WHAT CAN WE LEARN FROM THE CLASSICAL THEORY OF YANG–MILLS AND DIRAC FIELDS*

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(Received July 3, 1996)

Minimally coupled classical Yang–Mills and Dirac fields in the Minkowski space-time and in spatially bounded domains are investigated. The extended phase space, defined as the space of the Cauchy data admitting solutions of the evolution equations, is identified. The structure of the gauge symmetry group, defined as the group of all gauge transformations acting in the extended phase space is analysed. In the Minkowski space-time the Lie algebra of infinitesimal gauge symmetries has an ideal giving rise to the constraints. The quotient algebra, isomorphic to the structure algebra, labels the conserved colour charges. In the case of spatially bounded domains, each set of the boundary data gives rise to an extended phase space in which the evolution is Hamiltonian. The problem of a physical interpretation of the boundary data is discussed.

PACS numbers: 11.15. Kc

1. Introduction

This is a report on an ongoing German–Polish collaboration with a Canadian connection, based on the joint work with Günter Schwarz (Mannheim) and Jacek Tafel (Warsaw) on the classical theory of Yang–Mills and Dirac fields, [1–9]. The present state of knowledge in this field is comprehensively reviewed in [10]. It consists of a multitude of very interesting results. However, the assumptions under which these results are derived are often incompatible.

* Presented at the II German–Polish Symposium "New Ideas in the Theory of Fundamental Interactions", Zakopane, Poland, September 1995.

The immediate aim of our project is a systematic study of minimally interacting classical Yang–Mills and Dirac fields, starting with the extended phase space and continuing with the analysis of the gauge symmetries, the constraints, the reduced phase space and the dynamical variables. The extended phase space consists of the Cauchy data of solutions of the evolution equations of the theory. Hence, in order to determine possible extended phase spaces we had to establish the existence and uniqueness theorems for the evolution equations. Once we know what extended phase spaces are possible, we can choose one of them for our study, and this is the only choice we have. We cannot make any ad hoc assumptions at a later stage of the game.

Coupled Yang–Mills and Dirac fields are governed by non-linear partial differential equations. Many phenomena can be discovered just by analyzing the equations. This is well known in general relativity, where most of the progress comes from the study of the content of Einstein equations. We try to do in Yang–Mills theory what relativists have been doing for years. However, in order to get to physically observable phenomena, we have to get on step further and attempt to draw conclusions in the framework of the quantum theory.

Many physicists argue that, since the classical non-abelian Yang–Mills fields are not observed in nature, the classical gauge theory is irrelevant to physics. However, this belief does not stop them from writing the classical Lagrangian and deriving from it the equations of motion and the conservation laws.

The usual basic frameworks used in studying quantum non-abelian gauge fields are the perturbation expansion, the lattice gauge theory and the Feynman path integral. The perturbation expansion is not gauge invariant. Hence, in order to obtain gauge invariant results, one has to consider the sum over all diagrams in the same orbit of the gauge group. The lattice gauge theory is not relativistic, and the arguments based on the relativistic invariance of the theory require non-trivial limiting considerations. The Feynman path integral is both gauge invariant and relativistic. However, due to the difficulties with the definition of measures in the infinite dimensional spaces, precise statements are problematic.

2. Field equations

Let $M \subseteq \boldsymbol{R}^3$denote the region of the physical space accessible to the fields. We consider two cases: (i) $M = \boldsymbol{R}^3$, and (ii) M is a bounded contractible domain in $\boldsymbol{R}^3$ with smooth boundary ∂M. We consider minimally coupled Yang–Mills and Dirac fields in $\boldsymbol{R} \times M$. The product structure in $\boldsymbol{R} \times M$, leads to the splitting of the Yang–Mills potential A_μ into the scalar potential Φ and the vector potential $A = A_i dx^i$. It also allows for the representation of the field strength $F_{\mu\nu}$ in terms of the electric field E and the magnetic field B with the components

$$E_j = F_0 = \partial_0 A_j - \partial_j \Phi + [\Phi, A_j] \quad \text{and}$$
$$B_j = \tfrac{1}{2}\epsilon_{jmn} F^{mn} = (curl\ A)_j + [A\times, A]_j\,,$$

where $\times$ denotes the cross product, and we use the Euclidean metric in $\boldsymbol{R}^3$ to identify vector fields and forms. A, E and B are time-dependent forms on M with values in the Lie algebra g of the structure group G of the theory, which is assumed to be compact. The Dirac field Ψ is a time dependent spinor field on M with values in the space V of the fundamental representation of G.

