# QUANTUM GAUGE THEORIES AND NONCOMMUTATIVE GEOMETRY*

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I review results from recent investigations of anomalies in fermion — Yang –Mills systems in which basic notions from noncommutative geometry (NCG) where found to appear naturally. The general theme is that derivations of anomalies from quantum field theory lead to objects which have a natural interpretation as generalization of de Rham forms to NCG, and that this allows a geometric interpretation of anomaly derivations which is useful *e.g.* for making these calculations efficient. This paper is intended as selfcontained introduction to this line of ideas, including a review of some basic facts about anomalies. I first explain the notions from NCG needed and then discuss several different anomaly calculations: Schwinger terms in 1+1 and 3+1 dimensional current algebras, Chern–Simons terms from effective fermion actions in arbitrary odd dimensions. I also discuss the descent equations which summarize much of the geometric structure of anomalies, and I describe that these have a natural generalization to NCG which summarize the corresponding structures on the level of quantum field theory.

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## 1. Introduction

In the last 15 years or so spectacular progress in 1+1 dimensional quantum field theory has been made. This was closely related to developments in mathematics, especially the representation theory of infinite dimensional Lie algebras (affine Kac–Moody algebras, Virasoro algebra). It also was understood already quite some time ago that there are interesting relations to noncommutative geometry (NCG), a mathematical discipline due to Connes

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unifying and extending many results from geometry and functional analysis. Unfortunately the case of interest to particle physics: 3+1 dimensions, is much more complicated, and a picture similarly complete as in 1+1 dimensions is far from complete. Nevertheless there has been progress [MR, M1, L1], and there are now several examples that NCG provides useful mathematical tools also for higher dimensional quantum field theory [LM1, LM2, LM3].

In this contribution I try to explain in a simple way how NCG appears in quantum gauge theory and in what way it suggests efficient calculation tools there. I also review [L4] in which all Yang–Mills anomalies (Chern–Simons terms, axial anomalies, Gauss law anomalies *etc.* in all possible dimensions) and the descent equations connecting them [Z] are generalized to NCG, and realizations of all cyclic cocycles [C] through these generalized anomalies were found.

I should point out that there are several other recent, interesting results on NCG in connection with particle physics which I do not discuss here. Most of them concern gauge theory models on a classical (non-quantized) level. My concern is how to use NCG to make progress in understanding the *quantum* structure of gauge theories with all its problems like divergences, anomalies *etc.* Also when I refer to NCG I do not mean the full mathematical framework with all its deep results as *e.g.* summarized in a recent book by Connes [C] but rather a few notions and special examples within that general framework which can be understood without going in depth into the mathematical theory. As I see it (as far the the examples discussed here are concerned), NCG has provided a very useful way of looking at mathematical structures which appear in quantum field theory but were partly known to (mathematical) physicists before. The known examples demonstrate that NCG provides a very powerful language for these kinds of problems, and new insights in quantum field theory can be hoped for in the future.

The plan of this paper is as follows. In Section 2 I discuss the main notions from NCG needed. I then describe the construction of fermion current algebras in 1+1 and 3+1 dimensions and explain how NCG naturally appears there (Section 3). In Section 4 the generalization of the descent equations to NCG is explained, and a simple example of an anomaly calculation using ideas from NCG is also outlined there. Section 5 contains a few final remarks.

My discussion is restricted to explaining results of our group at KTH in Stockholm. I would like to thank Jouko Mickelsson for a very pleasant collaboration.

## 2. NCG: a few basic ideas and examples

A basis idea in NCG is to try to generalize notions and results form differential geometry to more general situations without underlying manifolds. The strategy for this is along the following lines: considering some manifold $M$, it is possible to encode the geometric information about $M$ in terms of the algebra $\mathcal{A}$ of complex valued functions on $M$ (product given by pointwise multiplication). Such an algebra $\mathcal{A}$ is commutative. Reformulating notions and results in ordinary geometry using only properties of $\mathcal{A}$ (and making no reference to the underlying manifold $M$), it is then often possible to generalize these through replacing $\mathcal{A}$ by some other, in general noncommutative, algebra $\hat{\mathcal{A}}$. The algebra $\hat{\mathcal{A}}$ is often naturally realized by operators on some Hilbert space $\mathcal{H}$ *i.e.* is a subalgebra of the bounded operators $B(\mathcal{H})$ acting on $\mathcal{H}$.

**De Rham forms.** Let me give an example for an important concept in geometry generalized in NCG, namely de Rham forms. Consider the manifold $\mathbb{R}^d$ (for simplicity) and the algebra $\mathcal{A} = C_0^\infty(\mathbb{R}^d)$ of smooth, compactly supported and $\mathbb{C}$–valued maps $X$ on $\mathbb{R}^d$. Then exterior differentiation d can be defined on $\mathcal{A}$ as usual, $\mathrm{d}X(x) = \partial^i X(x)\mathrm{d}x_i$ ($x = (x_1, \ldots, x_d)$, $x_i \in \mathbb{R}$, is a point in $\mathbb{R}^d$, and $\partial^i = \partial/\partial x_i$; summations in $i$ from 1 to $d$ understood). With that one can define de Rham forms on $\mathbb{R}^d$: Elements in $\mathcal{A}$ are defined to be 0–forms, 1–forms are linear combinations of elements $X_0\mathrm{d}X_1, \ldots$, $n$–forms linear combinations of elements $X_0\mathrm{d}X_1 \cdots \mathrm{d}X_n$ ($X_i \in \mathcal{A}$; the product here is the wedge product). Then the (wedge) product $\omega_n\omega_m$ of a $n$ form $\omega_n$ and a $m$–form $\omega_m$ is naturally defined and is a $(n+m)$–form. A crucial property is $\mathrm{d}^2 = 0$ (this is how d is defined), and this allows to naturally extend the definition of d to all $n$–forms $\omega_n$ such that $\mathrm{d}\omega_n$ is a $(n+1)$–form. There is another useful operation on these de Rham forms, namely integration: a $d$–form $\omega_d$ can always be written as $f(x)\mathrm{d}x_1 \cdots \mathrm{d}x_d$ with $f(x)$ in $C_0^\infty(\mathbb{R}^d)$, thus the integral $\int \omega_d$ can be naturally defined as $\int_{\mathbb{R}^d} f(x)\mathrm{d}x_1 \cdots \mathrm{d}x_d$. This definitions can be extended to all $n$–forms by setting $\int \omega_n = 0$ for all $n \neq d$. One can then show many nice relations, like $\mathrm{d}(\omega_n\omega_m) = \mathrm{d}(\omega_n)\omega_m + (-)^n\omega_n(\mathrm{d}\omega_m)$ (graded Leibniz rule), $\int \omega_n\omega_m = (-)^{nm} \int \omega_m\omega_n$ (cyclicity), $\int \mathrm{d}\omega_n = 0$ (Stokes' theorem) *etc.*

**Generalizing de Rham forms to NCG.** By inspection one can convince oneself that the algebra $\mathcal{A}$ being commutative or the fact that there is a manifold $\mathbb{R}^d$ are not essential in this example. It is therefore easy to generalize it to the following situation:[1] there is some given algebra $\hat{\mathcal{A}}$ and some linear mapping $\hat{\mathrm{d}}$ on $\hat{\mathcal{A}}$ (mapping $\hat{\mathcal{A}}$ to some vector space) such that $\hat{\mathrm{d}}^2 = 0$. Then one can construct the vector space $\hat{\mathcal{C}}$ which is the linear span

[1] throughout the paper a hat indicates generalizations to NCG

of monomials $u_0\hat{\mathrm{d}}u_1\cdots\hat{\mathrm{d}}u_n$ $(u_i\in\hat{\mathcal{A}})$, and this is naturally an algebra: for example $(v_i\in\hat{\mathcal{A}})$

$$(u_0\hat{\mathrm{d}}u_1)(v_0\hat{\mathrm{d}}\hat{v}_1\cdots\hat{\mathrm{d}}v_n)\equiv u_0\hat{\mathrm{d}}(u_1v_0)\hat{\mathrm{d}}v_1\cdots\hat{\mathrm{d}}v_n-(u_0u_1)\hat{\mathrm{d}}v_0\hat{\mathrm{d}}v_1\cdots\hat{\mathrm{d}}v_n$$

