HIGH-ORDER BEHAVIOUR AND SUMMATION METHODS IN PERTURBATIVE QCD*

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After reviewing basic facts about large-order behaviour of perturbation expansions in various fields of physics, I consider several alternatives to the Borel summation method and discuss their relevance to different physical situations. Then I convey news about the singularities in the Borel plane in QCD, and discuss the topical subject of the resummation of renormalon chains and its application in various QCD processes.

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1. Introduction. Some history

The problem of the high-order behaviour of the perturbation-expansion coefficients in field theory calculations has received new interest during the past two years. Particular attention has been paid to power corrections to QCD predictions for hard scattering processes. One can point out two reasons of this growing interest:

- The theoretical problem of how physical observables can be reconstructed from their (often divergent) power expansions.
- The pragmatic need to assess the usefulness of performing the extensive evaluations of multi-loop Feynman diagrams in QCD. Much effort has been devoted to the computation of higher-order QCD perturbative corrections; in some cases third-order approximations are known,

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and we seem presently to be at the border of what can be carried out analytically or numerically in high-order perturbative calculations. So, why next order? If the series is an asymptotic one, the next order may represent no improvement to the lower-order result. On the contrary, at a certain order it will lead to deterioration.

There are good reasons to believe that the vanishing convergence radius of perturbation expansions of physically relevant quantities is a general feature of quantum theory; it has been proved in most of the theories and models considered. Bender and Wu [1] showed in 1971 that perturbation theory for one-dimensional quantum mechanics with a polynomial potential is divergent, and they also discussed the very-large-order behaviour of the perturbation coefficients. In 1976, Lipatov [2] obtained the same results for massless renormalizable scalar field theories.

Brézin, Le Guillou and Zinn–Justin [3], applying independently the same method to anharmonic oscillations in quantum mechanics, were able to rederive and to generalize the results obtained by Bender and Wu.

In QED, after the pioneering work by Dyson [4], a number of papers appeared discussing the analyticity properties of the Green functions (some of them are cited in Refs. [5] and [6]). The growth like $n!$ (where n is the order of approximation) has two sources:

1. The number of diagrams grows like $n!$, each diagram giving a contribution of the order of 1.
2. There are types of diagrams for which the amplitude itself grows like $n!$ [7].

Gross and Periwal [8] proved in 1988 that perturbation theory for the bosonic string diverges for any value of the coupling constant and is not Borel summable.

The situation in QCD is particularly complex not only because the expansion coefficients behave like $n!$ and are of non-alternating sign, but also due to strong dependence of a truncated series on the renormalization prescription. Of particular interest are situations in which the kinematic regime allows one quantity, M say, (momentum, rest mass) to have a large value. Then the matrix element of a QCD operator $\mathcal{O}$ containing quark fields is represented by means of the operator product expansion in inverse powers of M,

$$\mathcal{O} = C_1(M/\mu)\mathcal{O}_1(\mu) + \frac{1}{M}C_2(M/\mu)\mathcal{O}_2(\mu) + O(1/M^2)\,, \qquad (1)$$

where μ is the renormalization scale, $\mathcal{O}_i$ are local operators of the theory (ordered by their dimension) and $C_i(M/\mu)$ are the expansion coefficients, to be calculated in perturbation theory. (As $\mathcal{O}$ is independent of the renormalization prescription, the renormalization-prescription dependence of the quantities on the right-hand side must mutually cancel.)

Relation (1) allows for separation of small and long-distance effects in the process. Perturbation theory does not allow (as will be discussed below) an unambiguous computation of the coefficients $C_i(M/\mu)$. This generates an ambiguity of the order of $1/M$ (see [9]) in the determination of $C_1(M/\mu)$, which implies that one should not include the $O(1/M)$ term in (1) until $C_1(M/\mu)$ has been correctly computed. Leaving aside for a while the question what a correct computation of $C_1(M/\mu)$ means we conclude that the problem of a perturbative determination of $C_1(M/\mu)$ has to be solved before passing to, or simultaneously with, higher terms in the expansion (1). In my talk I shall focus on the former subject, referring for the latter topic to the recent review talk [9] and references therein.

