CAUSAL PERTURBATION THEORY FOR MASSIVE VECTOR BOSON THEORIES*

F. KRAHE †

Institut für Theoretische Physik der Universität Zürich,
Winterthurerstrasse 190, CH-8057 Zürich

(Received July 3, 1996)

In these lectures we apply the method of causal perturbation theory to Yang–Mills theories with massive vector bosons. We show how the differential property of the BRS-charge leads to the introduction of scalar gauge fields. The general relationship between gauge invariance and unitarity is pointed out in detail by using Krein space techniques.

PACS numbers: 11.15. Bt

1. Introduction

Recently, massless Yang–Mills theory has been successfully studied in the framework of causal perturbation theory [1–4]. This work has been rewied by M. Duetsch at this conference. Here we study the application of causal peturbation theory to massive Yang– Mills theories. A very detailed presentation of the material covered in these lecture notes is given in [26].

Let us first briefly summarize some essential points in the construction of massless Yang–Mills theories. The central object in causal perturbation theory is the causal S-matrix

$$S[g] = 1 + \sum_{n=1}^{\infty} \int d^4x_1 \cdots d^4x_n T^{(n)}(x_1, \cdots, x_n) , \tag{1.1}$$

$T^{(1)}$ specifies the theory. For massless Yang–Mills theories it is given by

$$T^{(1)}(x) \overset{\text{def}}{=} -ief_{abc}\{\tfrac{1}{2} : A_{\mu a}A_{\nu b}F_c^{\mu\nu} : - : A_{\mu a}u_b\partial^\mu\tilde{u}_c :\}(x) , \tag{1.2}$$

---

* Presented at the II German–Polish Symposium "New Ideas in the Theory of Fundamental Interactions", Zakopane, Poland, September 1995.

† Partially supported by the Stiftung Für Deutsch–Polnische Zusammenarbeit Grant No. 984/94/LN.

e is the coupling constant and f_{abc} are the structure constants of a non-abelian semi-simple compact gauge group G. A_a^μ are the *free* gauge fields, defined by

$$\partial \cdot \partial A_a^\mu(x) = 0, \quad [A_a^\mu(x), A_b^\nu(y)]_- = i\delta_{ab} g^{\mu\nu} D_0(x-y), \tag{1.3}$$

where D_0 is the Pauli–Jordan commutation function for $m = 0$. $F_a^{\mu\nu}$ are the *free* field strenghts:

$$F_a^{\mu\nu} \stackrel{\text{def}}{=} \partial^\mu A_a^\nu - \partial^\nu A_a^\mu, \tag{1.4}$$

and u_a and $\tilde{u}_a$ are the *free* ghost fields:

$$\partial \cdot \partial u_a(x) = \partial \cdot \partial \tilde{u}_a(x) = 0, \quad \{u_a(x), \tilde{u}_b(y)\}_+ = -i\delta_{ab} D_0(x-y), \tag{1.5}$$

$$\{u_a(x), u_b(y)\}_+ = \{\tilde{u}_a(x), \tilde{u}_b(y)\}_+ = 0. \tag{1.6}$$

A detailed discussion of the algebraic properties of these ghost fields can be found in [5].

Differentiating (1.3) we get

$$[\partial_\mu A_a^\mu(x), \partial_\nu A_b^\nu(y)]_- = 0, \tag{1.7}$$

$$[\partial_\mu A_a^\mu(x), F_b^{\kappa\lambda}(y)]_- = 0. \tag{1.8}$$

Despite their simplicity, these equations have important consequences. For, let us consider the operator

$$Q \stackrel{\text{def}}{=} \int_{x_0=\text{const.}} d^3\vec{x}\, (\partial_\mu A_a^\mu(x)) \overleftrightarrow{\partial_0}\, u_a(x). \tag{1.9}$$

Using the Leibnitz rule for graded algebras gives

$$\begin{aligned} Q^2 = \frac{1}{2}\{Q, Q\}_+ = \frac{1}{2} \int_{x_0=\text{const.}} d^3\vec{x} \int_{y_0=\text{const.}} d^3\vec{y} \\ \times \Big\{ [\partial_\mu A_a^\mu(x), \partial_\nu A_b^\nu(y)]_- \overleftrightarrow{\partial}_{x^0} \overleftrightarrow{\partial}_{y^0} (u_a(x) u_b(y)) \\ + (\partial_\nu A_b^\nu(y) \partial_\mu A_a^\mu(x)) \overleftrightarrow{\partial}_{x^0} \overleftrightarrow{\partial}_{y^0} \{u_a(x), u_b(y)\}_+ \Big\} = 0. \end{aligned} \tag{1.10}$$

Thus Eqs. (1.6), (1.7) make Q a differential operator in the sense of homological algebra. This allows for standard homological notions [6,7]: Let $\mathcal{F} = \{F\}$ be the field algebra consisting of the polynomials in the (smeared) gauge and ghost fields and their Wick powers. Consider the ghost charge operator [5]

$$Q_g \stackrel{\text{def}}{=} \int\limits_{x_0=\text{const.}} d^3\vec{x} : \tilde{u}_a(x) \overleftrightarrow{\partial_0} u_a(x) : \tag{1.11}$$

and the corresponding derivation δ_g in $\mathcal{F}$

$$\delta_g F \stackrel{\text{def}}{=} [Q_g, F]_- . \tag{1.12}$$

We say an operator F has ghost charge z if

$$\delta_g F = zF . \tag{1.13}$$

Since Q_g has integer spectrum [5] we have $z \in \mathcal{Z}$. The operators F_z with ghost charge z form the subspace $\mathcal{F}_z$, and we obviously have

$$\mathcal{F} = \bigoplus_{z\in\mathcal{Z}} \mathcal{F}_z \tag{1.14}$$

which makes $\mathcal{F}$ a $\mathcal{Z}$-graded algebra. Consider the unitary operator [5]

$$E \stackrel{\text{def}}{=} (-1)^{Q_g}, \quad E^2 = 1 . \tag{1.15}$$

It induces the canonical involution ω in $\mathcal{F}$ by

$$\omega F \stackrel{\text{def}}{=} EFE, \quad \omega^2 = 1 . \tag{1.16}$$

We define the bosonic part F_b and the fermionic part F_f of an operator F by

$$F_{b(f)} \stackrel{\text{def}}{=} \tfrac{1}{2}(1 \underset{(-)}{+} \omega)F, \quad \Rightarrow \omega F_{b(f)} = \underset{(-)}{+} F_{b(f)} \tag{1.17}$$

and the graded bracket of two operators F and G by

$$[F,G] = [F_b+F_f, G_b+G_f] \stackrel{\text{def}}{=} [F_b, G_b]_- + [F_b, G_f]_- + [F_f, G_b]_- + \{F_f, G_f\}_+ . \tag{1.18}$$

We also define on $\mathcal{F}$ the operator d_Q by

$$d_Q F \stackrel{\text{def}}{=} [Q, F] = QF - (\omega F)Q . \tag{1.19}$$

This is a differential operator:

$$d_Q^2 = 0, \iff \{Q, [Q, F_b]_-\}_+ = [Q, \{Q, F_f\}_+]_- = 0 \tag{1.20}$$

and an antiderivation with respect to ω:

$$d_Q(FG) = (d_Q F)G + (\omega F)(d_Q G)\,. \tag{1.21}$$

The commutator relation

$$[Q_g, Q] = -Q \tag{1.22}$$

implies

$$[\delta_g, d_Q]_- = -d_Q\,, \quad \Rightarrow d_Q \mathcal{F}_z \subseteq \mathcal{F}_{z-1} \tag{1.23}$$

i.e. d_Q is a homogeneous homomorphism of degree (-1) over $\mathcal{F}$. This implies in particular that it anticommutes with the canonical involution:

$$\{d_Q, \omega\}_+ = 0\,. \tag{1.24}$$

We conclude that the quadruplet $\{\mathcal{F}, \delta_g, \omega, d_Q\}$ fits well into the definition of a graded differential algebra [6].