*etc.* (postulating a graded Leibniz rule for $\hat{\mathrm{d}}$ and using $\hat{\mathrm{d}}^2=0$). Moreover, setting

$$\hat{\mathrm{d}}(u_0\hat{\mathrm{d}}u_1\cdots\hat{\mathrm{d}}u_n)\equiv\hat{\mathrm{d}}u_0\hat{\mathrm{d}}u_1\cdots\hat{\mathrm{d}}u_n$$

naturally defines $\hat{\mathrm{d}}$ as linear mapping on $\hat{\mathcal{C}}$ such that $\hat{\mathrm{d}}^2=0$. This algebra $\hat{\mathcal{C}}$ together with the map $\hat{\mathrm{d}}$ has most of the algebraic properties of the de Rham forms on $\mathbb{R}^d$ and is what is called a *graded differential algebra* (GDA): it is a graded algebra, $\hat{\mathcal{C}}=\bigoplus_{n=0}^{\infty}\hat{\mathcal{C}}^{(n)}$ where $\hat{\mathcal{C}}^{(n)}$ is the vector space generated by elements $u_0\hat{\mathrm{d}}u_1\cdots\hat{\mathrm{d}}u_n$ $(u_i\in\hat{\mathcal{A}})$, $\hat{\mathrm{d}}$ maps $\hat{\mathcal{C}}^{(n)}$ to $\hat{\mathcal{C}}^{(n+1)}$, $\hat{\mathrm{d}}^2=0$, $\hat{\omega}_n\in\hat{\mathcal{C}}^{(n)}$ and $\hat{\omega}_m\in\hat{\mathcal{C}}^{(m)}$ implies that $\hat{\omega}_n\hat{\omega}_m$ is in $\hat{\mathcal{C}}^{(n+m)}$, the graded Leibniz rule holds *etc.* It is thus natural to also denote elements in $\hat{\mathcal{C}}^{(n)}$ as $n$–forms and $\hat{\mathrm{d}}$ as exterior differentiation. In some cases is is also possible to naturally define an integration $\hat{\int}$ *i.e.* a linear map $\hat{\mathcal{C}}\to\mathbb{C}$ such that cyclicity and Stokes' theorem holds. The triple $(\hat{\mathcal{C}},\hat{\mathrm{d}},\hat{\int})$ is what is called *cycle* by Connes.[2] If $\hat{\int}$ is concentrated on some $\hat{\mathcal{C}}^{(d)}$ (*i.e.* $\hat{\int}\hat{\omega}_n=0$ for all $n$–forms except $n=d$) it is possible to naturally assign a dimension to a cycle *viz.* dimension$=d$.

**Example 1.** This first example is important for Yang–Mills theory. Even though it starts with a noncommutative algebra, there is still an underlying manifold. I thus regard it as example for ordinary geometry which will then be generalized to NCG in Example 2 below.

Take as $\hat{\mathcal{A}}=\mathcal{A}_d$ the algebra of all complex $N\times N$ matrix–valued functions on $\mathbb{R}^d$ which are smooth and compactly supported, $\mathcal{A}_d=C_0^d(\mathbb{R}^d;\mathrm{gl}_N)$ (the product is pointwise matrix multiplication, $\mathrm{gl}_N$ is the complex algebra of $N\times N$ matrices). Since matrix multiplication is not commutative, $\mathcal{A}_d$ is not commutative, either. However, exterior differentiation d is naturally extended to $\mathrm{gl}_N$–valued functions such that $\mathrm{d}^2=0$. I denote as $\mathcal{C}_d$ the the GDA constructed from these data as explained above. There is also a natural integration $\int$: for all $d$–forms $\omega_d=f(x)\mathrm{d}x_1\cdots\mathrm{d}x_d$ with $f(x)$ a $\mathrm{gl}_N$ valued function on $\mathbb{R}^d$, thus $\int\omega_n$ can be defined as $\int_{\mathbb{R}^d}\mathrm{tr}_N(\omega_n(x))$ (ordinary integration) where $\mathrm{tr}_N$ is the usual $N\times N$ matrix trace. With that, $(\mathcal{C}_d,\mathrm{d},\int)$ naturally becomes a $d$–dimensional cycle. I will refer to it as (generalized) de Rham cycle.

**Example 2.** Consider a separable Hilbert space $\mathcal{H}$ which is decomposed in two orthogonal subspaces, $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$. The operator $F$ which is $\pm1$ on

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[2] see *e.g.* [C] p.183; my definition is slightly more general

$\mathcal{H}_\pm$ is a grading operator, $F^* = F^{-1} = F$, and $\mathcal{H}_\pm = \frac{1}{2}(1 \pm F)\mathcal{H}$ ($*$ is the Hilbert space adjoint). Then for all $u \in B = B(\mathcal{H})$ (bounded operators on $\mathcal{H}$) one can define

$$\hat{\mathrm{d}}u \equiv \mathrm{i}[F, u] \tag{1}$$

($[a, b] \equiv ab - ba$). Taking $\hat{\mathcal{A}} = B$ and this $\hat{\mathrm{d}}$ one can construct a GDA. Thus $n$–forms $\hat{\omega}_n$ are linear combinations of operators $(\mathrm{i})^n u_0[F, u_1] \cdots [F, u_n]$ ($u_i \in B$), and it is easy to see that $\hat{\mathrm{d}}\hat{\omega}_n = \mathrm{i}\,(F\hat{\omega}_n - (-)^n\hat{\omega}_n F)$ (check that $\hat{\mathrm{d}}^2 = 0$!).

It is now useful to restrict this GDA as follows: Denote as $B_1$ the trace class operators on $\mathcal{H}$ and as $B_p = \{a \in B | (a^*a)^{p/2} \in B_1\}$ (these are the so–called Schatten classes). I will need a few basic properties of these operator classes [S]: $B_p \subset B_q$ for $p < q$; $a \in B$, $b \in B_p$ and $c \in B_q$ implies $ab$ and $ba$ are in $B_p$ and $bc$ is in $B_r$ where $1/r = 1/p + 1/q$; moreover, the Hilbert space trace $\mathrm{Tr}\,(a)$ is only defined if $a \in B_1$, and $\mathrm{Tr}\,(bc) = \mathrm{Tr}\,(cb)$ if $1/p + 1/q \geq 1$. We can now define the following subalgebras of $B$,

$$\hat{\mathcal{A}}_d \equiv \{u \in B \mid [F, u] \in B_{d+1}\} = \hat{\mathcal{A}}_d(\mathcal{H}; F)\,, \tag{2}$$

where $d$ is a positive integer. (The reason for denoting this integer as $d$ will be explained below.)

Take as $\hat{\mathcal{A}} = \hat{\mathcal{A}}_d$ and $\hat{\mathrm{d}}$ as in (1). I denote as $(\hat{\mathcal{C}}_d, \hat{\mathrm{d}})$ the GDA constructed from these data. An integration $\hat{\int}$ can then be defined as follows ($\hat{\omega}_n \in \hat{\mathcal{C}}_d^{(n)}$),

$$\hat{\int} \hat{\omega}_n = \begin{cases} \mathrm{Tr}_C \left(\Gamma^{d-1}\hat{\omega}_n\right) & \text{for n= d} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

($\Gamma^{d-1} = \Gamma$ for $d$ even and 1 for $d$ odd) where $\Gamma$ is a grading operator on $\mathcal{H}$ such that $F\Gamma = -\Gamma F$ and $\mathrm{Tr}_C\,(a) \equiv \frac{1}{2}\mathrm{Tr}\,(a + FaF)$ is a conditional Hilbert space trace[3] (check that (3) is always well defined!). Cyclicity is obvious, and also Stokes' theorem holds. Thus $(\hat{\mathcal{C}}_d, \hat{\mathrm{d}}, \hat{\int})$ is a $d$–dimensional cycle[4].

