrangian

$$L = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}(\dot{y} - z)^2$$

which is invariant under time-dependent translations along the y-axis

$$y \longrightarrow y + b(t), \qquad z \longrightarrow z + \dot{b}(t)\,.$$

According to Dirac theory of constraints [4], there are two constraints that we write

$$p_z = 0 \quad \stackrel{\cdot}{\Longrightarrow} \quad p_y = 0 \qquad (p_y = \dot{y} - z)$$

in order to recall that the secondary constraint results from the primary one in the Hamiltonian formalism.

These constraints are first class. This means that they involve different degrees of freedom and therefore that the two pairs of variables (y, p_y) and (z, p_z) are unphysical. One can classify the gauge conditions according to the number of unphysical degrees of freedom they involve.

In class I gauges, only physical degrees of freedom are involved. The simplest but not the unique gauge condition of this type is $y = 0$. By time derivation, it implies $z = 0$.

$$y = 0 \quad \stackrel{\cdot}{\Longrightarrow} \quad z = 0\,.$$

For mathematicians, it is the unique way of fixing a gauge. For people working in the theory of constrained systems, this is known as the reduced phase space technic. Elementary particle physicists call it unitary gauge. One should emphasize that, in order to be of class I, a gauge condition must have Poisson brackets with the secondary constraint which are non vanishing. Morever, they should not involve the variables in order not to generate Gribov ambiguities. In our example, the class I gauge condition we take leads to the effective Lagrangian

$$L_{\text{eff}} = \tfrac{1}{2}\dot{x}^2$$

which shows that the theory describes the free motion of a unit mass particle along a line.

A class II gauge works with the y degree of freedom in addition to the physical one. It can be realized through the gauge condition $z = 0$ and the effective Lagrangian in this gauge is

$$L_{\text{eff}} = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\dot{y}^2 \, .$$

In order to recover the physical theory, the secondary constraint is, in quantum theory, imposed on states

$$p_y|\Psi_{\text{phys}} >= 0 \, .$$

If such a condition works for bound state problems, it does not garantee that unphysical states do not contribute as intermediate states in perturbative expansion.

Instead of eliminating any unphysical degree of freedom, a class III gauge condition gives a time evolution to unphysical degrees of freedom. It can be realized here as $\dot{z} = 0$. The presence of two unphysical degrees of freedom can be an advantage because their contributions to intermediate states can mutually cancel. Such a cancellation involves an indefinite metric and is garanted by the BRST formalism [5].

2.2. Choosing a gauge in Maxwell and Yang–Mills theories

The typical class I gauge in Maxwell theory is the Coulomb gauge

$$\partial_k A_k = 0 \, .$$

If it has the advantage of involving only physical degrees of freedom, it is not useful in perturbative field theory because the theory is non-local. Moreover, the lack of explicit covariance leads to complications in the calculations.

In Yang–Mills theory, the problem is even more crucial. Gribov ambiguities [6] occur. It can even be shown (Singer theorem [7] that if one compactifies the Minkowski space, a global section cannot be found. This means that no class I gauge condition can be found. These problems can be formulated in a simple way by noting that the Poisson brackets

$$\{\partial_k A_k(x), D^l \Pi^l(y)\}_{x_0=y_0} = \partial_k D^k \delta^{(3)}(\vec{x} - \vec{y})$$

are field dependent and that the operator $\partial_k D^k$ is not invertible.

One can also note that, if one tries to write the Coulomb gauge condition in a formally covariant form with the help of a fixed four-vector n, it reads

$$n \cdot \partial\, n \cdot A - \partial \cdot A = 0\,.$$

In any frame different from those where $n = (1, 0, 0, 0)$, it is not class I but class III. One can say that the frame in which the Coulomb gauge condition is $\partial_k A_k = 0$ is singular [8]. In this case, quantization in a singular frame leads to various troubles and should be avoided.

A typical class II gauge is the temporal gauge

$$A_0 = 0\,.$$

If it is useful in the search of classical solutions, though not explicitly covariant, in perturbative expansions, it is hard to find a propagator which is such that the unphysical longitudinal photon or gluon does not propagate. This problem is however outside the topics covered by this lecture.

The best representatives of class III gauges in Maxwell or Yang–Mills theories are the relativistic gauges

$$\partial_\mu A^\mu = aS\,.$$

They are manifestly covariant but involve unphysical degrees of freedom. An indefinite metric is fortunately present and it is well know since a long time how the cancellation occur. This is the Gupta-Bleuler formalism [9] in QED. It has been generalized to theories by the BRST formalism.