Let us study the action of d_Q on $\mathcal{F}$ more explicitly. We find

$$d_Q A_a^\mu(x) = i\partial^\mu u_a(x)\,, \tag{1.25}$$

$$d_Q u_a(x) = 0, \quad d_Q \tilde{u}_a(x) = -i\partial_\mu A_a^\mu(x)\,. \tag{1.26}$$

Eqs. (1.7), (1.8) immediately give two gauge invariants:

$$d_Q \partial_\mu A_a^\mu(x) = d_Q F_a^{\mu\nu}(x) = 0\,. \tag{1.27}$$

The above actions of d_Q on F may be called *free* or asymptotic BRS variations since the (formally defined) full BRS variations of interacting fields [7] reduce to them in the absence of interaction. It is exactly these free varations we are intersted in when-applying causal perturbation theory, since there we are looking for symmetries of the S-matrix which is defined in the Hilbert–Fock space H of free asymptotic fields. The algebra and homology of free BRS operators is well studied in [7, 8]. The variations induced by d_Q are also called operator valued gauge transformations in [1] since they emerge from the usual asymptotic gauge variations in QED [12] by replacing the gauge function χ with the ghost operator u. We will often simply call this asymptotic BRS variations gauge variations and their invariants gauge invariants.

The interaction (1.2) is gauge invariant, *i.e.* we have

$$d_Q T^{(1)}(x) = \partial_\mu T^{(1)\mu}(x),\ T^{(1)\mu}(x) \stackrel{\text{def}}{=} e f_{abc} : u_a\{A_{\nu a} F_c^{\mu\nu} + \tfrac{1}{2} u_b \partial^\mu \tilde{u}_c\} : (x). \tag{1.28}$$

The quintessence of causal perturbation theory is that all higher terms $T^{(n)}$, $n \geq 2$ in (1.1) are determined from $T^{(1)}$ by Poincaré invariance and causality [9–12]. This determination is unique up to some (finite!) normalization constants, which can be determined by the requirement of symmetries and (finitely many) normalization conditions. $T^{(n)}$ is given symbolically by

$$T^{(n)}(x_1, \cdots, x_n) = \Theta[T^{(1)}(x_1) \cdots T^{(n)}(x_n)], \tag{1.29}$$

where Θ means the time ordered product. This, however, *cannot* be constructed by multiplying with step functions, since this would lead to the well known UV-divergences[9, 10]. Instead one has to use the method of distribution splitting, developed by Epstein and Glaser [11] and applied to QED, for example, by Scharf [12]. Using exactly this construction Duetsch *et al.* [1–4] have shown that the Yang–Mills theory specified by (1.2) is gauge invariant in all orders, *i.e.* the following equations hold true:

$$d_Q T^{(n)}(x_1, \cdots, x_n)] = \sum_{l=1}^{n} \partial_{\mu_l} T^{(n)\mu_l}_{l}(x_1, \cdots, x_n), \tag{1.30}$$

$$T^{(n)\mu_r}_{r}(x_1, \cdots, x_n) \stackrel{\text{def}}{=} \Theta[T^{(1)}(x_1) \cdots T^{(1)\mu_r}(x_r) \cdots T^{(1)}(x_n)]. \tag{1.31}$$

Thus the gauge variation of the $T^{(n)}$ are total divergences. One would like to conclude from this the more convential form of gauge invariance:

$$\lim_{g \to 1} d_Q S[g] = 0. \tag{1.32}$$

While this adabatic limit is well controlled in massive theories [13] the situation is far more difficult in massless theories, where it generally fails to exist in the S-matrix elements [12, 14]. The strength of Eq. (1.30) is to give a formulation of gauge invariance which is completely independent of the infrared problems encountered when passing to the adiabatic limit. The importance of the gauge invariance (1.30) lies in the fact that it enables one to proof the unitary of the physical S-matrix S_{phys} defined in the physical subspace H_{phys} of the total Hilbert-Fock space H. The former one can be defined as the cohomology space of Q or, equivalently, as $\{\text{Ker}Q\} \ominus_\perp \{\text{Ran}Q\}$.

The interaction $T^{(1)}(x)$ in (1.2) admits gauge invariant generalizations [26]. A particular symmetric one is

$$T^{(1)}_s = ief_{abc} : \Big(-\tfrac{1}{2} A_{\mu a} A^{\nu}_b F^{\mu\nu}_c - \tfrac{1}{2} A_{\mu a} u_b \overleftrightarrow{\partial^\mu} \tilde{u}_c \Big) : \tag{1.33}$$

$$d_Q T^{(1)}_s = \partial_\mu T^{(1)}_s \mu, \; \mathcal{T}^{(1)}_s \mu = ef_{abc} : u_a \Big(A_{\nu b} F^{\mu\nu}_c + \tfrac{1}{2} A^{\mu}_b \partial_\nu A^{\nu}_c + \tfrac{1}{2} u_b \overleftrightarrow{\partial^\mu} \tilde{u}_c \Big) : \tag{1.34}$$

It is invariant under the ghost charge conjugation C_g. This unitary operator reflects the gauge charge:

$$C_g Q_g C_g^{-1} = -Q_g \tag{1.35}$$

and acts on the ghost fields in the following way:

$$C_g u_a(x) C_g^{-1} = i\tilde{u}_a(x)\,, \quad C_g \tilde{u}_a(x) C_g^{-1} = i u_a(x)\,. \tag{1.36}$$

This implies indeed:

$$C_g T C_g^{-1} = T\,. \tag{1.37}$$

$T_s^{(1)}$ is actually not only invariant under ghost charge conjugation; it is invariant under SU(1,1) - "rotations" in ghost space, too [5].

Let us also remember how matter fields couple to massless Yang–Mills fields [15]: They form conserved currents j_a^μ whose coupling to the gauge fields is given by:

$$T^{(1)}_{\rm matter} = iej_a^\mu A_{\mu a}\,. \tag{1.38}$$

This coupling is gauge invariant:

$$d_Q T^{(1)}_{\rm matter} = \partial_\mu\{-j_a^\mu u_a\}\,. \tag{1.39}$$

Here current conversation: $\partial_\mu j_a^\mu = 0$ has been used: Nonconserved currents cannot couple to massless Yang–Mills fields in a gauge invariant way.

The lectures at hand aim at the construction of Yang–Mills theories with massive gauge (and ghost) fields. This is usally done via the Higgs mechanism which is known to give a renormalizable, gauge invariant, and unitary perturbation series. While not questioning the validity of this result we here want to develop a different approach. This is for the following reason: The Higgs mechanism is mainly based on a classical picture. It extensively uses gauge transformations of interacting classical fields. Also the so called "vacuum expectation value" of the Higgs field is nothing but the minimum of the *classical* Energy as a functional of the field configurations. Causal perturbation theory, on the other hand, does not take any reference to classical fields and classical gauge transformations since it lives in the space of (free) quantized fields from the very beginning. Instead, (asymptotic) BRS-invariance and nilpotency of the BRS-charge are the cornerstones of the theory. So we will base our discussion of massive Yang–Mills theory on these fundaments.

2. Massive Yang–Mills fields and the algebraic introduction of scalar gauge fields

To construct a theory of massive Yang–Mills fields we have to use free asymptotic massive gauge and ghost fields:

$$(\partial \cdot \partial + M^2) A_a^\mu(x) = (\partial \cdot \partial + M^2) u_a(x) = (\partial \cdot \partial + M^2)\tilde{u}_a(x) = 0\,, \tag{2.1}$$

$$[A_a^\mu(x), A_b^\nu(y)]_- = i\delta_{ab} g^{\mu\nu} D_M(x-y)\,, \tag{2.2}$$

$$\{u_a(x), \tilde{u}_b(y)\}_+ = -i\delta_{ab} D_M(x-y), \;\; \{u_a(x), u_b(y)\}_+ = \{\tilde{u}_a(x), \tilde{u}_b(y)\}_+ = 0\,, \tag{2.3}$$

where D_M, the Pauli–Jordan commutation function for mass $M > 0$, appears. The free massive field strenghts $F_a^{\mu\nu}$ are defined as in (1.4). We have given all coloured fields the same mass, since we do not discuss breaking of the global group G here, while the ghost and the gauge fields have the same mass because they transform among each other under gauge transformations.