This latter cycle is important since it provides a natural generalization of the de Rham forms discussed in Example 1: there is a natural embedding of $(\mathcal{C}_d, \mathrm{d}, \int)$ in $(\hat{\mathcal{C}}_d, \hat{\mathrm{d}}, \hat{\int})$ which I now describe. As discussed in the next Section, it is this embedding which naturally appears in quantum field theory.

Consider the Hilbert space $\mathcal{H}_d \equiv L^2(\mathbb{R}^d) \otimes \mathbb{C}^\nu \otimes \mathbb{C}^N$ where $\nu = 2^{[d/2]}$ ($[d/2]$ is equal to $d/2$ for $d$ even and $(d-1)/2$ for $d$ odd; thus elements in $\mathcal{H}_d$

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[3] $\mathrm{Tr}_C$ is denoted as $\mathrm{Tr}'$ by Connes, see *e.g.* [C], p. 293

[4] $(\hat{\mathcal{C}}_d, \hat{\mathrm{d}}, \hat{\int})$ is called *cycle associated with the* $(d+1)$*–summable Fredholm module* by Connes, see *e.g.* [C], p. 292.

are functions $f_{\sigma,n}(x)$ on $\mathbb{R}^n$ carrying a 'spin index' $\sigma = 1, \ldots, \nu$ and a 'color index' $n = 1, \ldots, N$). Physically one can interpret this as Hilbert space of fermions on $\mathbb{R}^d$, and then it is natural to consider the free Dirac operator $\not{D}_0 = \gamma_i(-\mathrm{i}\partial^i)$ acting on $\mathcal{H}_d$; the matrices $\gamma_i$ act on $\mathbb{C}^\nu = \mathbb{C}^\nu_{spin}$ and are the usual $\nu \times \nu$ $\gamma$–matrices obeying $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}$. For even $d$ there exists an additional $\gamma$–matrix $\gamma_{d+1}$ which will be also needed. The Dirac operator naturally defines a grading operator $\varepsilon = |\not{D}_0|^{-1}\not{D}_0$ (zero modes of $\not{D}_0$ are taken care of if one sets $|x|^{-1}x = 1$ for $x \geq 0$ and $-1$ for $x < 0$). Now elements in $\mathcal{A}_d = C_0^\infty(\mathbb{R}^d; \mathrm{gl}_N)$ naturally act on $\mathcal{H}_d$ by bounded operators, namely by pointwise multiplication ($\mathrm{gl}_N$ acts on $\mathbb{C}^N = \mathbb{C}^N_{\mathrm{color}}$), and this defines an embedding of $\mathcal{A}_d$ in $B(\mathcal{H}_d)$. (For simplicity I will use the same symbol $X$ for an element in $\mathcal{A}_d$ and the corresponding operator in $B(\mathcal{H}_d)$.) Take $F = \varepsilon$. The embedding of $\mathcal{A}_d$ in $\hat{\mathcal{A}}_d$, *i.e.*

$$X \in \mathcal{A}_d \Rightarrow X \in \hat{\mathcal{A}}_d(\mathcal{H}_d; \varepsilon)\,, \tag{4}$$

can be proved by an explicit (not quite trivial) calculation (see *e.g.* [MR]). Moreover, $\hat{\int}$ generalizes integration of de Rham forms, *i.e.*

$$(\mathrm{i})^d \mathrm{Tr}_C\left(\Gamma X_0[\varepsilon, X_1]\cdots[\varepsilon, X_d]\right) = c_d \int_{\mathbb{R}} \mathrm{tr}_N\left(X_0 \mathrm{d}X_1 \cdots \mathrm{d}X_d\right) \tag{5}$$

$\forall X_i \in C_0^\infty(\mathbb{R}^d; \mathrm{gl}_N)$, with $\Gamma = 1$ for $d$ odd and $\Gamma = \gamma_{d+1}$ for $d$ even. I recently gave a proof of this quite nontrivial result determining also the constant

$$c_d = \frac{(2\mathrm{i})^{[d/2]} 2\pi^{d/2}}{d(2\pi)^d \Gamma(d/2)} \tag{6}$$

[L3]; my method of proof of this quite fundamental result in NCG (namely that $\mathrm{Tr}_C$ is a noncommutative generalization of integration of de Rham forms) is by direct calculation and was motivated by quantum field theory calculations discussed in the next Section.

I finally note that there is also a natural generalization of *partial integration* to arbitrary GDA which generalizes the notion of integration of de Rham forms over submanifolds to NCG [L4].

## 3. NCG and quantum field theory

Dimensional regularization [tHV] seems to be an early example for NCG being relevant for quantum gauge theories: it is done by formally considering the model on a $(4-\epsilon)$–dimensional spacetime $M^{4-\epsilon}$ which, of course, is not a manifold. It is tempting to conjecture that a deeper understanding of $M^{4-\epsilon}$, and thus dimensional regularization, should be possible in the framework of NCG.

In the following I discuss better understood examples for NCG appearing in quantum field theory.

**Fermion quantum field theory.** It has been realized already more than 20 years ago that several mathematical questions in quantum field theory (QFT) can be studied on a general, abstract level [SS] which turned out to be similar in the spirit to NCG. For many purposes, the appropriate QFT setting for relativistic[5] fermions can namely be completely characterized by, (*i*) the Hilbert space $\mathcal{H}$ of 1–particle states, (*ii*) a grading operator $F$ providing the splitting of $\mathcal{H}$ in positive– and negative energy subspaces, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with $F\mathcal{H}_\pm = \pm\mathcal{H}_\pm$. Then $\mathcal{H}$ determines the field algebra $CAR(\mathcal{H})$ of fermion fields: it can be constructed as $C^*$–algebra generated by elements $\psi^*(f)$ and $\psi(f) = \psi^*(f)^*$, $f \in \mathcal{H}$, such that $f \mapsto \psi^*(f)$ is linear and the canonical anticommutator relations (CAR) hold, $(\psi^*(f) + \psi(g))^2 = (f, g)$ (= inner product of $f, g \in \mathcal{H}$); moreover, $||\psi^*(f)|| = (f, f)^{1/2}$. Then $F$ determines the appropriate representation of $CAR(\mathcal{H})$ which corresponds to filling the negative energy states ('Dirac sea') so as to get a positive many particle Hamiltonian. This setting is appropriate for models for fermions in external fields characterized by a 1–particle Hamiltonian given by a self–adjoint operator $D$ acting on $\mathcal{H}$: then the choice $F = |D|^{-1}D$ guarantees that the many particle Hamiltonian is positive definite.

In this setting one can develop a general theory of Bogoliubov transformations [SS] (my description here is closer to Ref. [R]): such a transformation $\psi^*(f) \to \psi^*(Uf)$ is always given by a unitary operator $U \in B(\mathcal{H})$, and there is a unitary many particle operator $\Gamma(U)$ implementing it, $\psi^*(Uf) = \Gamma(U)^*\psi^*(f)\Gamma(U)$ $\forall f \in \mathcal{H}$, if and only if

$$[F, U] \in B_2\,, \tag{7}$$

this is the well–known *Hilbert-Schmidt condition.* In my notation above, (7) is equivalent to $U \in \hat{\mathcal{A}}_1$. Thus this is an early example where these operator algebras $\hat{\mathcal{A}}_d$, which now play a fundamental role in NCG, appear in QFT.