TABLE I

High-order behaviour of perturbation expansion coefficients

Theory	Notation and references	High-order behaviour
1 Disp. relation	$\mathrm{Im} f(z) \sim z^{-b} e^{-a/z}$	$f_n \sim (-a)^n \Gamma(n+b)$
2 Anharmonic oscillator	$E_m = \sum_{n=0}^{\infty} E_{m,n} g^n$ [1]	$E_{m,n} \sim (-\frac{3}{4})^n n!$
3 Anharmonic oscillator, ϕ^4	$I(g) \sim \int_{-\infty}^{\infty} e^{(-x^2/2 - gx^4/4)} dx$ [1, 11]	$I_n \sim (-1)^n \frac{4^{-n}}{n} n!$
4 Instantons, $D = \frac{2m}{m-2}$	$m \geq 6$, [2, 10]	$C_n(m) \sim (-a)^n e^{n(1-m/2)} n^{(m+D)/2}$
	$m = 4$, [2]	$C_n(4) \sim (\frac{n}{16\pi^2 e})^n n^4$
5 Field theories without fermions	$Z = \sum_{n=0}^{\infty} Z_n (-\frac{e^2}{4\pi})^n$ [3, 13–16]	$Z_n \sim C n^b A^{-n} n!$
6 Field theories	[11]	$f_n \sim \Gamma(n\frac{d-2}{d})$
with fermions	[17]	$f_n \sim (-\alpha)^n A^{n/2} \Gamma(\frac{1}{2}n)$
7 Yukawa theories	[11]	$Z_n \sim n^{-\alpha} A^{-n} \cos(\frac{2\pi n}{d}) \Gamma(n\frac{d-2}{d})$
$d = 2$	[18]	$Z_n \sim A^{-n} (\ln n)^n$
8 QED	[17, 19]	$Z_n \sim (-1)^n A^n \Gamma(\frac{1}{2}n)$
9 QCD	[20]	$\sim A^n n^{\gamma} n!$
10 Bosonic strings	h is # of handles, [8]	$\sim h!$

Table I gives a survey of the large-order behaviour of the expansion coefficients in some typical theories and models. It is intended for first information and should not be used for systematic analyses because some important conditions or restrictions are not mentioned. (Let me also point out that references in the Table as well as in the talk as a whole are often made not to the original papers but rather to reviews or more recent papers.

As a result, a number of relevant valuable papers are not quoted; I apologize to their authors.) For details the reader is referred to the original papers and to the anthology by LeGuillou and Zinn–Justin [6](which contains a list of references till 1990).

The series are divergent in all the cases indicated, with vanishing radius of convergence. In this connection, two remarks are in order. The first is that the estimates are based only on certain subclasses of higher-order diagrams which, in the case of QED (QCD), are obtained by inserting an arbitrary number of fermion loops into the photon (gluon) lines of the lowest-order radiative correction. It is not known what additional contributions come from other diagrams; whether they are negligible or give the same or even a greater contribution, or finally whether they cause cancellations in the original estimate.

The second remark I want to make is as follows. In most of the items of Table I, the coefficients exhibit factorial large-order growth (items 1–5, 9 and 10). It would however be misleading to conclude that all these theories face the same divergence and ambiguity problems, which could be treated in the same way. Knowledge of the high-order behaviour shows only one part of the problem of a perturbation expansion, the other ones being those of summability and of uniqueness of the summability prescription. Some details are discussed in Sections 2 and 3.

2. Useful facts on power series

Certain important facts on power series are overlooked in physical considerations. It may therefore be useful to recall some of them here because spontaneous intuition is often misleading.

1. The divergence of a perturbation expansion does not signal an inconsistency of the theory. (See an analogue in item 5.)

The problem is not that of convergence or divergence, but whether the expansion uniquely determines the function or not. The method of Feynman diagrams allows one to find, at least in principle, all coefficients of the perturbation series, which may determine the function uniquely even if it is divergent and may not do so even if it is convergent. This depends on additional conditions.

2. The requirement of asymptoticity of a perturbation series,

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^n , \qquad (2)$$

is not a formal assumption. It has physical content.

When a perturbation series is divergent, it is usually re-interpreted as an asymptotic series. This is a weaker assumption, but not a technical one.

It is by no means trivial; it physically means that there is a very smooth transition between the system with interaction and the system without it. For certain classes of observables the perturbation series is believed to be an asymptotic expansion.

3. If $f(z)$ is singular at the origin, its asymptotic expansion (2) may be a convergent series.

Consider for example the function $f(z) = g(z) + A\mathrm{e}^{-\alpha/z}$ with A real and α positive, where $g(z)$ is analytic at the origin: the asymptotic expansion of the singular function $f(z)$ in the right complex halfplane is a convergent series. This convergent series is asymptotic to many functions (most of which are singular at the origin), but its values converge only to one of them, $g(z)$. They do so inside the Taylor circle, which extends to the nearest singularity of $g(z)$.

4. A very violent behaviour of the expansion coefficients a_n at $n \to \infty$ might make us expect that no function with the property $f(z) \sim \sum_{n=0}^{\infty} a_n z^n$ would exist. This fear is not justified; it was proved by Borel and Carleman (see [21] for details) that there are analytic functions corresponding to arbitrary asymptotic power series.