The nonvanishing of the mass M has simple but far reaching consequences: While (1.8) remains true,

$$[\partial_\mu A_a^\mu(x), F_b^{\kappa\lambda}(y)]_- = 0\,. \tag{2.4}$$

(1.7) is altered to

$$[\partial_\mu A_a^\mu(x), \partial_\nu A_b^\nu(y)]_- = iM^2 \delta_{ab} D_M(x-y)\,. \tag{2.5}$$

The nonvanishing of this commutator forbids us to define the gauge charge in the same way as in the massless case. For, the operator

$$q \stackrel{\text{def}}{=} \int\limits_{x_0=\text{const.}} d^3\vec{x}\, (\partial_\mu A_a^\mu(x))\, \overleftrightarrow{\partial_0}\, u_a(x) \tag{2.6}$$

is not a differential operator:

$$\begin{aligned} q^2 = \frac{1}{2}\{q,q\}_+ = \frac{1}{2} \int\limits_{x_0=\text{const.}} d^3\vec{x} \int\limits_{y_0=\text{const.}} d^3\vec{y} \\ \times \Big\{ [\partial_\mu A_a^\mu(x), \partial_\nu A_b^\nu(y)]_- \overleftrightarrow{\partial}_{x^0} \overleftrightarrow{\partial}_{y^0} (u_a(x) u_b(y)) \\ + (\partial_\nu A_b^\nu(y) \partial_\mu A_a^\mu(x)) \overleftrightarrow{\partial}_{x^0} \overleftrightarrow{\partial}_{y^0} \{u_a(x), u_b(y)\}_+ \Big\} \end{aligned} \tag{2.7}$$

and this is due to the nonvanishing commutator (2.5) unequal to zero. Instead it is given by

$$q^2 = iM^2 Q_u \quad , \quad Q_u \stackrel{\text{def}}{=} i \int\limits_{x_0=\text{const.}} d^3\vec{x}\, u_a(x)\, \overleftrightarrow{\partial^0}\, u_a(x)\,. \tag{2.8}$$

The charge Q_u has been discussed in the framework of the ghost charge algebra in [5]. The condition that the gauge (:=BRS) charge has vanishing square is, however, indispensable. For, as will be shown in the next chapter, it is one of the cornerstones in the proof of unitarity of the physical S-matrix. Thus q cannot be the right choice.

To construct a gauge operator with vanishing square we have to replace the divergences of the Yang–Mills fields by fields which commute with themselves. This is done in the following way: We introduce $\dim G$ hermitean free quantized Klein–Gordon fields $h_a(x)$ which are, like the gauge fields A_a^μ and the ghosts fields u_a and $\tilde{u}_a$, in the adjoint representation of G and which have the same mass as these fields. They obey

$$(\partial \cdot \partial + M^2) h_a(x) = 0\ , \ [h_a(x), h_b(y)]_= - i\delta_{ab} D_M(x-y)\,. \tag{2.9}$$

Their commutuator has the opposite sign to (2.5). It follows that the fields $\partial_\mu A_a^\mu(x) + M h_a(x)$ have vanishing commutators with themselves:

$$[\partial_\mu A_a^\mu(x) + M h_a(x), \partial_\nu A_b^\nu(y) + M h_b(y)]_- = 0\,. \tag{2.10}$$

This suggests the following definition for Q:

$$Q \stackrel{\text{def}}{=} \int\limits_{x_0=\text{const.}} d^3\vec{x}\, (\partial_\mu A_a^\mu(x) + M h_a(x))\, \overleftrightarrow{\partial_0}\, u_a(x)\,. \tag{2.11}$$

For, this implies

$$\begin{aligned} Q^2 = \frac{1}{2} & \int\limits_{x_0=\text{const.}} d^3\vec{x} \int\limits_{y_0=\text{const.}} d^3\vec{y} \\ & \times \Big\{ [\partial_\mu A_a^\mu(x) + M h_a(x), \partial_\nu A_b^\nu(y) + M h_b(y)]_- \overleftrightarrow{\partial_{x^0}}\, \overleftrightarrow{\partial_{y^0}} (u_a(x) u_b(y)) \\ & + \big(\partial_\nu A_b^\nu(y) + M h_b(y)\big)\big(\partial_\mu A_a^\mu(x) \\ & + M h_a(x)\big)\, \overleftrightarrow{\partial_{x^0}}\, \overleftrightarrow{\partial_{y^0}} \{u_a(x), u_b(y)\}_+ \Big\} = 0 \end{aligned} \tag{2.12}$$

i.e. Q is indeed an admissible gauge charge.

The gauge variations of the elementary fields are given by

$$d_Q A_a^\mu(x) = i\partial^\mu u_a(x)\,, \quad d_Q h_a(x) = iMu_a(x)\,,$$
$$d_Q u_a(x) = 0\,, \quad d_Q \tilde{u}_a(x) = -i\big(\partial_\mu A_a^\mu(x) + Mh_a(x)\big)\,. \qquad (2.13)$$

Consequently,

$$d_Q\partial_\mu A_a^\mu(x) = -iM^2 u_a(x)\,, \quad d_Q F_a^{\mu\nu}(x) = d_Q\big(\partial_\mu A_a^\mu(x) + Mh_a(x)\big) = 0 \qquad (2.14)$$

We see that the scalar fields $h_a(x)$ are effected by the gauge variation d_Q and that they appear in the gauge variations of other fields. Hence it is appropriate to call them scalar gauge fields. We will see in chapter six that these fields are unphysical, *i.e.* their projections onto $H_{\rm phys}$ vanish.

We now have to study the possible gauge invariant interactions $T^{(1)}(x)$. This has been done in great generality in [26]. As in the massless case, different choiches for $T^{(1)}(x)$ are possible. The most symmetric one is given by:

$$T^{(1)} = ief_{abc} : \Big\{ -\tfrac{1}{2}A_{\mu a}A_{\nu b}F_c^{\mu\nu} - \tfrac{1}{2}A_{\mu a}u_b\,\overleftrightarrow{\partial^\mu}\,\tilde{u}_c + \tfrac{1}{4}A_{\mu a}h_b\,\overleftrightarrow{\partial^\mu}\,h_c\Big\}\,, \qquad (2.15)$$

The last term describes the interaction of the scalar gauge fields h_a with the Yang–Mills fields A_a^μ. This interaction is a consequence of gauge invariance, which is expressed as

$$d_Q T^{(1)} = \partial_\mu T^{(1)\mu}\,,$$
$$T^{(1)}\mu = ef_{abc} : u_a\Big\{A_{\nu b}F_c^{\mu\nu} + \tfrac{1}{2}u_b\,\overleftrightarrow{\partial^\mu}\,\tilde{u}_c + \tfrac{1}{2}A_b^\mu\partial_\nu A_c^\nu - \tfrac{1}{4}h_b\,\overleftrightarrow{\partial^\mu}\,h_c\Big\} : \qquad (2.16)$$

Gauge invariance also determines the interaction of the matter currents j_a^μ:

$$T^{(1}_{\rm matter} = \big\{J_a^\mu A_{\mu a} + M^{-1}\partial_\mu J_a^\mu h_a\big\}\,, \quad d_Q T^{(1)}_{\rm matter} = \partial_\mu\left(-J_a^\mu u_a\right)\,. \qquad (2.17)$$

Let us interpret this result. In the case of conserved currents the matter fields couple only to the Yang–Mills fields and this interaction has the same form as the coupling of matter fields to massless Yang–Mills fields [15]. More interesting is the case of nonconserved currents: There the matter fields couple to the scalar gauge fields, too. We conclude that the scalar gauge fields are a very important part of the whole theory: They allow a consistent treatment of massive Yang–Mills fields and of nonconserved currents at the same time. We also notice that the coupling of the nonconseved currents to the scalar gauge fields is proportional to the inverse mass of the gauge fields. That is only possible if this mass does not vanish, which is in striking agreement with physical reality: The conserved strong vector currents couple to massless (though confined) gluons while the nonconserved weak axial currents couple to the massive weak bosons!