**Current algebras in 1+1 dimensions.** Lundberg [Lb] studied fermion currents in the general, abstract setting above (for a detailed presentation of this formalism I recommend [CR]): Given a 1–particle observable *i.e.* self–adjoint operator $u \in B(\mathcal{H})$, one tries to construct the corresponding many particle observable $\mathrm{d}\Gamma(u)$ which should obey $[\mathrm{d}\Gamma(u), \psi^*(f)] = \psi^*(uf)$ $\forall f \in \mathcal{H}$. These observables $\mathrm{d}\Gamma(u)$ are called *currents* since for operators $u$ given by a (possibly matrix valued) integral kernel, $(uf)(x) = \int \mathrm{d}y\, u(x, y)f(y)$

[5] by this I mean that there is a 1–particle Hamiltonian not bounded from below

$\forall f \in \mathcal{H}$, one formally has $\mathrm{d}\Gamma(u) = :\int \mathrm{d}x\, \psi^*(x)u(x,y)\psi(y):$ , where the double dots symbolize normal ordering and the $\psi^*(x)$ are fermions fields (*i.e.* operator valued distributions such that $\psi^*(f) = \int \mathrm{d}x\, \psi^*(x)f(x)$) as considered usually by physicists. Again one can prove that these currents $\mathrm{d}\Gamma(u)$ exists as self-adjoint many particle operator if and only if the Hilbert–Schmidt condition $u \in \hat{\mathcal{A}}_1$ holds. Moreover, Lundberg found that these currents obey the relations

$$[\mathrm{d}\Gamma(u), \mathrm{d}\Gamma(v)] = \mathrm{d}\Gamma([u,v]) + \hat{S}_1(u,v) \quad \forall u,v \in \hat{\mathcal{A}}_1\,, \tag{8}$$

where[6]

$$\hat{S}_1(u,v) = \tfrac{1}{2}\mathrm{Tr}_C\,(u[F,v])\,. \tag{9}$$

This second term here is a $\mathbb{C}$–number which results from normal ordering. In the physics literature such a term is usually referred to as *Schwinger term.* This term (9) is a *2–cocycle* from a mathematical point of view: it has to obey non–trivial cocycle relations due to basic properties of the commutator $[\cdot,\cdot]$ *i.e.* change of sign under exchange of arguments (= antisymmetry) and the Jacobi identity. Since normal ordering of the currents $\mathrm{d}\Gamma(u)$ is not unique but can be modified by finite terms, $\mathrm{d}\Gamma(u) \to \mathrm{d}\Gamma(u) - b(u)$, this cocycle is only unique up to trivial terms and can be modified, $\hat{S}_1(u,v) \to \hat{S}_1(u,v) + b([u,v]) \sim \hat{S}_1(u,v)$ where $b : \hat{\mathcal{A}}_1 \to \mathbb{C}$ is some (smooth) linear function. Such a trivial term $b([u,v])$ is usually denoted as *coboundary.* It is thus only the cohomology class of this cocycle (= equivalence class under $\sim$ above) which is important here, at least as far as ultraviolet divergences are concerned; $\hat{S}_1$ can be shown to be nontrivial (*i.e.* there is no normal ordering prescription yielding no Schwinger term). There might be other reasons to choose some specific normal ordering prescription, however. For example, the normal ordering prescription leading to the form (9) of this cocycle is quite special since only in this form its interpretation in the framework of NCG, as explained below, becomes obvious.

These general, abstract results above are directly applicable only in 1+1 dimensions. For fermions on spacetime $\mathbb{R}^d \times \mathbb{R}$ coupled to external Yang–Mills fields (*e.g.*), one is interested in the special case $\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathbb{C}^\nu_{spin} \otimes \mathbb{C}^N_{color}$ equal to our $\mathcal{H}_d$ above (to be specific, I assume that the gauge groups is $\mathrm{SU}(N)$ in the fundamental representation); if the (time independent) external Yang–Mills field is $A = A^i \mathrm{d}x_i$, then the appropriate 1–particle operator is $D = \not{D}_0 + \not{A}$ where $\not{A} = \gamma_i A^i$, and $F = |D|^{-1}D$. Then an interesting class of observables are the generators of gauge transformations which, on the 1–particle level, are given by elements in $u = X \in C_0^\infty(\mathbb{R}^d; \mathrm{gl}_N)$. Now as discussed above for $A = 0$ (Eq. (4)) — and one can show that this is

[6] Lundberg's formula looks different but is equivalent to this

true also for $A \neq 0$ — such infinitesimal gauge transformation obey the Hilbert–Schmidt condition only for $d = 1$, and only in this case Lundberg's construction applies and provides the currents $\rho(X) = \mathrm{d}\Gamma(X)$ which, in this case, are just the time component of the chiral fermion current smeared by the test function $X$. The Schwinger term (9) in this case becomes

$$S_1(X,Y) = -\frac{\mathrm{i}}{2\pi} \int_{\mathbb{R}} \mathrm{tr}_N\,(X\mathrm{d}Y)\ . \tag{10}$$

To show this involves a nontrivial calculation. After our discussion in the last Section this is, however, trivial since $\hat{S}_1(X,Y) = S_1(X,Y)$ for $X, Y \in \mathcal{A}_1$ is just is a special case of (5).

Note that the relations

$$[\rho(X), \rho(Y)] = \rho([X,Y]) + S(X,Y) \quad \forall X, Y \in \mathcal{A}_1 \tag{11}$$

define what is known as *affine Kac–Moody algebra*[7] and (10) is the well-known Kac–Moody cocycle. This algebra has been known since quite some time (see *e.g.* [BH]) and is usually called *1+1 dimensional current algebra* by physicists. The abstract current algebra (8) can be regarded as a generalization of this to NCG. Especially Lundberg's cocycle $\hat{S}_1$ corresponds is the noncommutative generalization of the Kac–Moody cocycle $S_1$. To my knowledge, this was first pointed out in [CH][8].

At this point a further remark on the role of (11) in physical models might be helpful. The fermion currents of Dirac fermions in $d+1$ dimensions formally are $j^a_\nu(x) = :\bar{\psi}(x)\gamma_\nu T^a \psi(x): = :\psi^*(x)\gamma^0\gamma_\nu T^a\psi(x):$ where $x \in \mathbb{R}$, $\nu = 0, 1, \ldots, d$, $\gamma_\nu$ are the $\gamma$–matrices, and $T^a$ generators of the Lie algebra of the gauge group which I assume as $N \times N$ matrices (here I assume a Minkowski metric with signature $(+, -, \ldots, -)$). For the present case $d = 1$, these currents can be written in terms of the chiral fermion charges, $j_0 = \rho_+ + \rho_-$ and $j_1 = \rho_+ - \rho_-$ where $\rho^a_\pm(x) = :\bar{\psi}(x)\gamma_0\frac{1}{2}(1 \pm \gamma_3)T^a\psi(x):$ are the chiral currents (note that $\gamma_3 = \gamma_0\gamma_1$). The relation (11) now is for the chiral currents $\rho = \rho_+$, and one gets similar ones for $\rho_-$ but with opposite sign of the Schwinger term. From these one obtains the commutator relations of the currents $j_\nu$ where a Schwinger terms appears only in the $j_0$-$j_1$–commutators. (See *e.g.* [LS] for a more detailed discussion of this.) More generally, the abstract formalism leading to (8) provides all 'hard' mathematical analysis necessary to construct the fermion observables

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[7] to be precise, it is the corresponding algebra on the circle $S^1$ and not on the real line $\mathbb{R}$ as here

[8] A.L. Carey pointed out to me that A. Connes was aware of this before

interesting in several 1+1 dimensional quantum field theory models, at least in case of massless fermions. In addition to the fermion currents discussed above, it also gives the energy–momentum tensor (which in special cases lead to a Virasoro algebra). A full construction of 1+1 dimensional QCD with massless quarks (space compact) in this spirit was given in [LS][9]. I also note that in this case, the requirement of gauge invariance eliminates the above mentioned freedom in the normal ordering prescription for constructing the fermion currents.

**Current algebras in higher dimensions.** From Eq. (4) it is clear that the abstract current algebra (8) is of no use in higher dimensions. This is not surprising from a physics point of view: the Schatten ideal condition (4) can be regarded as a precise characterization of ultraviolet divergences, and these are worse in higher dimensions. Lundberg's construction amounts to giving a precise mathematical meaning to normal ordering which is the only regularization necessary to deal with the ultraviolet divergences occurring in 1+1 dimensions; this is not sufficient, however, for the more severe divergences occurring in higher dimensions.