5. Borel non-summability of a perturbation expansion alone does not signal an inconsistency or ambiguity in the theory.

The Borel procedure is just one of many possible summation methods and need not be applicable always and everywhere. The problem is to find a method (not necessarily the Borel one) which is appropriate for the case considered.

6. If the a_n behave very violently at $n \to \infty$ (so that the Borel series $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ has zero convergence radius) one might expect that it would be sufficient to replace $n!$ in the denominator by a sequence b_n that grows faster than $n!$, in order to reach a more efficient suppression of the a_n. This can of course be done, but the price to pay for this is that stronger conditions on analyticity will be required for the summation procedure to be unambiguous. Analyticity of the ction expanded must be examined simultaneously with the asymptotic expansion, otherwise the same series can be summed to different functions.

We can conclude (and will elaborate below) that uniqueness of a summation procedure (in other words, recoverability of a function by means of its asymptotic series, see [5]) requires a balance between high-order behaviour of the series and the analyticity domain of $f(z)$. A violent high-order behaviour can lead to a unique definition only if "enough analyticity" is available.

3. Analyticity vs. high-order behaviour: balance for uniqueness

How to deal with divergent series and under what conditions a power series can uniquely determine the expanded function are questions of fundamental importance in quantum theory. Power expansions are badly needed in physics but to ensure that they have precise meaning additional conditions are required. These additional conditions should reflect some physical features of the system.

The fact that the expansion coefficients, which grow asymptotically like $n!$, are of constant, non-alternating sign, is the origin of most problems connected with the uniqueness of perturbative expansions in QCD.

Let us discuss a simple example to illustrate this crucial point. Consider a generic quantity D, calculated in perturbation theory with coupling z,

$$D(z) = \sum_{n=0}^{\infty} a_n z^n \,. \tag{3}$$

This can be re-written as

$$D(z) = \sum_{n=0}^{\infty} a_n z^n (1/n!) \int_0^{\infty} \mathrm{d}t e^{-t} t^n \,. \tag{4}$$

If the series (3) has a non-vanishing convergence radius r, the integration in (4) can be exchanged with the sum. If, on the other hand, the convergence radius is zero, $r = 0$, we can give the series meaning by exchanging the order of integration and summation. In either case we obtain

$$D(z) = \int_0^{\infty} \mathrm{d}t e^{-t} \sum_0^{\infty} a_n \frac{(zt)^n}{n!} = \int_0^{\infty} \mathrm{d}t e^{-t} B(zt) \,, \tag{5}$$

where $B(zt)$ is the Borel transform of $D(z)$. Taking $a_n = n!$ (finite-order coefficients are irrelevant for the character of singularities) we obtain

$$D(z) = \int_0^{\infty} \mathrm{d}t e^{-t} \frac{1}{1 - zt} \,. \tag{6}$$

This integral does not exist for z positive, nor is defined the Borel sum of such a series. The summation can be defined in many ways; there are infinitely many functions with the asymptotic expansion $\sum_{n=0}^{\infty} n! z^n$.

Note that non-uniqueness of the summation prescription is not a mathematical difficulty; it rather signals lack (or insufficient use) of physics in the theory. There are two kinds of conditions for a function $f(z)$ to be uniquely determined by its asymptotic expansion $\sum_0^\infty a_n z^n$. The function must

(i) have a sufficiently large analyticity domain K

(ii) satisfy upper bounds (uniform in z and N) on the remainder $f(z) - \sum_0^N a_n z^n$ for each N above a certain value. When K has a small opening angle at the origin, the inequalities must be sufficiently restrictive in order to reach uniqueness. If the angle is large enough, the condition *(ii)* may be weakened.

Let us sketch how this works in the case of the Borel summation method. The series (2) is called Borel summable if

a) its Borel transform,

$$B(t) = \sum_{n=0)}^{\infty} a_n t^n/n!\,, \tag{7}$$

converges inside some circle, $|\, t \,| < \delta\,,\ \delta > 0$;

b) $B(t)$ has an analytic continuation to a neighbourhood of the positive real semiaxis Re $t \geq 0$, and

c) the integral

$$g(z) = \frac{1}{z}\int_0^\infty \mathrm{e}^{-t/z} B(t)\mathrm{d}t\,, \tag{8}$$

called the Borel sum, converges for some $z \neq 0$.