2.3. The BRST formalism

The BRST formalism is now a classical stuff in textbooks on QCD. Let us therefore only summarize the main steps. It rests on the existence of the BRST symmetry [10] which is a global symmetry associated with gauge invariance and which involves nilpotent transformations. Starting from the gauge transformations in Yang–Mills theory (internal symmetry indices are understood)

$$\delta A_\mu = D_\mu \theta(x)\,,$$

one sets

$$\theta(x) = c(x)\,\delta\lambda\,,$$

where c, $\delta\lambda$ are objects of odd Grassmann parity. $c(x)$ is called the ghost field [15] while δ does not depend on the space-time point. It is a global parameter. Imposing the nilpotency $\delta(D_\mu c) = 0$ leads to the transformation law of the ghost fields

$$\delta c = \tfrac{1}{2} c \times c\,\delta\lambda\,.$$

The **class III** gauge condition is introduced in the Lagrangian through a Lagrange multiplier field $S(x)$ which is invariant under the BRST transformation $\delta S = 0$. The Lagrangian becomes

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \partial_\mu S A^\mu - \tfrac{1}{2} a\, S^2\,.$$

It is not invariant under BRST transformations

$$\delta\mathcal{L} = -\partial_\mu S\, D^\mu c\,\delta\lambda\,.$$

In order to restore the invariance, one adds the ghost term

$$\mathcal{L}_{gh} = \partial_\mu \bar{c}\, D^\mu c$$

with the antighost field $\bar{c}(x)$ submitted to the transformation

$$\delta\bar{c} = -S\,\delta\lambda\,.$$

The total Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \partial_\mu S A^\mu - \tfrac{1}{2} a\, S^2 + \partial_\mu \bar{c}\, D^\mu c$$

is BRST invariant. Through Noether theorem, a conserved nilpotent charge, the BRST charge Q_B is associated with this invariance. The physical states are the cohomology classes of this operator *i.e.*

$$\begin{aligned} &Q_B|\Psi_{\rm phys}> = 0\,, \\ &\quad |\Psi'_{\rm phys}> \equiv |\Psi_{\rm phys}> + Q_B|\Psi_0>\ . \end{aligned}$$

In perturbation theory, BRST invariance implies Ward–Takahashi –Slavnov–Taylor identities [11]. They assure unitarity (only physical states contribute) and renormalizability of the theory.

3. Massive vector boson theories

3.1. Abelian vector boson

Let us now try to apply these ideas on gauge theories to massive vector bosons. The simplest case is of course the abelian vector boson. It is usually described by the Proca Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \tfrac{1}{2}m^2 A_\mu A^\mu$$

for which the propagator is

$$D_{\mu\nu}(k) = -\frac{i}{k^2 - m^2 + i\epsilon}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right] .$$

It is not renormalizable by power counting. It is however known since more than twenty years that the theory is nevertheless renormalizable when it is coupled to a conserved current. It is possible to formulate this fact by using a gauge theory.

It is usually claimed in almost all textbooks that the presence of a mass term breaks gauge invariance. This allegation is wrong because it does not take into account all the features of a gauge theory. One has seen that a gauge theory involves two unphysical degrees of freedom, those constrained by the first class constraints. A massive vector boson has three physical polarizations. Adding the two gauge degrees of freedom implies that a gauge theory of the massive vector boson involves five fields which can be gathered into a four-vector and a scalar. It is then an easy task to build up a gauge invariant Lagrangian [12]. Indeed,

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}(mA_\mu - \partial_\mu K)^2$$

is invariant under

$$A_\mu \longrightarrow A_\mu + \partial_\mu \theta, \quad K \longrightarrow K + m\theta .$$

The constraint analysis of this theory leads to

$$\Pi_0 = 0 \quad \stackrel{\cdot}{\Longrightarrow} \quad \partial^k \Pi^k + m\,\Pi = 0 ,$$

where Π^μ and Π are the canonical momenta associated respectively with A_μ and K. These two constraints are first class as expected in a gauge theory.

In order to quantize the theory, one chooses a gauge. Here a class I relativistic gauge can be found. Indeed, if one chooses as gauge condition $K = 0$, it implies by time derivation $\Pi + m\,A_0 = 0$ and if $m \neq 0$, K, A_0

and their canonically conjugate momenta can be eliminated. The effective Lagrangian resulting from this gauge choice is the Proca Lagrangian which is, of course, no longer gauge invariant because it is the Lagrangian in a given gauge.