The analog of (8) for 3+1 dimensions was found by Mickelsson and Rajeev [MR, M1], and their construction can be easily extended to other dimensions [FT]. It gives a precise mathematical meaning to multiplicative regularization required in higher dimensions and can be motivated by physical considerations: consider chiral fermions coupled to external Yang–Mills fields $A$ as above. Again one takes chiral fermions as motivation since for them one expects a nontrivial Schwinger term (see below); the Schatten ideal conditions are the same for Dirac fermions, however, and the abstract construction will therefore apply to this case also.

In 1+1 dimensions the regularization of fermions currents can be chosen independent of $A$ and is always equivalent to the one for the simplest case $A = 0$. In higher dimensions this is not true and one gets currents $\rho(X;A)$ depending on $A$ through regularization. Instead of the currents one then has to consider generators of gauge transformations $G(X;A) = \mathcal{L}_X + \rho(X;A)$ where $\mathcal{L}_X$ accounts for the action of the infinitesimal gauge transformation on the Yang–Mills field,

$$\mathcal{L}_X(\cdots)(A) \equiv \frac{\mathrm{d}}{\mathrm{d}s}(\cdots)(A + s(\mathrm{d}X + \mathrm{i}[A,X]))\Big|_{s=0} \tag{12}$$

(Lie derivative). These $G(X;A)$ are usually called Gauss' law generators (their vanishing on physical states is Gauss' law).

[9] I use this occasion to note that some of the results in [LS] (not discussed here) were obtained earlier by Bos using different methods [B]. I thank R. Jackiw for pointing this out to me.

The Dirac sea corresponding to the external field $A$ is characterized by $F_A = |\not{D}_A|^{-1}\not{D}_A$ where $\not{D}_A = \not{D}_0 + \not{A}$, and it has the important property that $[\varepsilon, F_A] \in B_{d+1}$ where $\varepsilon = F_0$ (no external field) [MR]. Thus on the abstract level, it is natural to fix a grading operator $\varepsilon$ and introduce the sets

$$Gr_d = \{F = F^* = F^{-1} \in B \mid [\varepsilon, F] \in B_{d+1}\} \tag{13}$$

in addition to $\hat{\mathcal{A}}_d$. The generalization of (8) to higher dimensions is then

$$[\hat{G}(u;F), \hat{G}(v;F)] = \hat{G}([u,v];F) + \hat{S}_d(u,v;F) \quad \forall u,v \in \hat{\mathcal{A}}_d\,, \quad F \in Gr_d \tag{14}$$

and is an abstract generalization of the algebra of Gauss' law generators. For $d = 3$ Mickelsson and Rajeev found the Schwinger term

$$\hat{S}_3(u,v;F) = -\tfrac{1}{8}\mathrm{Tr}_C\left((F-\varepsilon)[[\varepsilon,u],[\varepsilon,v]]\right) . \tag{15}$$

which is a 2–cocycle due to the antisymmetry and the Jacobi identity of $[\cdot,\cdot]$ as discussed above.

It is worth noting how (8) can be interpreted as a special case of this: for $d = 1$, $\mathrm{d}\Gamma(u;F) = \mathrm{d}\Gamma(u;\varepsilon)$ can be chosen independent of $F \in Gr_1$, thus (8) is equivalent to (14) if one interprets $\hat{G}(u;F) = \mathcal{L}_u + \mathrm{d}\Gamma(u;\varepsilon)$ where $\mathcal{L}_u(\cdots)(F) \equiv \mathrm{d}(\cdots)(\mathrm{e{-}isuFeisu})/\mathrm{ids}|_{s=0}$ is the obvious generalization of the Lie derivative above (thus $\hat{S}_1(u,v;F) = \hat{S}_1(u,v)$ (10) for $F = \varepsilon$).

I note that the original construction of Eqs. (14), (15) [MR, M1] was in a framework quite different from the one discussed here. A construction in the present framework (which perhaps is more closer to physicists) was given in [L1]. It was pointed out in this paper that the fermion currents in $\hat{G}(u;F)$ are not operators but only sesquilinear forms, and that $[\cdot,\cdot]$ on the l.h.s. of Eq. (14) is not just a commutator but involves a nontrivial regularization depending on dimension.

As described, the motivation for this construction was to find an explicit field theory construction of the Gauss' law operators $G(X;A)$ for chiral fermions in the Hamiltonian framework. There were cohomological arguments [FS, M2] that these should obey the relations ($d = 3$)

$$[G(X;A), G(Y;A)] = G([X,Y];A) + S_3(X,Y;A) \quad \forall X,Y \in \mathcal{A}_3\,, \ A \in YM_3 \tag{16}$$

($YM_3 \subset \mathcal{C}^{(1)}(\mathcal{A}_3)$ is the set of all YM–connections) with a Schwinger term

$$S_3(X,Y;A) = \frac{1}{24\pi^2}\int_{\mathbb{R}^3} \mathrm{tr}_N\left(A[\mathrm{d}X, \mathrm{d}Y]\right) \tag{17}$$

which can be interpreted as manifestation of the gauge anomaly in the Hamiltonian framework. Thus (14), (15) provides as a special case these relations if $\hat{S}_3(X,Y;F_A)$ is equivalent (cohomologous) to $S_3(X,Y;A)$. This was shown by explicit calculation in [LM1]. An essential step was the insight that (15) is the natural generalization of (17) to NCG, and the rule for this generalization is

$$X \to u\,, \quad \mathrm{d}X \to \mathrm{i}[\varepsilon, u]\,, \quad A \to F - \varepsilon\,, \quad \int \mathrm{tr}_N \to \frac{1}{c_d}\mathrm{Tr}_C \qquad (18)$$

which now, after our discussion in the last Section, is quite obvious (the second rule is natural since, for a pure gauges $A = -\mathrm{i}U^{-1}\mathrm{d}U$ with $U \in \mathcal{A}_1$ ($\mathrm{SU}(N)$–valued map on $\mathbb{R}^d$), one gets $\not{D}_A = U^{-1}\not{D}_0 U$ and thus $F_A - \varepsilon = U^{-1}\varepsilon U - \varepsilon = U^{-1}[\varepsilon, U]$). Especially for pure gauges $A$, $\hat{S}_3(X,Y;F_A) = S_3(X,Y;A)$ is just a special case of (5).

## 4. Noncommutative descent equations

**Yang–Mills setting and anomalies.** On a classical (non–quantized) level, a Yang–Mills (YM) field $A$ is given by a $\mathrm{su}(N)$–valued 1–form on a manifold $M^d$ (I assume $\mathrm{su}(N)$ is the Lie algebra of the gauge group); $M^d$ can be spacetime (Euclidean framework) or space (Hamiltonian framework; time fixed). Infinitesimal gauge transformation are given by $\mathrm{su}(N)$–valued functions on $M^d$ and act on functions of $A$ by Lie derivative $\mathcal{L}_X$ (12), especially as $\mathcal{L}_X A = -\mathrm{i}\mathrm{d}X + [A, X]$. Then the components of the YM field strength, $\mathcal{F}_{ij} = \partial_i A_j - \partial_j A_i + \mathrm{i}[A_i, A_j]$, can be collected in a 2–form, $\mathcal{F}_A = -\frac{1}{2}\mathrm{i}\mathcal{F}_{ij}\mathrm{d}x^i\mathrm{d}x^j$. In compact notation, $\mathcal{F}_A = -\mathrm{i}\mathrm{d}(A) + A^2$, and $\mathcal{L}_X\mathcal{F}_A = [\mathcal{F}_A, X]$.