We now have carefully studied the interactions of *quantized* Yang–Mills fields, ghost fields, scalar gauge fields, and matter fields. Though only working in first order T^1 we have discovered very interesting structures. To complete the theory, we would have to study gauge invariance in all orders, Eq. (1.30). Before we take on this Herculean task we like to know what we get if we succeed. It is the unitarity of S_{phys}. This is shown in the next chapter.

3. Gauge invariance and unitarity of the physical S-matrix

Unitarity of the physical S-matrix in the case of massless Yang–Mills fields was proven in [4]. Here we treat the massive case, *i.e.* the interaction constructed in the two preceding chapters.

Let us begin with discussing the Krein structure [5, 8, 16, 17] in the Hilbert–Fock space of the gauge fields. The massive Yang–Mills fields are quantized as

$$A_a^\mu(x) = \sum_{\lambda=0}^{3} \int dk \left\{ \epsilon_\lambda^\mu(k) a_{\lambda,a}(k) e^{-ikx} + \epsilon_\lambda^\mu(k) a_{\lambda,a}^K(k) e^{ikx} \right\} \tag{3.1}$$

k is always on the mass shell $\mathcal{M}$:

$$\begin{aligned} &k \stackrel{\text{def}}{=} (k_0, \vec{k}) \ , \ k_0 \stackrel{\text{def}}{=} +[(\vec{k})^2 + M^2]^{\frac{1}{2}} \ , \\ &dk \stackrel{\text{def}}{=} \frac{d^3\vec{k}}{2k_0(2\pi)^3} \ , \ \delta(k-k') \stackrel{\text{def}}{=} 2k_0(2\pi)^3 \delta^{(3)}\left(\vec{k} - \vec{k}'\right) \end{aligned} \tag{3.2}$$

ϵ_λ^μ are four polarisation vectors satisfying

$$\begin{aligned} &\epsilon_0^\mu(k) \stackrel{\text{def}}{=} \frac{k^\mu}{M} \ , \ g_{\mu\nu}\epsilon_\lambda^\mu(k)\epsilon_\kappa^\nu(k) = g_{\kappa\lambda} \ , \\ &\sum_{\lambda=1}^{3} \epsilon_\lambda^\mu(k)\epsilon_\lambda^\nu(k) = -\left[g^{\mu\nu} - \frac{k^\mu k^\nu}{M^2}\right] \ , \\ &\sum_{\lambda=0}^{3} g_{\lambda\lambda}\epsilon_\lambda^\mu(k)\epsilon_\lambda^\nu(k) = g^{\mu\nu} \ , \ \overline{\epsilon_\lambda^\mu(k)} = \epsilon_\lambda^\mu(k) \end{aligned} \tag{3.3}$$

$a_{\lambda,a}(k)$ are $4(\dim G)$ standard (distributional) bosonic annihilation operators [12,17,18,19] acting in the Hilbert–Fock space

$$H_A = \bigoplus_{n=0}^{\infty} \left\{ \bigvee^{n} \left\{ \bigoplus_{\lambda=0}^{3} \bigoplus_{a=1}^{\dim G} \left[L^2(\mathcal{M}, dk)\right]_{\lambda,a} \right\} \right\} \tag{3.4}$$

which is equipped with the standard positive scalar product $(\underline{a}, \underline{b})_A$. The operator O^+ denotes the adjoint of O with respect to this scalar product. The Fock space operators fulfil

$$[a_{\lambda,a}(k), a^+_{\kappa,b}(k')]_- = \delta_{\lambda\kappa}\delta_{ab}\delta(k-k') \,. \tag{3.5}$$

The number operators for a given polarization λ are defined by

$$N_\lambda \stackrel{\text{def}}{=} \int dk \, a^+_{(\lambda),a}(k) a_{(\lambda),a}(k) \,. \tag{3.6}$$

The Krein operator J_A in H_A [5, 8, 16, 17] is defined by

$$J_A \stackrel{\text{def}}{=} (-1)^{N_0} \,. \tag{3.7}$$

It defines a pseudo-conjugation O^K [4, 5, 12, 16, 17] of an operator O by

$$O^K \stackrel{\text{def}}{=} J_A O^+ J_A \,. \tag{3.8}$$

Sometimes the form $< \underline{a}, \underline{b} >_A := (\underline{a}, J_A \underline{b})_A$ is called an indefinite scalar product. We will not follow this terminology here. The word orthogonal (hermitean, unitary) will always mean orthogonal (hermitean, unitary) with respect to the positive inner product. Otherwise we say pseudo-orthogonal (pseudo-hermitean, pseudo-unitary).

The gauge invariant *physical* Yang–Mills fields A_{phys} have only the three transversal polarisations:

$$\begin{aligned} (A_{\text{phys}})^\mu_a(x) &= \sum_{\lambda=1}^{3} \int dk \, \left\{ \epsilon^\mu_\lambda(k) a_{\lambda,a}(k) e^{ikx} + \epsilon^\mu_\lambda(k) a^+_{\lambda,a}(k) e^{-ikx} \right\} \,, \\ d_Q A_{\text{phys}} &= 0 \end{aligned} \tag{3.9}$$

and the commutator

$$[(A_{\text{phys}})^\mu_a(x), (A_{\text{phys}})^\nu_b(y)]_- = -\left[g^{\mu\nu} + \frac{\partial^\mu \partial^\nu}{M^2}\right] \delta_{ab}(-i) D_m(x-y) \,. \tag{3.10}$$

The *unphysical* Yang–Mills A_{unphys} fields are given by

$$(A_{\text{unphys}})^\mu_a(x) \stackrel{\text{def}}{=} A^\mu_a(x) - (A_{\text{phys}})^\mu_a(x) = \frac{-1}{M^2}\partial^\mu\partial_\nu A^\nu_a(x)\,. \qquad (3.11)$$

The following conjugation properties are easily checked:

$$A = A^K\ ,\ A_{\text{phys}} = A^K_{\text{phys}} = A^+_{\text{phys}}\ ,\ A_{\text{unphys}} = A^K_{\text{unphys}} = -A^+_{\text{unphys}}\,. \qquad (3.12)$$

We also note

$$F^{\mu\nu}_a \stackrel{\text{def}}{=} \partial^\mu A^\nu_a - \partial^\nu A^\mu_a = \partial^\mu(A_{\text{phys}})^\nu_a - \partial^\nu(A_{\text{phys}})^\mu_a = (F^{\mu\nu}_a)^K = (F^{\mu\nu}_a)^+\,, \qquad (3.13)$$

$$\partial_\mu A^\mu_a = \partial_\mu(A_{\text{unphys}})^\mu_a = (\partial_\mu A^\mu_a)^K = -(\partial_\mu A^\mu_a)^+\,. \qquad (3.14)$$

The representation of the proper Poincaré group $P^\uparrow_+$ in H_A is defined by

$$U_A(a,\Lambda)A^\mu_a(x)U_A(a,\Lambda)^{-1} = \Lambda^\mu{}_\nu A^\nu_a(\Lambda x + a)\ ,\ U_A(a,\Lambda)\Omega_A = \Omega_A\,, \qquad (3.15)$$

where Ω_A is the vacuum in H_A. It is pseudo-unitary:

$$U_A(a,\Lambda)U_A(a,\Lambda)^K = 1 \qquad (3.16)$$

and, since it commutes with J_A:

$$J_A U_A(a,\Lambda)J_A = U_A(a,\Lambda) \qquad (3.17)$$

unitary as well:

$$U_A(a,\Lambda)U_A(a,\Lambda)^+ = 1\,. \qquad (3.18)$$

The last two equations fail in the massless case where $\epsilon^\mu_0(k)$ cannot be chosen covariantly.