The field equations split into the evolution equations

$$\partial_t A = E + \text{grad}\ \Phi - [\Phi, A]\,,$$
$$\partial_t E = curl B - [A\times, B] - [\Phi, E] + J\,,$$
$$\partial_t \Psi = -\gamma^0(\gamma^j \partial_j + im + \gamma^0 \Phi + \gamma^j A_j)\Psi\,,$$

and the constraint equation

$$\text{div} E + [A; E] = \rho\,.$$

The source terms can be described in terms of an orthonormal basis $\{T_a\}$in the Lie algebra g,

$$\rho = \rho^a T_a = \Psi^\dagger T^a \Psi T_a\,,$$
$$J^k = J^{ka} T_a = \Psi^\dagger(\gamma^0\gamma^k) \otimes T^a \Psi T_a\,,$$

where the latin indices are lifted in terms of an ad-invariant metric on g.

3. Gauge conditions

Since the linear part of the evolution equations contains the operator $curl^2$ which is not elliptic in the space $L^2(M)$ of square integrable vector fields on M, we have to decompose the fields A and E into their longitudinal and transverse parts,

$$A = A^L + A^T \ E = E^L + E^T, ,$$

where the longitudinal parts A^L and E^L are gradients, and

$$\text{div}(A^T) = \text{div}(E^T) = 0..$$

If M is a bounded domain with non-empty boundary ∂M, one also requires vanishing on ∂M of the normal components nA^L and nE^L of A^L and E^L, respectively.

The evolution equations do not involve the time derivative of the scalar potential Φ. In order to get a uniqueness of the evolution we have to choose a gauge condition relating Φ to the dynamical variables A and E. Here we use the condition:

$$\operatorname{grad} \Phi = -E^L .$$

In order to determine Φ uniquely, we have to add a normalization condition. If M is a bounded domain in $\boldsymbol{R}^3$, the normalization condition is

$$\int_M \Phi \ d_3x = 0 .$$

For $M = \boldsymbol{R}^3$we use a weighted normalization condition

$$\int_{\boldsymbol{R}^3} \Phi(1+x^2)^{-2} \ d_3x = 0 .$$

4. Existence and uniqueness theorems

We denote by $H^k(\boldsymbol{R}^3, g)$ and $H^k(\boldsymbol{R}^3, V \otimes \boldsymbol{C}^4)$ the spaces of g-valued forms and spinor fields, respectively, which are square integrable together with its derivatives up to order k. The existence results in the Minkowski space-time are given by

Theorem 1. *For every* $(A_0, E_0, \Psi_0) \in H^2(\boldsymbol{R}^3, g) \times H^1(\boldsymbol{R}^3, g) \times H^2(\boldsymbol{R}^3, V \otimes \boldsymbol{C}^4)$, *there exist maximal* $\tau^+ > 0$, $\tau^- > 0$, *and the unique classical solution*

$$(A(t), E(t), \Psi(t)) \in H^2(\boldsymbol{R}^3, g) \times H^1(\boldsymbol{R}^3, g) \times H^2(\boldsymbol{R}^3, V \otimes \boldsymbol{C}^4)$$

of the evolution equations, defined for $t \in (-\tau^-, \tau^+)$, *which satisfies the initial conditions* $A(0) = A_0$, $E(0) = E_0$, $\Psi(0) = \Psi_0$.

For a bounded domain, the initial data have to be supplemented by the boundary conditions. We specify on the boundary the tangential component $t(curl A)$ of the *curl* of A, $\Gamma(\Psi) = \frac{1}{2}(Id - i\gamma^k n_k)\Psi|\partial M$, and $\Gamma(\mathcal{D}\Psi)$, where $D = -\gamma^0(\gamma^k \partial_k + im)$ is the free Dirac operator in $\boldsymbol{R}^3$. For each $A \in H^2(M, g)$, $t(\text{curl } A) \in H^{1/2}(\partial M, g)$. Similarly, for $\Psi \in H^2(M, V \otimes \boldsymbol{C}^4)$,

$\Gamma(\Psi) \in H^{3/2}(\partial M, V \otimes \boldsymbol{C}^4)$ and $\Gamma(\mathcal{D}\Psi) \in H^{1/2}(\partial M, V \otimes \boldsymbol{C}^4)$. Moreover, the range of Γ is characterized by the condition $(Id + i\gamma^k n_k)\Gamma = 0$.