For $M^d = \mathbb{R}^d$, this YM setting naturally is realized in the GDA $(\mathcal{C}_d, \mathrm{d})$ of Section 2, Example 1: $A \in \mathcal{C}_d^{(1)}$ are YM fields, $X \in \mathcal{C}_d^{(0)}$ infinitesimal gauge transformation, and then $\mathcal{F}_A \in \mathcal{C}_d^{(2)}$. An important observation now is that all what was needed here are basic operations available in any GDA. Thus one can naturally define a generalized YM setting for arbitrary GDA $(\hat{\mathcal{C}}, \hat{\mathrm{d}})$: $\hat{A} \in \hat{\mathcal{C}}^{(1)}$ are (generalized) YM fields, $u \in \hat{\mathcal{C}}^{(0)}$ infinitesimal gauge transformations acting on functions of $\hat{A}$ by Lie derivative

$$\mathcal{L}_u(\cdots)(\hat{A}) \equiv \frac{\mathrm{d}}{\mathrm{d}s}(\cdots)(\hat{A} + s(\hat{\mathrm{d}}u + \mathrm{i}[\hat{A}, u]))\bigg|_{s=0}\,, \qquad (19)$$

and $\mathcal{F}_{\hat{A}} = -\mathrm{i}\hat{\mathrm{d}}(\hat{A}) + \hat{A}^2 \in \hat{\mathcal{C}}^{(2)}$ obeys then $\mathcal{L}_u\mathcal{F}_{\hat{A}} = [\mathcal{F}_{\hat{A}}, u]$ *etc.* To my opinion, this is an insight from NCG which is fundamental in applications to quantum gauge theories. It provides *e.g.* flexible regularization schemes which still are gauge invariant in a natural way. I will come back to this latter and first continue with a discussion of anomalies in the usual YM setting (see [J] for more details).

Anomalies are special topological terms which are local functions of such YM fields $A$ and infinitesimal gauge transformations $X_i$. They are integrals of de Rham forms and thus do not depend on the metric of $M^d$. The Schwinger terms $S_1$ and $S_3$ encountered in the last section are examples of anomalies[10],

$$S_1 = \int \bar{\omega}_1^2\,, \quad \bar{\omega}_1^2(X_1, X_2) = -\frac{\mathrm{i}}{2\pi}\mathrm{tr}_N\left(X_1 \mathrm{d}X_2\right) \quad (d=1)$$
$$S_3 = \int \bar{\omega}_3^2\,, \quad \bar{\omega}_3^2(X_1, X_2; A) = \frac{1}{24\pi^2}\mathrm{tr}_N\left(A[\mathrm{d}X_1, \mathrm{d}X_2]\right) \quad (d=3) \tag{20}$$

(here $\int$ is integration of $\mathbb{C}$-valued de Rham forms). There are many more examples: for all integers $n$ and $k$ such that $0 \leq 2n-k-1 \leq d$, there are special de Rham forms $\bar{\omega}^k_{2n-k-1} \in \mathcal{C}_d^{(2n-k-1)}$ depending on $k$ infinitesimal gauge transformations $X_i \in \mathcal{C}_d^{(0)}$ and $A \in \mathcal{C}_d^{(1)}$ giving rise to anomalies $\int \bar{\omega}^k_{2n-k-1}$ obeying certain cocycle relations discussed in more detail below (the integration here can be over any $2n-k-1$ dimensional boundaryless submanifold of $\mathbb{R}^d$). For example, $CS_{2n-1}(A) = \int \bar{\omega}^0_{2n-1}$ with (I ignore normalization constants here)

$$\bar{\omega}^0_{2n-1}(A) \propto \int_0^1 \mathrm{d}t\, \mathrm{tr}_N\left((\mathcal{F}_{tA})^{n-1}\, A\right) \tag{21}$$

are the *Chern–Simons terms*, $Anom_{2n-2}(X, A) = \int \bar{\omega}^0_{2n-1}$ with

$$\bar{\omega}^1_{2n-2}(X; A) \propto \int_0^1 \mathrm{d}t \sum_{n_1+n_2=n-2} \mathrm{tr}_N\left((\mathcal{F}_{tA})^{n_1}\, \mathrm{d}X\, (\mathcal{F}_{tA})^{n_2}\, A\right) \tag{22}$$

the *axial anomalies*, $S_{2n-3} = \int \bar{\omega}^2_{2n-3}$ the *Gauss' law anomalies* (Schwinger terms) *etc.*; the lower index refers to the dimension where these anomalies can be relevant.

These forms $\bar{\omega}^k_{2n-k-1}(X_1, \ldots, X_k; A)$ are very special. Firstly, they are $k$*-chains*, *i.e.* they are linear in the the $X_i$, polynomial in $A$, and antisymmetric *i.e.* they change sign under exchanges $X_i \leftrightarrow X_j$. Moreover, there are

[10] to be concrete one can assume $M^d = \mathbb{R}^d$ as before, but all formulas in the following immediately generalize to other manifolds $M^d$, of course

interesting relations among these chains which can be conveniently written in terms of an operator $\delta$ mapping $(k-1)$–chains to $k$–chains and which is defined as follows,

$$\delta f^{k-1}(X_1,\cdots,X_k;A)=\sum_{\mu=1}^{k}(-)^{\mu-1}\mathcal{L}_{X_\mu}f^{k-1}(X_1,\cdots,\not{X}_\mu,\cdots,X_k;A)$$
$$+\sum_{\substack{\nu,\mu=1\\ \nu<\mu}}^{k}(-)^{\nu+\mu}f^k([X_\nu,X_\mu],X_1,\cdots,\not{X}_\nu,\cdots,\not{X}_\mu,\cdots,X_k;A)\,; \qquad (23)$$

here $\not{X}_\mu$ means that $X_\mu$ is omitted and $\mathcal{L}_X$ is the Lie derivative (12). These relations start with the so–called *Chern character* $ch_{2n}(\mathcal{F}_A)=\mathrm{tr}_N\,(\mathcal{F}_A)^n$, which can be written as

$$\mathrm{tr}_N\,(\mathcal{F}_A)^n=\mathrm{d}\bar{\omega}^0_{2n-1} \qquad (24)$$

(exterior derivative of the Chern–Simons form). From that one derives the *descent equations* [Z]

$$\delta\bar{\omega}^{k-1}_{2n-k}+\mathrm{d}\bar{\omega}^k_{2n-k-1}=0 \qquad (25)$$

for $k=0,1,\ldots,2n$.

These relations immediately imply $\delta\int\bar{\omega}^k_{2n-k-1}=0$ which are the co-cycle relations of the anomalies mentioned. They are essential properties of anomalies [J], *e.g.* in case of the Schwinger terms they guarantee the validity of the Jacobi identity in the algebra of Gauss' law operators as discussed in the last Section.

**Descent equation in NCG.** Anomalies arise as manifestations of the non-trivial quantum nature of quantized gauge theory models, and usually are understood as remnants of regularizations necessary to deal with divergences (for review see [J]).

It is an interesting question how these terms $c^k_{2n-k-1}=\int\bar{\omega}^k_{2n-k-1}$ with their rich differential geometric structure arise from such explicit field theory calculations. These involve calculating Feynman diagrams, introducing cut–offs *etc.* and in the end exactly these local, metric independent anomalies appear. How come? To my opinion, a hint towards a general answer to this question comes from the abstract current algebras described in the last Section: the abstract Schwinger terms $\hat{S}_1$ and $\hat{S}_3$ arise from a field theory construction of fermion currents but turned out to have a natural interpretation as generalizations of the Schwinger terms $S_1$ and $S_3$ to NCG and to which they becomes equal in a special case. It thus seems that here, the differential geometric structure of anomalies is already present of the level of Hilbert space operators on which the regularizations are performed, and NCG seems to provide the natural language for making this precise. I believe that this is true for all field theory derivations of anomalies.

This suggests that all anomalies $c^k_{2n-k-1}$ have similar generalizations to NCG. We first note that the notation of $k$–chains and the operator $\delta$ (23) naturally generalize to arbitrary GDA. One natural conjecture for a generalization

$$c^k_{2n-k-1} = \int \bar{\omega}^k_{2n-k-1}(X_1, \ldots, X_k; A) \to \hat{c}^k_{2n-k-1}$$

then is to use (18). However, it is highly non–trivial to check that the resulting generalized anomalies indeed obey the appropriate cocycle relations $\delta\hat{c}^k_{2n-k-1} = 0$ (a proof of this by direct calculation is already quite lengthy for $\hat{S}_3 = \hat{c}^2_3$ [L1]).