Nevanlinna [22] gave the following criterion of Borel summability:

Let $f(z)$ be analytic in the domain $K(\eta)$ defined by the inequality $\mathrm{Re}\frac{1}{z} > \frac{1}{\eta}$ (with η positive), a disc of radius $\frac{1}{2}\eta$ bisected by the positive real semiaxis and tangent to the imaginary axis (see Fig. 1(a)), and let $f(z)$ have the asymptotic expansion (2). If the remainder $R_N(z)$ after subtracting N terms from $f(z)$,

$$R_N(z) = f(z) - \sum_{n=0}^{N-1} a_n z^n \tag{9}$$

is bounded by the inequality

$$|\, R_N(z) \,| < A\sigma^N N!\, |\, z \,|^N \tag{10}$$

uniformly for all $z \in K(\eta)$ and all N above some value N_0, then the sum is determined uniquely and has the form (8) for all $z \in K(\eta)$.

For other types of regions, similar theorems hold with modified regions. Details are exposed, along with references, in the review [23]; a survey of

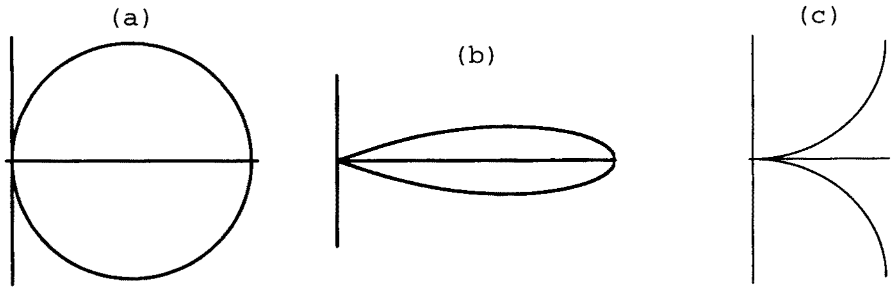

Fig. 1. Three summation methods for three different analyticity and boundedness domains: (a) the disc $K(\eta)$, (b) the drop $K(\eta,\rho)$, and (c) the wedge W. Crucial is not the size of the domain, but the opening angle at the origin.

typical cases is given in Table II, together with conditions for a unique determination of $f(z)$ from its asymptotic expansion.

TABLE II

Analyticity vs. high-order behaviour: a balance is needed for uniqueness.

(z) at $z=0$	Uniform bound on $R_N(z)$	Transform	Summation
1 analytic			$\sum_{n=0}^{\infty} a_n z^n$ is convergent
2 singular, opening angle $=\pi$	$A\sigma^N N! \mid z \mid^N$ in $K(\eta)$ and $N > N_0$	$B(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$	$g(z) = \frac{1}{z}\int_0^{\infty} e^{-t/z} B(t)dt$
3 singular, opening angle > 0	$A\sigma^N (N!)^{\rho} \mid z \mid^N$ in $K(\eta,\rho)$, $N > N_0$	$B_{\rho}(t) = \sum_0^{\infty} \frac{a_n t^n}{\Gamma(n\rho+1)}$	$g_{\rho}(z) = \frac{1}{\rho}\int_0^{\infty} t^{1/\rho-1} \exp(-t^{1/\rho}) B_{\rho}(tz)dt$
4 singular, opening angle $=0$	$A\mu(N) \mid z \mid^N$ in wedge W, $N > N_0$	$M(t) = \sum_{n=0}^{\infty} \frac{a_n}{\mu(n)} t^n$	$g_m(z) = \int_0^{\infty} M(tz) \exp(-e^t)dt$

The function $\mu(n)$ used in Table II is defined as follows (see [25] for theorems relevant to the wedge-shaped analyticity region)

$$\mu(n) = \int_0^{\infty} \exp(-e^t) t^n dt \,. \tag{11}$$

Note the double exponential function in the integrand, which implies a $(\ln n)^n$ behaviour of the function $\mu(n)$ at large n.

The disc $K(\eta)$, the "drop" $K(\eta, \rho)$ and the wedge W are depicted in Figs. 1 (a), (b), and (c) respectively. The general rule for the use of Table 2 is as follows: To ascertain uniqueness in the determination of a function $f(z)$ out of its asymptotic series (2), one has to check if the conditions in all columns in one row are satisfied for the case considered; otherwise the function is either overdetermined or not uniquely determined. In QCD, where the function is, according to [24], analytic only in the wedge W (see row 4 of Table II), but the a_n behave like $n!$ at large orders (row 2 of column 2), the function is not uniquely determined by its perturbation expansion and the situation calls for additional (nonperturbative) information.