Theorem 2. *Let (λ, μ, ν) be a differentiable curve of boundary data in $H^{1/2}(\partial M, g) \times H^{3/2}(\partial M, V \otimes \boldsymbol{C}^4) \times H^{1/2}(\partial M, V \otimes \boldsymbol{C}^4)$ such that $\lambda = \lambda_1 + \text{grad}\,\chi$, for λ_1 and χ in $H^{3/2}(\partial M, g)$, and (μ, ν) satisfy the conditions*

$$(Id + i\gamma^k n_k)\mu = 0 \quad \text{and} \quad (Id + i\gamma^k n_k)\nu = 0 .$$

For every $(A_0, E_0, \Psi_0) \in H^2(\boldsymbol{R}^3, g) \times H^1(\boldsymbol{R}^3, g) \times H^2(\boldsymbol{R}^3, V \otimes \boldsymbol{C}^4)$ satisfying

$$t(\text{curl } A_0) = \lambda(0) , \quad \Gamma(\Psi_0) = \mu(0) \quad \Gamma(\mathcal{D}\Psi_0) = \nu(0) ,$$

there exist maximal $\tau^+ > 0$, $\tau^- > 0$, and the unique classical solution

$$(A(t), E(t), \Psi(t)) \in H^2(\boldsymbol{R}^3, g) \times H^1(\boldsymbol{R}^3, g) \times H^2(\boldsymbol{R}^3, V \otimes \boldsymbol{C}^4)$$

of the evolution equations, defined for $t \in (-\tau^-, \tau^+)$, which satisfies the initial conditions $A(0) = A_0$, $E(0) = E_0$, $\Psi(0) = \Psi_0$, and the boundary conditions

$$t(\text{curl } A(t)) = \lambda(t) , \quad \Gamma(\Psi(t)) = \mu(t) , \quad \Gamma(\mathcal{D}\Psi(t)) = \nu(t) .$$

5. Constraints and reduction

The constraint equations are preserved by the dynamics. Hence, in order to obtain a description of the space of solutions, it suffices to study the constraint equation at the initial Cauchy surface. On the basis of available information we conjecture that, for the Minkowski space theory and for the type of boundary data considered here, the constraint equation define a manifold in the space of the initial data. It seems that in this case the reduced phase space is a smooth manifold endowed with an exact weakly symplectic form.

The bag boundary conditions $nA = 0$, $nE = 0$, $t(curl\ A) = 0$, $\Gamma(\Psi) = 0$ and $\Gamma(\mathcal{D}\Psi) = 0$ define an invariant manifold in the extended phase space. Its intersection with the constraint set is not a manifold. We have shown in [4] that in this case the resulting reduced phase space is the union of exact weakly symplectic manifolds parameterized by the conjugacy classes of gauge isotropy groups of the Cauchy data. Each symplectic manifold in the reduced phase space corresponds to a definite form of symmetry breaking which does not lead to vector bosons.

6. Colour charges

Colour charges are the conserved quantities corresponding to infinitesimal gauge symmetries of the theory. In the Minkowski space theory, the Lie algebra gs of infinitesimal gauge symmetries has a direct sum decomposition

$$gs = gs_0 \oplus g\,,$$

where gs_0 is the completion in the Beppo Levi topology of the Lie algebra of smooth compactly supported maps $\xi : \boldsymbol{R}^3 \to g$, and the second summand g is interpreted as constant maps from $\boldsymbol{R}^3$to g. It should be noted that gs_0 is an ideal in gs. In the language used by physicists gs_0 corresponds to "local gauge transformations", while the second summand g corresponds to "global gauge transformations". However, the constant maps from $\boldsymbol{R}^3$ to g do not form an ideal is gs. Hence, the notion of a "global gauge transformation" has an invariant meaning only as an element of the quotient algebra gs/gs_0.

For each $\xi \in gs$, the corresponding conserved quantity is denoted by J_ξ. Since the constraint equation is equivalent to $J_\xi = 0$ for all $\xi \in gs_0$, the colour charges of states satisfying the constraint equation are parameterized by elements of the algebra gs/gs_0, which we refer to as the *colour algebra.* The colour charge J_ξ of a classical state (A, E, Ψ) satisfying the constraint equation is given by

$$J_\xi(A, E, \Psi) = \lim_{r\to\infty} \int_{S_r} nE\xi\; dS\,,$$

where S_r is the sphere of radius r centered at the origin and dS denotes the element of the surface area. The map $\xi : \boldsymbol{R}^3 \to g$ appears on the right hand side only through its asymptotic behaviour at infinity which determines the projection of ξ to the quotient gs/gs_0. We have shown in [5] that a gs_0 invariant local charge density of the colour charge $J_\xi(A, E, \Psi)$, which depends locally on the fields (A, E, Ψ), can be defined only if the projection of ξ to the quotient gs/gs_0 is in the centre of gs/gs_0. This is a classical analogue of non-central colour charge confinement phenomenon in quantum gauge theory.