Recently I found a generalization of the descent equations (25) to every GDA $(\hat{\mathcal{C}}, \hat{\mathrm{d}})$ which then provides all generalizations of anomalies to NCG which automatically obey the cocycle relations. This result is very general since it not only applies to the GDA $(\hat{\mathcal{C}}, \hat{\mathrm{d}})$ of Example 2 in Section 2, but if fact to any other one which one might find useful in the future for performing gauge invariant regularizations in quantum gauge theories.

The starting point was the observation made above that every GDA naturally gives rise to a generalized YM setting: $A \in \hat{\mathcal{C}}^{(1)}$ can be regarded as generalized YM fields, $X_i \in \hat{\mathcal{C}}^{(0)}$ as gauge transformations acting by Lie derivative (19) *etc.* For general GDA, however, the analog of the matrix trace $\mathrm{tr}_N$ does not exist, thus I first tried to recast (25) in a form not using $\mathrm{tr}_N$. This turned out to be the crucial step. Instead of the $\mathbb{C}$–valued de Rham forms $\bar{\omega}^k_{2n-k-1}$ I had to find descent equations for the $\mathrm{gl}_N$–valued forms $\omega^k_{2n-k-1}$ without $\mathrm{tr}_N$ taken (*i.e.* $\bar{\omega}^k_{2n-k-1} = \mathrm{tr}_N\omega^k_{2n-k-1}$). These involve additional terms which vanish under $\mathrm{tr}_N$, and they have a natural generalization to all GDA.

Before writing down the generalized descent equations, I would like to illustrate them by giving a detailed proof for the generalization of (24) to NCG. For $n$–forms $\omega_n$, the distinction of $\hat{\mathrm{d}}\hat{\omega}_n$ from $\hat{\mathrm{d}}(\hat{\omega}_n)$ is important in the following since I interpret the former as operator $\hat{\mathrm{d}}(\hat{\omega}_n) + (-)^n\hat{\omega}_n\hat{\mathrm{d}}$ acting according to the graded Leibniz rule.

For arbitrary GDA one can construct $ch_{2n}(\hat{A}) = \left(\mathcal{F}_{\hat{A}}\right)^n$ and write this as $\int_0^1 \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}_{t\hat{A}}\right)^n$ where $\mathcal{F}_{t\hat{A}} = -\mathrm{i}t\hat{\mathrm{d}}(\hat{A}) + t^2\hat{A}^2$. Then $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{t\hat{A}} = -\mathrm{i}\hat{\mathrm{d}}(\hat{A}) + 2t\hat{A}^2 = \{-\mathrm{i}\hat{\mathrm{d}} + t\hat{A}, \hat{A}\}$ ($\{a, b\} \equiv ab + ba$), thus $\left(\mathcal{F}_{\hat{A}}\right)^n = \int_0^1 \mathrm{d}t \sum_{m=0}^{n-1} \left(\mathcal{F}_{t\hat{A}}\right)^{n-1-m} \{-\mathrm{i}\hat{\mathrm{d}} + t\hat{A}, \hat{A}\} \left(\mathcal{F}_{t\hat{A}}\right)^m$ . Using now that $\mathcal{F}_{t\hat{A}}$ commutes with

$-\mathrm{i}\hat{\mathrm{d}}+t\hat{A}$ (since $\mathcal{F}_{t\hat{A}} = (-\mathrm{i}\hat{\mathrm{d}}+t\hat{A})^2$) I obtain $\left(\mathcal{F}_{\hat{A}}\right)^n = \int_0^1 \mathrm{d}t\,\{\hat{\mathrm{d}} + \mathrm{i}t\hat{A}, \nu^0_{2n-1}\}$, where

$$\nu^0_{2n-1}(A) = -\mathrm{i}\sum_{m=0}^{n-1} \left(\mathcal{F}_{t\hat{A}}\right)^{n-1-m} \hat{A}\left(\mathcal{F}_{t\hat{A}}\right)^m . \tag{26}$$

I thus get

$$\left(\mathcal{F}_{\hat{A}}\right)^n = \hat{\mathrm{d}}\left(\omega^0_{2n-1}\right) + \mathrm{i}\{\hat{A}, \tilde{\omega}^0_{2n-1}\}, \tag{27}$$

where

$$\omega^0_{2n-1} = \int_0^1 \mathrm{d}t\,\nu^0_{2n-1}, \quad \tilde{\omega}^0_{2n-1} = \int_0^1 \mathrm{d}t\,t\,\nu^0_{2n-1} \tag{28}$$

which generalizes the starting point (24) of the descent equations and is valid for arbitrary GDA. (I used $\{\hat{\mathrm{d}}, \omega^0_{2n-1}\} = \hat{\mathrm{d}}\left(\omega^0_{2n-1}\right)$). $\omega^0_{2n-1}$ is thus the natural generalization of the Chern-Simons form to NCG. As promised, the additional term as compared to (24) vanishes under $\mathrm{tr}_N$ in case of the GDA of Example 1 in Section 2. More generally, it vanishes under partial integrations $\hat{\int}_{part}$ (all one needs for this is cyclicity of $\hat{\int}_{part}$).

To proceed one can calculate $\delta\omega^0_{2n-1}$. Similarly as above one can get by direct calculation that

$$\delta\omega^0_{2n-1}(u;A) + \hat{\mathrm{d}}\left(\omega^1_{2n-2}\right) = \mathrm{i}[\omega^0_{2n-1}, u] - \mathrm{i}[\hat{A}, \tilde{\omega}^1_{2n-2}],$$

where $\omega$ and $\tilde{\omega}$ are given by a formula similar to (28) with $\nu^1_{2n-2}$ proportional to

$$\mathrm{i}(t-1)\sum\Big(\left(\mathcal{F}_{t\hat{A}}\right)^{n_1}\hat{\mathrm{d}}(u)\left(\mathcal{F}_{t\hat{A}}\right)^{n_2}\hat{A}\left(\mathcal{F}_{t\hat{A}}\right)^{n_3} - \left(\mathcal{F}_{t\hat{A}}\right)^{n_1}\hat{A}\left(\mathcal{F}_{t\hat{A}}\right)^{n_2}\hat{\mathrm{d}}(u)\left(\mathcal{F}_{t\hat{A}}\right)^{n_3}\Big);$$

the sum here is over all positive integers $n_\nu$ such that $n_1 + n_2 + n_3 = n-2$ (I ignore normalization constants here for simplicity; they can be found in [L4]). This gives the generalized axial anomaly. We now observe that this can be compactly written as

$$\nu^1_{2n-2} \propto \int \mathrm{d}\theta_1 \mathrm{d}\theta_0 \left(\mathcal{F}_{t\hat{A}} + \theta_0\hat{A} - \mathrm{i}(t-1)\theta_1\hat{\mathrm{d}}(u)\right)^n,$$

where the $\theta_\nu$ are Grassmann numbers and $\int \mathrm{d}\theta_\nu$ Grassmann integrations as usual.

At this point it is natural to conjecture that

$$\omega^k_{2n-k-1} \propto \int_0^1 \mathrm{d}t \int \mathrm{d}\theta_0 \cdots \mathrm{d}\theta_k \left(\mathcal{F}_{t\hat{A}} + \theta_0\hat{A} - \mathrm{i}(t-1)\sum_{\nu=1}^{k}\theta_\nu\hat{\mathrm{d}}(u_\nu)\right)^n \tag{29}$$

with $\theta_\nu$ Grassmann numbers, which indeed turns out to be correct for $k = 0, 1, \ldots, n-1$. In this case one can prove [L4]

$$\delta\omega^{k-1}_{2n-k} + \hat{\mathrm{d}}\left(\omega^{k}_{2n-k-1}\right) = \begin{cases} \mathcal{J}\omega^{k-1}_{2n-k} - \mathrm{i}[\hat{A}, \tilde{\omega}^{k}_{2n-k-1}] & \text{for k odd} \\ \mathcal{J}\omega^{k-1}_{2n-k} - \mathrm{i}\{\hat{A}, \tilde{\omega}^{k}_{2n-k-1}\} & \text{for k even} \end{cases} \tag{30}$$