Next we come to the hermitean scalar gauge fields which are quantized in the usual way:

$$h_a(x) = \int dk\,\left\{b_a(k)e^{-ikx} + b^+_a(k)e^{ikx}\right\} = h^+_a(x) \qquad (3.19)$$

as operators in the Hilbert–Fock space

$$H_h = \bigoplus_{n=0}^{\infty}\left\{\bigvee^{n}\left\{\bigoplus_{a=1}^{\dim G}\left[L^2(\mathcal{M},dk)\right]_a\right\}\right\} \qquad (3.20)$$

with standard positive scalar product $(\underline{a}, \underline{b})_h$. We have

$$[b_a(k), b_b^+(k')]_- = \delta_{ab}\delta(k-k') . \tag{3.21}$$

We do not introduce an additional Krein structure in H_h. This is equivalent to saying that $J_h = 1$ and that the two conjugations agree over H_h: $O^K = O^+$. The same is true for the two forms over H_h: $< \underline{a}, \underline{b} >_h = (\underline{a}, \underline{b})_h$. The representation of $P_+^\uparrow$ in H_h: $U_h(a, \Lambda)$ is unitary. The total number of scalar gauge particles is given by

$$N_h = \int dk\, b_a^+(k) b_a(k) . \tag{3.22}$$

Now we consider the ghost fields. They have been extensively studied in [5]. So we summarize only the most important formulae here. The ghost fields

$$u_a(x) = \int dk \left\{ c_{1,a}(k) e^{-ikx} + c_{-1,a}^+(k) e^{ikx} \right\} ,$$
$$\tilde{u}_a(x) = \int dk \left\{ - c_{-1,a}(k) e^{-ikx} + c_{1,a}^+(k) e^{ikx} \right\} \tag{3.23}$$

are defined in the Hilbert–Fock space

$$H_g = \bigoplus_{n=0}^{\infty} \left\{ \bigwedge_n \left\{ \bigoplus_{i=\pm 1} \bigoplus_{a=1}^{\dim G} \left[L^2(\mathcal{M}, dk) \right]_{i,a} \right\} \right\} \tag{3.24}$$

with positive inner product $(\underline{a}, \underline{b})_g$. The index i distinguishes ghost from antighost particles. We have

$$\{c_{i,a}(k), c_{j,b}^+(k')\}_+ = \delta_{ij}\delta_{ab}\delta(k-k') . \tag{3.25}$$

The Krein operator in H_g is defined by

$$J_g = i^{N_g - \Gamma} . \tag{3.26}$$

Here N_g denotes the total number of ghost and antighost particles:

$$N_g = \int dk\, c_{i,a}^+(k) c_{i,a}(k) \tag{3.27}$$

while Γ is defined by

$$\Gamma = \int dk\, \left\{ c_1^+(k) c_{-1}(k) + c_{-1}^+(k) c_1(k) \right\} . \tag{3.28}$$

Again one considers the indefinite form $< \underline{a}, \underline{b} >_g := (\underline{a}, J_g \underline{b})$ and defines $O^K = J_g O^+ J_g$. The representation of $P_+^\uparrow$ in H_g: $U_g(a, \Lambda)$ is unitary and, since it commutes with J_g, pseudo-unitary as well [5].

The scalar and Dirac matter fields are quantized in their own Hilbert–Fock space H_{matter} in the usual way. The scalar product in this space is again positive and the Krein structure J_{matter} is the unit operator. The representation $U_{\text{matter}}(a, \Lambda)$ is unitary.

The Hilbert space H of the total system is the tensor-product of the spaces above:

$$H = H_A \otimes H_h \otimes H_g \otimes H_{\text{matter}} \,. \tag{3.29}$$

The Krein operator and the representation of $P_+^\uparrow$ factorize accordingly:

$$J = J_A \otimes J_h \otimes J_g \otimes J_{\text{matter}} \,, \tag{3.30}$$

$$U(a, \Lambda) = U_A(a, \Lambda) \otimes U_h(a, \Lambda) \otimes U_g(a, \Lambda) \otimes U_{\text{matter}}(a, \Lambda) \,. \tag{3.31}$$

This U is unitary and pseudo-unitary, since it commutes with J. The positive scalar product in H is denoted by $(\underline{a}, \underline{b})$ and $||\underline{a}|| := (\underline{a}, \underline{a})$, $< \underline{a}, \underline{b} > := (\underline{a}, J\underline{b})$, $O^K := JO^+J$.

Our next task is to study more closely the gauge charge Q It is expressed in momentum space as

$$Q = M \int dk \, \left\{ c^+_{-1,a}(k) \left[a_{0,a}(k) + i b_a(k) \right] - \left[a^+_{0,a}(k) + i b^+_a(k) \right] c_{1,a}(k) \right\} \,. \tag{3.32}$$

Its adjoint Q^+ is given by:

$$Q^+ = M \int dk \, \left\{ c^+_{1,a}(k) \left[-a_{0,a}(k) + i b_a(k) \right] + \left[a^+_{0,a}(k) - i b^+_a(k) \right] c_{-1,a}(k) \right\} . \tag{3.33}$$

Q and Q^+ are both pseudo-hermitean $P_+^\uparrow$ invarinat differential operators:

$$Q^2 = \left(Q^+\right)^2 = 0 \,, \; Q = Q^K \,, \; \left(Q^+\right)^K = Q^+ \,, \; U(a, \Lambda) Q^{(+)} U(a, \Lambda)^{-1} = Q^{(+)} \,. \tag{3.34}$$

We now follow Razumov and Rybkin [8] who showed that the physical Hilbert space of a gauge theory with quadratic BRS charge Q can be defined as

$$H_{\text{phys}} \stackrel{\text{def}}{=} \text{kernel} \, \{Q, Q^+\}_+ \,. \tag{3.35}$$

Razumov showed the equivalence of this definition with the more conventional one using equivalent classes in semidefinite metric spaces [7]. Razumovs definition is advantageous because it realizes H_{phys} as a concrete subspace of the Hilbert space H which has a clear particle interpretation. To work this out we only have to calculate the above anticommutator. We find:

$$\{Q, Q^+\}_+ = 2[N_0 + N_h + N_g] \stackrel{\text{def}}{=} 2N \tag{3.36}$$

i.e. N is the number of longitudinal Yang–Mills fields plus the number of scalar gauge fields plus the number of ghost and antighost particles. Thus all these particles are unphysical. The only physical particles are the transverse quanta of the Yang–Mills fields and the matter particles.

The spectrum of the number operator N are the natural numbers and 0:

$$N = \sum_{n=0}^{\infty} n P_n \,, \tag{3.37}$$

where P_n is the orthogonal projector on the subspace with n unphysical particles. (3.35) means that the orthogonal projector on H_{phys} is given by P_0:

$$H_{\text{phys}} = P_0 H \,. \tag{3.38}$$

The operator N can be inverted on the orthogonal complement of its kernel:

$$N^{\sim 1} \stackrel{\text{def}}{=} 0 P_0 + \sum_{n=1}^{\infty} n^{-1} P_n \;,\; N^{\sim 1} N = N N^{\sim 1} = (1 - P_{\text{phys}}) \,. \tag{3.39}$$

Since Q and Q^+ are $P_+^\uparrow$ invariant, so are N, $N^{\sim 1}$, and P_n:

$$U(a,\Lambda)\left\{N; N^{\sim 1}; P_n\right\} U(a,\Lambda)^{-1} = \left\{N; N^{\sim 1}; P_n\right\} \,. \tag{3.40}$$

We also note that N, P_n, and $N^{\sim 1}$ commute with Q and Q^+:

$$[Q^{(+)}, N]_- = [Q^{(+)}, P_n]_- = [Q^{(+)}, N^{\sim 1}]_- = 0 \,. \tag{3.41}$$