7. Hamiltonian evolution

If Φ were a function of the space-time variables only, $\Phi = \Phi(x, t)$, then the evolution equations would be Hamiltonian with the Hamiltonian

$$\begin{aligned} H_\Phi(A, E, \Psi) = &\int_M \{\tfrac{1}{2}(E^2 + B^2) + \Psi^\dagger\gamma^0[\gamma^k(i\partial_k + A_k) + m]\Psi\}d_3x \\ &+ \int_M \{-E(\operatorname{grad}\Phi + [A, \Phi]) + \Psi^\dagger\Phi\Psi\}d_3x\,, \end{aligned}$$

and the symplectic form

$$\omega = d\theta\,,$$

where θ is given by

$$\langle \theta(A, E, \Psi) | (\delta A, \delta E, \delta \Psi)\rangle = \int\limits_M (E \cdot \delta A + \Psi^\dagger \delta \Psi) d_3 x\,.$$

With the scalar potential Φ depending on the dynamical variable E, the Hamiltonian $H_\Phi(A, E, \Psi)$ generates the equations

$$\partial_t A = E + \text{grad}\;\; \Phi - [\Phi, A] + \int\limits_M \left\{ -E(\text{grad}\; \frac{\delta \Phi}{\delta E} + [A, \frac{\delta \Phi}{\delta E}]) + \Psi^\dagger \frac{\delta \Phi}{\delta E} \Psi \right\} d_3 x\,.$$

$$\partial_t E = \text{curl}\; B - [A\times, B] - [\Phi, E] + J\,,$$
$$\partial_t \Psi = -\gamma^0(\gamma^j \partial_j + im + \gamma^0 \Phi + \gamma^j A_j)\Psi\,.$$

In order to prove the existence and uniqueness theorems for these evolution equations we have to modify our gauge condition and choose Φ to be given by

$$\text{grad}\; \Phi = -2E^L\,,$$

and the same normalization conditions as in Section 3. The statements of results are the same as in Theorems 1 and 2.

In the Minkowski space theory, we have the usual Hamiltonian evolution. However, in the case of a bounded domain the Hamiltonian structure is more complicated. Let $\beta = (\lambda, \mu, \nu)$ be a curve of the boundary data in $H^{1/2}(\partial M, g) \times H^{3/2}(\partial M, V \otimes \boldsymbol{C}^4) \times H^{1/2}(\partial M, V \otimes \boldsymbol{C}^4)$. For each time t, let $\mathcal{P}_{\beta(t)}$ be the subspace of $H^2(\boldsymbol{R}^3, g) \times H^1(\boldsymbol{R}^3, g) \times H^2(\boldsymbol{R}^3, V \otimes \boldsymbol{C}^4)$ consisting of the fields (A, E, Ψ) which satisfy the boundary conditions

$$t(\text{curl}\; A) = \lambda(t)\,,\; \Gamma(\Psi) = \mu(t)\,,\; \Gamma(\mathcal{D}\Psi) = \nu(t)\,.$$

It is an affine subspace of $H^2(\boldsymbol{R}^3, g) \times H^1(\boldsymbol{R}^3, g) \times H^2(\boldsymbol{R}^3, V \otimes \boldsymbol{C}^4)$ with the associated vector space $\mathcal{P}_0$ corresponding to the homogeneous boundary conditions, i.e. the vanishing of the boundary data. The weakly symplectic form ω in $H^2(\boldsymbol{R}^3, g) \times H^1(\boldsymbol{R}^3, g) \times H^2(\boldsymbol{R}^3, V \otimes \boldsymbol{C}^4)$, defined above, pulls back to a weakly symplectic form $\omega_{\beta(t)}$ in $\mathcal{P}_{\beta(t)}$. Let $\mathcal{E}_\beta$ be the union of $\mathcal{P}_{\beta(t)}$ over all t,

$$\mathcal{E}_\beta = \bigcup_t \mathcal{P}_{\beta(t)}\,.$$

It is the evolution space corresponding to the chosen boundary data β.It can be alternatively described as the affine subspace of $H^2(\boldsymbol{R}^3, g) \times H^1(\boldsymbol{R}^3, g) \times$

$H^2(\boldsymbol{R}^3, V \otimes \boldsymbol{C}^4) \times \boldsymbol{R}$ consisting of (A, E, Ψ, t) such that (A, E, Ψ) satisfy the above boundary conditions at time t. The projection onto the time axis $\boldsymbol{R}$ is a fibration of $\mathcal{E}_\beta$ with weakly symplectic fibres.