(as above, the formulas for the $\tilde{\omega}$ are obtained from the ones for $\omega$ by replacing $\int_0^1 \mathrm{d}t$ through $\int_0^1 \mathrm{d}t\, t$) where $\mathcal{J}$ is the following operator mapping $(k-1)$–chains to $k$–chains,

$$\mathcal{J} f^{k-1}(u_1, \cdots, u_k; A) = \mathrm{i} \sum_{\nu=1}^{k} (-)^{\nu-1} [f^{k-1}(u_1, \cdots, \not{u}_\nu, \cdots, u_k; A), u_\nu] \,. \tag{31}$$

For $k = n, n+1, \ldots, 2n-1$ one gets

$$\omega^{k}_{2n-k-1} \propto \int_0^1 \mathrm{d}t \int \mathrm{d}\theta_k \cdots \mathrm{d}\theta_0$$
$$\times \left( -\mathrm{i} \sum_{\nu=0}^{k} t\theta_\nu \mathrm{d}(u_\nu) + \sum_{\substack{\nu,\mu=1 \\ \nu<\mu}}^{k} (t^2 - t)\theta_\nu\theta_\mu[u_\nu, u_\mu] + \sum_{i=\nu}^{k} \theta_0\theta_\nu u_\nu \right)^n \tag{32}$$

independent of $\hat{A}$, and the descent equations are somewhat simpler,

$$\delta\omega^{k-1}_{2n-k} + \hat{\mathrm{d}}\left(\omega^{k}_{2n-k-1}\right) = \mathcal{J}\omega^{k-1}_{2n-k} \tag{33}$$

(for $k = 2n-1$ one has $\delta\omega_0^{2n-1} = \mathcal{J}\omega_0^{2n-1}$). As mentioned, these equations (30), (33) are equal to (25) up to terms which vanish under partial integrations. They imply that $\hat{c}^{k}_{2n-k-1} = \hat{\int}\omega^{k}_{2n-k-1}$ all are $k$–cocycles for any integration $\hat{\int}$ satisfying Stokes' theorem and cyclicity, $\delta\hat{c}^{k}_{2n-k-1} = 0$. These thus are natural generalizations of the anomalies to NCG.

**Efficient anomaly calculations.** I now want to illustrate how explicit field theory calculations of anomalies can be simplified using notions from NCG as discussed above. In [LM2] a short derivation of the axial anomaly in all even dimensions (Euclidean framework) was given using a method inspired by NCG. Similar derivations of the axial– and Gauss law commutator anomalies (Schwinger terms) using the Hamiltonian framework can be found in [M3, LM3]. Here I sketch a short derivation of the Chern–Simons term from the imaginary part of the effective action of Dirac fermions in all odd dimensional spacetimes (Euclidean framework).

I consider effective actions $S(A)$ of massless Dirac fermions on odd dimensional spacetime. I denote as $\not{D}_0$ the free Dirac operator; for spacetime $\mathbb{R}^d$, *e.g.* $\not{D}_0 = \gamma_i(-\mathrm{i}\partial^i)$ as before, $\not{A} = \gamma_i A^i$ where $A = A^i \mathrm{d}x_i$ (de Rham 1–form) is the usual Yang–Mills field.

The effective action for fermions in the external Yang–Mills field $A$ formally is the determinant of the Dirac operator $\not{D}_0 + \not{A}$, or equivalently, the trace of its logarithm. Due to ultraviolet and infrared divergences some regularization of this trace is necessary (one has to add a small massterm and a large momentum cutoff, *e.g.*). I find it convenient to write this determinant as $\mathrm{Tr}_{\mathrm{reg}} \int_0^1 \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t} \log(\not{D}_0 + t\not{A})$, which formally is equivalent to

$$S(A) = \mathrm{Tr}_{\mathrm{reg}} \int_0^1 \mathrm{d}t (\not{D}_0 + t\not{A})^{-1} \not{A} \, ; \tag{34}$$

I take this as definition of the effective fermion action.

I now *define* $\hat{A} = |\not{D}_0|^{-1}\not{A}$, $\varepsilon = |\not{D}_0|^{-1}\not{D}_0$ and $\mathcal{F}_{t\hat{A}} = t\{\varepsilon, \hat{A}\} + t^2\hat{A}$ (possible zero modes of $\not{D}_0$ are taken care of by an infra red regulator not further specified here); at this point this can be regarded as useful notation. Then I can write

$$(\not{D}_0 + t\not{A})^{-1}\not{A} = (\varepsilon + t\hat{A})^{-1}\hat{A} = (\varepsilon + t\hat{A})(1 + \mathcal{F}_{t\hat{A}})^{-1}\hat{A} \tag{35}$$

(since $(\varepsilon + t\hat{A})^2 = 1 + \mathcal{F}_{t\hat{A}}$), and the imaginary part of the action becomes

$$\mathrm{Im}\, S(A) = \mathrm{Im}\, \mathrm{Tr}_{\mathrm{reg}} \int_0^1 \mathrm{d}t \sum_{m=0}^{\infty} (-)^m (\varepsilon + t\hat{A})(\mathcal{F}_{t\hat{A}})^m \hat{A} \, . \tag{36}$$

This should be equal, up to a constant, to the Chern–Simons term

$$\int \mathrm{tr}_N \int_0^1 \mathrm{d}t \, (\mathcal{F}_{tA})^n A \, , \tag{37}$$

where $\mathcal{F}_{tA} = -\mathrm{i}t\mathrm{d}(A) + t^2 A^2$ (de Rham 2–form) and $d = 2n+1$. This result is now very plausible by notation. To prove it, however, requires a nontrivial calculation: namely to show that $\mathrm{Im}\, \mathrm{Tr}_{\mathrm{reg}}$ of $\hat{A}(\mathcal{F}_{t\hat{A}})^m\hat{A}$ is always zero, and $\mathrm{Im}\, \mathrm{Tr}_{\mathrm{reg}}$ of $\varepsilon(\mathcal{F}_{t\hat{A}})^m\hat{A}$ is always zero except for $m = n$ where it is proportional to $\int \mathrm{tr}_N(\mathcal{F}_{tA})^n A$. This latter result can be regarded as a mathematical result in NCG similar to (5). Note also that all manipulations from (34) until Eq. (36) did not use any property of the operators $\not{D}_0$ and $\not{A}$ except that they are self-adjoint operators on some Hilbert space, and thus everything is valid also for much more general situations. The place where specific properties of these operators and the Hilbert space we are using enters is the step from (36) to (37). Especially only here the dimension of the underlying spacetime manifold enters: it is only the (regularized) trace that distinguishes the different dimensions, and it picks up exactly one term *viz.* the anomaly.

## 5. Final comments

In this paper I only discussed quantum field theory of fermions, thus an obvious question is: what about bosons? There is an analog of the abstract theory of fermion Bogoliubov transformations described above for bosons [R], and in my PhD thesis I worked out a natural $\mathbb{Z}_2$–graded version of the abstract current algebra (8) containing bosons and fermions and extending it to include super currents mixing bosons and fermions [GL]. A motivation for this were supersymmetric quantum field theory models. Again this formalism is sufficient only for 1+1 dimensions, and it provides as special cases super versions of the affine Kac–Moody algebras and the Virasoro algebra. In [L2] I outlined the construction of a boson version of (14) and found the boson analog of the Mickelsson–Rajeev cocycle (15) (up to as sign it is identical to the fermion cocycle). As far as I know, otherwise little is known about boson–analogs of results described here.

In all field theory calculations described in these paper anomalies could be represented as regularized trace $\mathrm{Tr}_{\mathrm{reg}}$ of some Hilbert space operator. A main point was that, on the level of Hilbert space operators (before $\mathrm{Tr}_{\mathrm{reg}}$ is taken), there is an interesting algebraic structure which naturally can be interpreted in the framework of NCG and can be exploited to make calculations efficient. Then, of course, a precise understanding of $\mathrm{Tr}_{\mathrm{reg}}$ is necessary. There are interesting mathematical properties of such regularized traces which are also related to NCG but were not discussed here (some discussion can be found in [LM2]).

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