We now follow again [8] and introduce the following subspaces of H:

$$H_K \stackrel{\text{def}}{=} \operatorname{kernel} Q \,,\; H_{K^+} \stackrel{\text{def}}{=} \operatorname{kernel} Q^+ \,,\; H_R \stackrel{\text{def}}{=} \operatorname{range} Q \,,\; H_{R^+} \stackrel{\text{def}}{=} \operatorname{range} Q^+ \,. \tag{3.42}$$

Let us study the relations between these spaces and H_{phys}. Let $\underline{a}_0 \in H_{\text{phys}}$. By

$$0 = \left(\underline{a}_0, \{Q, Q^+\}_+ \underline{a}_0\right) = ||Q^+ \underline{a}_0||^2 + ||Q \underline{a}_0||^2 \tag{3.43}$$

we find

$$H_{\text{phys}} = H_K \cap H_{K^+} \,. \tag{3.44}$$

Since $Q^2 = (Q^+)^2 = 0$ we have

$$H_R \subseteq H_K \;,\; H_{R^+} \subseteq H_{K^+} \,. \tag{3.45}$$

Let $\underline{a}_K \in H_K$, $\underline{b}_{R^+} \in H_{R^+}$. Since

$$(\underline{a}_K, \underline{b}_{R^+}) = (\underline{a}_K, Q^+ \underline{b}) = (Q \underline{a}_K, \underline{b}) = 0 \,. \tag{3.46}$$

H_K and H_{R^+} are orthogonal to each other, and replacing Q by Q^+ shows that the same holds true for H_R and H_{K^+}:

$$H_K \perp H_{R^+} \; , \; H_R \perp H_{K^+} \, . \tag{3.47}$$

Combining this with (3.46) gives

$$H_R \perp H_{R^+} \tag{3.48}$$

while (3.44) now implies

$$H_{\text{phys}} \perp H_R \; , \; H_{\text{phys}} \perp H_{R^+} \, . \tag{3.49}$$

We conclude that the three spaces H_{phys}, H_R, and H_{R^+} are all mutually orthogonal. Let now $\underline{a} \in H$. Then we can write

$$\begin{aligned} \underline{a} &= P_0\underline{a} + (1 - P_0)\underline{a} = P_0\underline{a} + NN^{\sim 1}\underline{a} \\ &= P_0\underline{a} + \tfrac{1}{2}QQ^+N^{\sim 1}\underline{a} + \tfrac{1}{2}Q^+QN^{\sim 1}\underline{a} \stackrel{\text{def}}{=} \underline{a}_0 + \underline{a}_R + \underline{a}_{R^+} \, , \end{aligned} \tag{3.50}$$

where $\underline{a}_0 \in H_{\text{phys}}$, $\underline{a}_R \in H_R$, and $\underline{a}_{R^+} \in H_{R^+}$. This shows that the Hilbert space H is the direct orthogonal sum of the three spaces H_{phys}, H_R, and H_{R^+}:

$$H = H_{\text{phys}} \oplus_\perp H_R \oplus_\perp H_{R^+} \, . \tag{3.51}$$

Since the first two of this spaces are subspaces of H_K and the third is orthogonal to it we can also write:

$$H = H_K \oplus_\perp H_{R^+} \; , \; H_K = H_{\text{phys}} \oplus_\perp H_R \tag{3.52}$$

i.e. the physical Hilbert space is also given by

$$H_{\text{phys}} = H_K \ominus_\perp H_R \, . \tag{3.53}$$

Moreover, since H_{phys} and H_{R^+} are subspaces of H_{K^+} and this space is orthogonal to H_R one can also write

$$H = H_{K^+} \oplus_\perp H_R \; , \; H_{K^+} = H_{\text{phys}} \oplus_\perp H_{R^+} \tag{3.54}$$

i.e. we get one more characterization of H_{phys} as

$$H_{\text{phys}} = H_{K^+} \ominus_\perp H_{R^+} \, . \tag{3.55}$$

The orthogonal decompositions above were already given in [8], and, in the specific context of massless Yang–Mills theories, in [4]. We denote the orthogonal projections on $\{H_{\text{phys}}\, ; H_K\, ; H_R\, ; H_{K^+}H_{R^+}\}$ by

$\{P_0; P_K; P_R; P_{K^+}; P_{R^+}\}$ and the vectors in these spaces by $\{\underline{a}_0; \underline{a}_K; \underline{a}_R; \underline{a}_{K^+}; \underline{a}_{R^+}\}$. From the preceding equations we find:

$$P_0 P_R = P_0 P_{R^+} = P_R P_{R^+} = 0\,,\ P_0 + P_R + P_{R^+} = 1\,,$$
$$P_0 = P_0^+\,,\ P_R = P_R^+\,,\ P_{R^+} = P_{R^+}^+ \tag{3.56}$$
$$P_R = \tfrac{1}{2} Q Q^+ N^{\sim 1}\,,\ P_{R^+} = \tfrac{1}{2} Q^+ Q N^{\sim 1}\,. \tag{3.57}$$

Let us now study the strucure of some important operators with respect to the orthogonal decomposition (3.51). The operators Q and Q^+ map the complements of their kernels onto their range. This gives:

$$Q = P_R Q P_{R^+}\,,\ Q^+ = P_{R^+} Q^+ P_R\,. \tag{3.58}$$

Then (3.36) implies that the decomposition of N and $N^{\sim 1}$ are given by

$$N^{(\sim 1)} = P_R N^{(\sim 1)} P_R + P_{R^+} N^{(\sim 1)} P_{R^+}\,. \tag{3.59}$$

Let now I be a gauge invariant operator, *i.e.* $d_Q I = 0$. Then H_K and H_R are stable under the action of I, that is

$$I P_K = P_K I P_K\,,\ I P_R = P_R I P_R\,. \tag{3.60}$$

Thus we get

$$I = P_0 I P_0 + P_0 I P_{R^+} + P_R I P_0 + P_R I P_R + P_R I P_{R^+} + P_{R^+} I P_{R^+}\,. \tag{3.61}$$

Next we use the pseudo-hermiticity of Q and Q^K to get information about the Krein operator J. Let $\underline{a}_K \in H_K$, $\underline{b}_R = Q\underline{c} \in H_R$. Then we have

$$(\underline{a}_K, J\underline{b}_R) = <\underline{a}_K, Q\underline{c}> = <Q\underline{a}_K, \underline{c}> = 0\,. \tag{3.62}$$

This means:

$$P_K J P_R = 0\,. \tag{3.63}$$

Taking the adjoint gives:

$$P_R J P_K = 0\,. \tag{3.64}$$

Using Q^+ instead of Q in the argument above gives

$$P_{K^+} J P_{R^+} = P_{R^+} J P_{K^+} = 0\,. \tag{3.65}$$

A direct inspection of J in (3.30) gives the additional information that J agrees on $H_{\rm phys}$ with the unit operator:

$$P_0 J P_0 = P_0\,. \tag{3.66}$$

The last four equations are summarized in

$$J = P_0 + P_R J P_{R+} + P_{R+} J P_R \,. \tag{3.67}$$

The second of Eqs. (3.52) means that H_K can be interpreted as a linear fiber bundle: $H_{\rm phys}$ is the base space and the fibers are the elements of H_R. Eqs. (3.63), (3.64) show that the fibers are pseudo-orthogonal to any vector in H_K. Moreover, writing $\underline{a}_K = \underline{a}_0 + \underline{a}_R$ according to the orthogonal decomposition (3.52) gives

$$\langle \underline{a}_K, \underline{b}_K \rangle = (\underline{a}_0, \underline{b}_0) \,. \tag{3.68}$$