The situation here is analogous to the Newtonian dynamics, where we have the natural projection from the space-time onto the absolute time. Since the space is not absolute, in order to write the equations of motion in the Hamiltonian form, we have to introduce the frame of reference (an observer). Here, the role of a frame of reference is provided by any background field $(a(t), \psi(t))$satisfying the boundary conditions $t(\text{curl } a(t)) = \lambda(t)$, $\Gamma(\psi(t)) = \mu(t)$, $\Gamma(\mathcal{D}\Psi(t)) = \nu(t)$ for all t. This give an isomorphism between $\mathcal{E}_\beta$ and $\mathcal{P}_0 \times \boldsymbol{R}$ given by $(A, E, \Psi, t) \longmapsto (A - a(t), E, \Psi - \psi(t), t)$. With respect to such a trivialization, the evolution equations for $A(t) - a(t)$, $E(t)$ and $\Psi(t) - \psi(t)$ are Hamiltonian with the Hamiltonian $H_\Phi(A, E, \Psi)$ given above. Note that $H_\Phi(A, E, \Psi)$ as a function of $A(t) - a(t)$, $E(t)$ and $\Psi(t) - \psi(t)$ may be time-dependent if $a(t)$ and $\psi(t)$ depend on t, that is if the boundary conditions are time-dependent.

8. Physical interpretation of the boundary data

Yang–Mills fields are commonly studied in the whole space-time. The resulting quantum theory is sensitive to the global topological properties of the space-time. On the other hand, the non-abelian gauge fields are supposed to be carriers of strong and weak interactions on sub-atomic level. All experiments are made in laboratories of finite spatial extent and they last for a finite period of time. The experimental data are obtained from the reading of the detectors surrounding the region in which the reaction takes place. This suggests that the appropriate theoretical description should take into account the finiteness of the observational domain, and the behaviour of the fields on its boundary.

The evolution of the boundary data can be prescribed arbitrarily. Hence, they cannot describe dynamical degrees of freedom. In order to describe realistically the observed quantum processes one should form a corresponding quantum field theory in which the physical role role of the boundary data is taken into account.

A possible interpretation of the boundary data is that they describe the experimental setting. For example, they may describe the interaction of the observed system with the detectors which, according to the Copenhagen interpretation of quantum mechanics, should be treated on the classical level. In this case, they could induce the symmetry breaking leading to the splitting of the Yang–Mills field into the vector boson fields and the residual gauge field. In the classical theory the vector boson fields obtained in such a way are massless. Upon quantization of such a theory, the mass of vector bosons might appear as an anomaly.

REFERENCES

[1] J. Śniatycki, *Commun. Math.* **141**, 593 (1991).
[2] J. Śniatycki, G. Schwarz, *Commun. Math. Phys.* **159**, 593 (1994).
[3] G. Schwarz, J. Śniatycki, *Commun. Math. Phys.* **168**, 441 (1995).
[4] J. Śniatycki, G. Schwarz, L. Bates, *Commun. Math. Phys.* **176**, 95 (1996).
[5] J. Śniatycki, G. Schwarz, *Rep. Math. Phys.* **34**, 311 (1994).
[6] G. Schwarz, J. Śniatycki, Yang-Mills and Dirac fields in the Minkowski space-time, preprint.
[7] J. Tafel, J. Śniatycki, Singular perturbations of linear semigroups in Yang-Mills theory, preprint.
[8] G. Schwarz, J. Śniatycki, J. Tafel, Yang-Mills and Dirac fields with inhomogeneous bag boundary conditions, preprint.
[9] G. Schwarz, J. Śniatycki, *Acta Phys. Pol.* **B27**, 881 (1996).
[10] L. Lusanna, Dirac observables for classical Yang-Mills theory with fermions, preprint DDF 201/1/94, Dipartamento di Fisica, Universita degli Studi di Firenze.