Like in the massless case, let us now try to find a class III gauge in which the propagator does not exhibit the bad ultra-violet behaviour. This job is not new because it was already done by Stueckelberg [13]. Many gauge choices lead to the Stueckelberg formalism. The simplest one is

$$\partial_\mu A^\mu + amK = aS\,.$$

It is a class III gauge condition because it gives an evolution to A_0. Such a gauge condition is realized with the Lagrangian

$$\mathcal{L} = \mathcal{L}_{inv} + S(\partial_\mu A^\mu + amK) - \tfrac{1}{2}aS^2\,,$$

where S is a Lagrange multiplier.

After elimination of the S field with the help of its field equation which is nothing other than the gauge condition, the field equations of this gauge are

$$\partial^\mu F_{\mu\nu} + m^2 A_\nu + \tfrac{1}{a}\partial_\mu\partial_\nu A^\mu = j_\nu\,,$$

$$(\partial_\mu\partial^\mu + am^2)K = 0\,.$$

Interactions with a matter field are introduced through the current j_ν which is conserved in virtue of gauge invariance. One notes that, even in the presence of interactions, the fifth field K is a free field which is therefore only a spectator in the game.

The propagator in this gauge is

$$D_{\mu\nu}(k) = -\frac{i}{k^2 - m^2 + i\epsilon}\left[g_{\mu\nu} + \frac{(a-1)k_\mu k_\nu}{k^2 - am^2 + i\epsilon}\right]\,.$$

Because the gauge condition is linear, the vertices of the theory are unchanged and this gauge is renormalizable. The choice $a = 1$, the Feynman gauge, eliminate the $k_\mu k_\nu$ term, a fact leading to a further simplification.

It remains to prove that the theory is unitary in this gauge. For, it is sufficient to find a nilpotent conserved BRST operator. Like in the massless case, the BRST transform

$$\delta A_\mu = \partial_\mu c\,\delta\lambda, \quad \delta K = m\,c\,\delta\lambda \quad \delta c = 0, \quad \delta S = 0, \quad \delta\bar{c} = -S\,\delta\lambda$$

which are obviously nilpotent lead to uncoupled Faddeev–Popov ghosts [15]. The Lagrangian

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}(mA_\mu-\partial_\mu K)^2-\partial_\mu S A^\mu+amSK-\tfrac{1}{2}aS^2+\partial_\mu\bar{c}\partial^\mu c-am^2\bar{c}\,c$$

is indeed invariant under these BRST transformations.

Though non-renormalizable by power counting, the Proca Lagrangian is renormalizable when coupled to a conserved current because it is gauge equivalent to a renormalizable description, the Stueckelberg formalism.

3.2. Abelian Higgs mechanism

It can be shown that when the mass is introduced with the help of the Higgs mechanism [14], the conclusion is similar. Indeed, the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}D_\mu\phi_\alpha D^\mu\phi^\alpha - \tfrac{a}{2}\phi_\alpha\phi^\alpha - \tfrac{\lambda}{4}(\phi_\alpha\phi^\alpha)^2\,,$$

where $a < 0$ and $D^\mu\phi^\alpha = \partial^\mu\phi + eA^\mu\epsilon^{\alpha\beta}\phi_\beta$ is invariant under

$$\phi_\alpha \longrightarrow \phi_\alpha - e\,\omega(x)\epsilon_{\alpha\beta}\phi^\beta\,,$$
$$A_\mu \longrightarrow A_\mu + \partial_\mu\omega\,.$$

Setting

$$\phi_1 = \rho\cos\theta, \quad \phi_2 = \rho\sin\theta$$

the Lagrangian can be rewritten as

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\partial_\mu\rho\partial^\mu\rho + \tfrac{1}{2}\rho^2(\partial_\mu\theta - eA_\mu))^2 - \tfrac{a}{2}\rho^2 - \tfrac{\lambda}{4}\rho^4\,.$$

The constraints are

$$\Pi^0 = 0 \quad \Longrightarrow \quad \partial^k\Pi^k + e\Pi_\theta = 0\,.$$

Again the gauge choice $\theta = 0$ is class I. It leads to the unitary gauge where only physical degrees of freedom are involved. The mass of the vector boson is generated through the non-vanishing of the minimum of the ρ-field. This gauge is again not renormalizable by power counting.

Again, the class III gauge condition $\partial_\mu A^\mu = aS$ leads to a renormalizable gauge which is also unitary if uncoupled Faddeev-Popov are introduced.