This shows that that the form $<,>$ agrees on $H_{\rm phys}$ with the positive form $(,)$, that it is positive semidefinite in H_K, and that its kernel as a quadratic form in this space are the fibers:

$$H_R = \text{kernel}\, \langle , \rangle_K \,, \tag{3.69}$$

where $\langle , \rangle_K$ means the restriction of the form $\langle , \rangle$ to H_K. So we get another expression for$H_{\rm phys}$:

$$H_{\rm phys} = H_K \ominus_\perp \text{kernel}\, \langle , \rangle_K \,. \tag{3.70}$$

The form $\langle , \rangle_K$ is constant along the fibers in both arguments separately:

$$\langle \underline{a}_K + \underline{a}_R, \underline{b}_K + \underline{b}_R \rangle_K = \langle \underline{a}_K, \underline{b}_K \rangle_K \tag{3.71}$$

and the same holds true for the matrix elements of any gauge invariant operator I with respect to this form:

$$\langle \underline{a}_K + \underline{a}_R, I\,(\underline{b}_K + \underline{b}_R) \rangle_K = \langle \underline{a}_K, I \underline{b}_K \rangle_K \,. \tag{3.72}$$

This allows to choose any linear cross section H_S in H_K, *i.e.* any subspace of H which is a (pseudo-orthogonal but generally not orthogonal) complement of H_R (in H_K) as a realization of the physical Hilbert space. The scalar product in H_S is the restriction of the form $<,>$ to this space, and there it is positive definite. All this spaces are unitarily equivalent, and the matrix elements of gauge invariant operators do not depend on the section chosen. So one might also consider the equivalence class of all this spaces and that is what is usually done in the literature [7]. We prefer to use $H_{\rm phys}$ as a concrete realization of the physical Hilbert space since it is the only section which is orthogonal to the fibers and which allows for a simple interpretation of the quanta of the elementary fields as physical or unphysical particles. The projections onto the sections along the fibers are also called gauge transformations. For H_{phys}, and only for it, they agree with the orthogonal projection.

We now consider again operators A, B, $C\cdots$ over H. We define the orthogonal projection of A on H_{phys}: A_0 by

$$A_0 \stackrel{\text{def}}{=} P_0 A P_0 \,. \tag{3.73}$$

A_0 is still an operator from H to$H$. It is zero on the orthogonal complement of H_{phys}: $H^{\perp}_{\text{phys}}$. Since this zero is certainly not very interesting we define A_{phys} to be the restriction of A_0 to H_{phys}:

$$A_{\text{phys}} \stackrel{\text{def}}{=} (A_0)_{\downarrow H_{\text{phys}}} \,. \tag{3.74}$$

The map $A \longrightarrow A_{\text{phys}}$ is certainly linear:

$$(\alpha A)_{\text{phys}} = \alpha A_{\text{phys}} \;, \; (A+B)_{\text{phys}} = A_{\text{phys}} + B_{\text{phys}} \,. \tag{3.75}$$

More interesting is the projection of the product of two operators. We calculate:

$$\begin{aligned} P_0 A B P_0 &= P_0 A (P_0 + P_R + P_{R^+}) B P_0 \\ &= P_0 A P_0 P_0 B P_0 + P_0 A \tfrac{1}{2} Q Q^+ N^{\sim 1} B P_0 + P_0 A \tfrac{1}{2} N^{\sim 1} Q^+ Q B P_0 \,. \end{aligned} \tag{3.76}$$

Let us concentrate on the second summand: X. Since P_0Q=0 we can replace AQ by $\{A,Q\}_\pm$. We take the anticommutator if A is fermionic and the commutator if it is bosonic. This gives

$$X = P_0 \tfrac{1}{2} \{A,Q\}_\pm Q^+ N^{\sim 1} B P_0 \,. \tag{3.77}$$

Now we use $P_0 Q^+ = 0$ to replace that by

$$X = P_0 \tfrac{1}{2} \left\{ \{A,Q\}_\pm, Q^+ \right\}_\mp N^{\sim 1} B P_0 \,. \tag{3.78}$$

And finally we use $P_0 N^{\sim 1} = 0$ to write

$$X = P_0 \tfrac{1}{2} \left[\left\{ \{A,Q\}_\pm, Q^+ \right\}_\mp, N^{\sim 1} \right]_- B P_0 \,. \tag{3.79}$$

There are always two commutators and one anticommutator in this expression. Thus it can be uniquely written as

$$X = P_0 (\mathcal{T} A) B P_0 \,, \tag{3.80}$$

where the *triple variation* $\mathcal{T}$ is defined by

$$\mathcal{T} \stackrel{\text{def}}{=} \tfrac{1}{2}\delta_{(N^{\sim 1})} d_{(Q^+)} d_Q \, . \tag{3.81}$$

Here $\delta_{(N^{\sim 1})}$ is the derivation induced by $N^{\sim 1}$, and d_Q and $d_{(Q^+)}$ are the antiderivations induced by Q and Q^+, repectively (see chapter 1). Note that the triple variation of gauge invariant operators vanishes. In the same way the third summand in (3.77): Y is written as

$$Y = P_0 A(\mathcal{T} B) P_0 \, . \tag{3.82}$$

We thus have found the important *projection formula*:

$$A_{\text{phys}} B_{\text{phys}} = (AB)_{\text{phys}} - \{(\mathcal{T} A)B + A(\mathcal{T} B)\}_{\text{phys}} \, . \tag{3.83}$$

This implies

Theorem I: *The product of the physical projections of two operators with vanishing triple variation, especially of two gauge invariant operators, is identical to the physical projection of their product. The physical projection of a group (of an algebra) of operators with vanishing triple variation, especially of gauge invariant operators, is a representation of this group (algebra).*

Next we consider the physical projection of the pseudo-adjoint A^K *of an operator* A*. So we have to study*

$$P_0 A^K P_0 = P_0 J A^+ J P_0 \, . \tag{3.84}$$

Now we use that (3.67) implies

$$P_0 J = J P_0 = P_0 J P_0 = P_0 \tag{3.85}$$

to conclude:

$$P_0 A^K P_0 = P_0 A^+ P_0 = (P_0 A P_0)^+ \tag{3.86}$$

which means

$$\left(A^K\right)_{\text{phys}} = (A_{\text{phys}})^+ \, . \tag{3.87}$$

Thus we have found

Theorem II: *The physical projection of the pseudo-adjoint operator is identical to the adjoint of the physical projection of this operator. The physical projection of a pseudo-hermitean operator is hermitean.*

Combining the two theorems above gives:

Theorem III: *The physical projection of a pseudo-unitary operator with vanishing triple variation, especially of a pseudo-unitary gauge invariant operator, is an unitary operator.*

Now we are well equipped to tackle unitarity of the physical S-matrix. The interaction $T^{(1)}(x)$ constructed in the preceding chapters is anti-pseudo-hermitean:

$$\left(T^{(1)}(x)\right)^K = -T^{(1)}(x)\,. \tag{3.88}$$

This guarantees the pseudo-unitarity of $S[g]$ [11, 12, 15]:

$$S[g]S^K[g] = 1\,. \tag{3.89}$$

This is [11] equivalent to

$$\sum_{I\oplus J=N} T(I)T^K(J) = 0,\ \ \forall N \neq \emptyset\,. \tag{3.90}$$

Here $T(I)$ means $T^{(r)}(x_{i_1},\cdots,x_{i_r})$, $T^K(J)$ means $\left(T^{(s)}(x_{j_1},\cdots,x_{j_s})\right)^K$, where $r+s=n$, and the sum Σ runs over all direct decompositions of the set $N=\{1,\cdots,n\}$ into two subsets $I=\{i_1,\cdots,i_r\}$ and $J=\{j_1,\cdots,j_s\}$. Let us now *assume* that gauge invariance holds true in all orders, *i.e.* we have

$$d_Q T^{(n)} = \sum_{k=1}^{4n} \partial^k T_k^{(n)}\,. \tag{3.91}$$

Taking the pseudo-conjugate of this equation gives:

$$d_Q\left(T^{(n)}\right)^K = -\sum_{k=1}^{4n} \partial^k \left(T_k^{(n)}\right)^K\,. \tag{3.92}$$