It can be noted that the limit of infinite Higgs mass leads to the previous theory.

3.3. Non-abelian Higgs mechanism

It is well known that the above argumentation can be extended to the non-abelian case in presence of the Higgs mechnism. Because of lack of space, the calcultaions will not be elaborated here.

3.4. Non-abelian massive vector boson without Higgs

Let us now try to extend the argumentation developped here above to the non-abelian theory. In order not to have to take care with the position of internal symmetry indices, one assumes that the global gauge group is compact. The first point is to build up a gauge invariant Lagrangian [16]. For, the mass term $1/2m^2(A^a_\mu)^2$ must be made invariant under

$$A^a_\mu \longrightarrow A^a_\mu + (D_\mu\theta)^a$$

with the help of an additional scalar field K^a. The most general transformation on the K^a field can be written

$$K_a(x) \longrightarrow K_a(x) + L_{ab}\theta^b .$$

On the other hand, four possible non-abelian generalizations of the combination $m\,A_\mu - \partial_\mu K$ are *a priori* possible. One can check that the combinations $m\,A^a_\mu - \partial_\mu K^a$, $m\,A^a_\mu - (D_\mu K)^a$ and $m\,A^a_\mu - N^{ab}(x)(D_\mu K)_b$ do not work. Only

$$\tfrac{1}{2}[m\,A^a_\mu - N^{ab}(x)\partial_\mu K_b]^2$$

can be made invariant. This invariance leads to conditions on the unknow functions of the K^a fields, N_{ab} and K^{ab}

$$N^{ab}L_{bc} = m\delta^a_c \,,$$

$$\partial^b N^a_d L_{bc} + N^a_b \partial^d L_{bc} = g\, f^a_{\ bc} N^b_d \,,$$

where

$$\partial^a = \frac{\partial}{\partial K_a} .$$

A solution of these equations can be obtained in a perturbative way. It reads

$$N^a_b = \left[E\left(\frac{g}{m}K^c T_c\right)\right]^a_b ,$$

where

$$E(\xi) = \sum_{n=0}^{\infty} \frac{(i\xi)^n}{(n+1)!} .$$

The constraint analysis of this Lagrangian leads to the chain

$$\Pi^0_a = 0 \quad \Longrightarrow \quad (D^k\Pi^k)_a + \Pi^b L_{ba} = 0 \,,$$

where Π^μ_a and Π_b are the momenta canonically conjugate to A^a_μ and K^b respectively.

The class I gauge condition

$$K_a = 0 \quad \Longrightarrow \quad m^2 A_a^0 + \Pi_b L_a^b = 0$$

is such that

$$\{K_a(x), (D^k\Pi^k)_b(y) + \Pi_c(y)L_{cb}(y)\}_{ET} = L_{ab}\delta(\vec{x} - \vec{y}) \approx m\delta_{ab}\delta(\vec{x} - \vec{y})\,.$$

so that, if $m \neq 0$, it is a good class I gauge for which the effective Lagrangian is

$$\mathcal{L}_{\text{eff}} = -\tfrac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + \tfrac{1}{2}m^2 A_\mu^a A_a^\mu\,.$$

Il leads to the vector boson propagator

$$D_{\mu\nu}^{ab}(k) = -\frac{i\delta^{ab}}{k^2 - m^2 + i\epsilon}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right]$$

which is non-renormalizable by power counting.

To the question "Is it possible to find a class III gauge which is renormalizable by power counting ?", the answer is here dramatically "No" because the K field is, when $K \neq 0$, present in N and N introduces non-renormalizable couplings.

Any attempt to generalize the Stueckelberg formalism without non-renormalizable couplings leads to a formalism which cannot be proved to be unitary [16]. Either no BRST symmetry can be found in the physical sector or a BRST symmetry can be found, it is not nilpotent. This fact is obvious because any class III gauge will always involve the K field. Unfortunately it does not decouple here.

As a conclusion, it is impossible to find a formulation of non-abelian vector boson theories without Higgs boson which is renormalizable by power counting. Let us emphasize again that this does not prove that such theories are not renormalizable at all. Because the perturbative expansion is a linearization of a theory which is governed by an evolution operator which is hyperbolic either with or without a mass term, one does not see why a mass term would dramatically break the formalism. Therefore massive vector theories without Higgs are expected to be renormalizable [18] but not by power counting.

An other argument against their perturbative use is that they are not unitary order by order in the perturbative expansion [17]. Again this is not a prove that they are not unitary at all and, in fact, they are.

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