Then (3.83), (3.87) and (3.90)-(3.92) imply

$$\begin{aligned} \sum_{I\oplus J=X} T_{\text{phys}}(I)(T_{\text{phys}}(J))^+ &= \sum_{k=1}^{4n} \partial^k W_k^{(n)}(X)\,,\\ W_k^{(n)}(X) \stackrel{\text{def}}{=} \tfrac{1}{2} \sum_{I\oplus J=X} \Big\{ &-\theta_I(k)(\delta_{(N\sim1)}d_{(Q+)}T_k(I))T^K(J)\\ &+\theta_J(k)T(I)(\delta_{(N\sim1)}d_{(Q+)}T_k^K(J))\Big\}_{\text{phys}}\,, \end{aligned} \tag{3.93}$$

where

$$\theta_I(k) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } k\in I\\ 0, & \text{if } k\notin I \end{cases}\,. \tag{3.94}$$

This is the exact *perturbative, pre-adiabatic* expression for the unitarity of the physical S-matrix. *If* the adiabtic limit:

$$S_{\text{phys}} = \lim_{g \to 1} (S[g])_{\text{phys}} \tag{3.95}$$

exists and has the same analytic properties as in the saclar theory discussed in [13], and *if* the boundary terms $\int \partial^k W_k^{(n)}$ vanish, (3.93) will imply

$$S_{\text{phys}}(S_{\text{phys}})^+ = 1\,. \tag{3.96}$$

We note, however, that this adiabatic limit may have additional subtleties if the heavy gauge particles are coupled to light matter, since then the gauge fields become unstable, *i.e.* are not really asymptotic fields.

Let us end this long chapter with a short discussion of $P_+^\uparrow$-invariance. The remark after (3.31) and eq. (3.40) immediately imply that $U(a, \Lambda)_{\text{phys}}$ is an unitary representation of $P_+^\uparrow$, while the $P_+^\uparrow$-invariance of $S[g]$:

$$U(a,\Lambda)S[g]U(a,\Lambda)^{-1} = S[g_{a,\Lambda}]\ ,\ g_{a,\Lambda}(x) \stackrel{\text{def}}{=} g\left(\Lambda^{-1}(x-a)\right) \tag{3.97}$$

and (3.41) lead direct to

$$U(a,\Lambda)S[g]_{\text{phys}}U(a,\Lambda)^{-1} = S[g_{a,\Lambda}]_{\text{phys}} \tag{3.98}$$

which is the $P_+^\uparrow$-invariance of the physical S-matrix.

The situation is different in the massless case where J is not $P_+^\uparrow$-invariant. However, the three theorems above and the projection formula (3.83) hold true in this case, too. Since Q is $P_+^\uparrow$-invariant also in the massless case, the theorems show that $U(a,\Lambda)_{\text{phys}}$ is indeed an unitary representation of $P_+^\uparrow$, while the $P_+^\uparrow$-invariance of $S[g]$ and (3.83), (3.87) implie that (3.98) is violated only by boundary terms. The latter *should* vanish in physical quantities like cross sections, for example.

4. Discussion

We have based our causal construction of massive Yang–Mills theories on asymptotic gauge invariance and on the differential property of the gauge charge. The latter one has led naturally to the introduction of scalar gauge fields. All these scalar fields are unphysical, as follows from the proof of unitarity in the last chapter. In this way we have arrived at a *Higgs-free model for massive Yang–Mills fileds*. However, our construction is not yet complete, since we do not have shown that gauge invariance holds true in all orders. Actually, very recent calculations by Hurth [20] point out the appearence of anomalies already in second order. The impact of these anomalies on a consistent interpretation of the model we have proposed here remains — in the opinion of the author — to be clarified. The important results of Section 3, *i.e.* the projection formula (3.83) and the theorems I–III remain certainly true, since they are derived by using only certain algebraic properties of the gauge charge and the Krein structure in the Hilbert space of the theory while not refering to specific properties of $T^{(n)}$.

Higgs-free models of massive Yang–Mills theories have also been studied in the Lagrangian framework. There they are known as Stueckelberg theories [21–23]. A very interesting Higgs-free model which gives a unification of the standard model and gravity has recently proposed by Raczka and Pawlowski [24, 25]. We plan to generalize the methods presented here such that they can be applied to this model, too.

I would like to thank Prof. Raczka for inviting me to this very interesting conference where I had the opportunity to present my results and to discuss them with other participants. Fruitful discussions with Profs. K. Fredenhagen, B. Schroer, and A. Burnel are especially acknowledged.

REFERENCES

[1] M. Duetsch, T. Hurth, F. Krahe, G. Scharf, *Nuovo Cimento* **A106**, 1029 (1993).
[2] M. Duetsch, T. Hurth, F. Krahe, G. Scharf, *Nuovo Cimento* **A107**, 375 (1994).
[3] M. Duetsch, T. Hurth, G. Scharf, *Nuovo Cimento* **A108**, 679 (1995).
[4] M. Duetsch, T. Hurth, G. Scharf, *Nuovo Cimento* **A108**, 737 (1995).
[5] F. Krahe, On The Algebra of Gauge Fields, preprint DIAS-STP-95-02, hep-th-9502097.
[6] W.H. Greub, *Linear Algebra*, 3rd ed., Springer, New York 1967.
[7] N. Nakanishi, I. Ojima, *Covariant Operator Formalism Of Gauge Theories And Quantum Gravity*, World Scientific, Singapore 1990.
[8] A.V. Razumov, G.N. Rybkin, *Nucl. Phys.* **332**, 209 (1990).
[9] N.N. Bogolubov, D.V. Shirkov, *Introduction To The Theory Of Quantized Fields*, Interscience, New York 1959.
[10] N.N. Bogolubov, D.V. Shirkov, *Quantum Fields*, Benjamin-Cummings, Reading (Mass), 1983.
[11] H. Epstein, V. Glaser, *Ann. Inst. H. Poincaré* **19**, 211 (1973).
[12] G. Scharf, *Finite QED*, Springer, Berlin 1995.
[13] H. Epstein, V. Glaser, *Adiabatic Limit in Peruturbation Theory*, in *Renormalization Theory*, G. Velo and A. S. Wightmann, Eds., D. Reidel, Dordrecht (NL), 1976, pp. 269–297.
[14] P. Blanchard, R. Sénéor, *Ann. Inst. H. Poincaré* **23**, 147 (1975).

[15] M. Duetsch, On Gauge Invariance of Yang-Mills Theories with Matter Fields, preprint ZU-TH-10-95.
[16] J. Bognar, *Indefinite Inner Product Spaces*, Springer, Berlin 1974.
[17] N.N. Bogolubov, A.A.Logunov, A.I. Oksak, I.I. Todorov, *General Principles Of Quantum Field Theory*, Kluwer, Dordrecht (NL), 1990.
[18] S. S. Schweber, *Relativistic Quantum Field Theory*, Harper and Row, New York 1961.
[19] F. A. Berezin, *The Method Of Second Quantization*, Academic Press, New York, 1966.
[20] T. Hurth, Higgs-free Massive Nonabelian Gauge Theories, preprint ZU-TH-25-95, hep-th-9511176.
[21] R. Delbourgo, S. Twisk, G. Thompson, *Int. J. Mod. Phys.* **A3**, 435 (1988).
[22] A. Burnel, *Phys. Rev.* **D33**, 2981 (1986).
[23] A. Burnel, *Phys. Rev.* **D33**, 2985 (1986).
[24] M. Pawlowski, R. Raczka, A Higgs-Free Model For Fundamental Interactions, Part I, preprint ILAS/EP-3-1995, hep-ph-9503269.
[25] M. Pawlowski, R. Raczka, A Higgs-Free Model For Fundamental Interactions, Part II, preprint Ilas/EP-4-1995, hep-ph-9503270.
[26] F. Krahe, A Causal Approach to Massive Yang Mills Theories, prepint DIAS-STP-95-02, hep-th-9508038.