A simultaneous use of Table I and Table II can tell us to what extent physical observables and Green's functions can be reconstructed from the asymptotic series (2) in a theory. Taking, for instance, the QCD large-order behaviour, $A^N N^\gamma N!$, from Table I (item 9), we obtain in Table II the uniform bound in the 2nd row to be valid on the whole disc $K(\eta)$ of the complex coupling constant plane as the condition for uniqueness. Of course, we do not expect uniqueness in perturbative QCD and it is therefore not a surprise that the actual region of analyticity of two-point Green functions in QCD is much smaller, having the form of a horn with zero opening angle at the origin [1], *i.e.*, row 4 of Table II and Fig. 1(c).

Table II also shows that the large-order behaviour of the coefficients a_n is not the only criterion of the (Borel or some other) summability of the series (2). *Even a series with a very tame behaviour of the coefficients a_n may not be Borel summable*, in spite of the radical suppression of the a_n by the Borel factors $n!$. An example is discussed in [26].

4. Generalized Borel transforms

1. The functions $B_\rho(t)$ and $M(t)$ defined in Table II are generalizations of the Borel transform, which can be used in the various situations listed in Table I to reduce non-uniqueness, provided some additional information is available. More about the properties of $B_\rho(t)$ and $M(t)$ can be found in [25–27, 23] and in references therein.

[1] This nonperturbative result (see 't Hooft [24]) is obtained by combining analyticity and unitarity of two-point Green functions in the complex momentum squared plane with asymptotic freedom. Of interest is a remark by Moroz [26] that the use of the Callan-Symanzik equation is not crucial in 't Hooft's argument, which works also if theoretical evidence of asymptotic freedom is replaced by the experimental one (if available). Then the argument is free of the problem of a suitable definition of the coupling constant.

2. I will now discuss another type of generalization of the notion of Borel transform, [28–30], which makes use of specific structures of singularities that are typical for QED and QCD.

Let me first make a general remark. Until now I have been mostly exposing mathematical methods. Now I will pass to some practical aspects of my talk, assuming that two-point Green's functions have special singularities, the renormalons, in the Borel plane, a structure that is now almost universally adopted. These ideas are based on various mathematical models developed in late 70's and early 80's, but nowadays these features are often considered as true features of Nature. With this reservation I am passing to this subject.

The electromagnetic current-current correlation function is a useful example to explain a typical structure of singularities in the complex coupling-constant plane. Denoting this function $\Pi^{\mu\nu}$,

$$\Pi^{\mu\nu} = \mathrm{i}\int \mathrm{d}^4 x \mathrm{e}^{-\mathrm{i}qx} < 0 \mid \mathrm{T}(j^\mu(x) j^\nu(0)) \mid 0 > \tag{12}$$

$$= (g^{\mu\nu} q^2 - q^\mu q^\nu)\Pi(-q^2) \tag{13}$$

and taking R, the ratio of the total cross section for $\mathrm{e^+e^-} \to$ hadrons to that for $\mathrm{e^+e^-} \to$ muon pairs, which is related to its imaginary part,

$$R(s) = 12\pi \mathrm{Im}\Pi(s + \mathrm{i}0^+)\,, \tag{14}$$

one introduces a modified quantity $\tilde{\Pi}$ defined as

$$\tilde{\Pi}(Q^2) = -4\pi^2 Q^2 \left(\frac{\mathrm{d}}{\mathrm{d}Q^2}\right)\Pi(Q^2)\,, \tag{15}$$

(where $Q^2 = -q^2$), to avoid inessential logarithmic terms. The perturbation expansion of Π in the powers of the coupling constant has the form

$$\tilde{\Pi} \sim 1 + \frac{\alpha_s(Q^2)}{\pi} \sum_{n=0}^{\infty} \tilde{\Pi}_n (\alpha_s(Q^2))^n. \tag{16}$$

The dependence of $\tilde{\Pi}$ on $\alpha_s(Q^2)$ exhibits a complex structure of singularities in the coupling constant complex plane at and around the origin [24]. Their nature can be conveniently displayed when studied in the Borel plane as singularities of the corresponding Borel transform. Perturbation theory suggests the following structure of the singularities of the Borel transform [29]:

(1) *Instanton-antiinstanton pairs* [2, 3] generate equidistant singularities along the positive real axis starting at $t = 4$, for $t = 4l, l = 1, 2, \ldots$ Balitsky

[31] calculated the behaviour of $R_{e^+e^-\to\mathrm{hadrons}}$ near $t = 4$ and found the leading $\mathrm{I} - \bar{\mathrm{I}}$ singularity to be a branch point of strength $\frac{11}{6}(N_\mathrm{f} - N)$, where N and N_f is the number of colours and of flavours respectively.

(2) *Ultraviolet renormalons* are generated by contributions behaving as $c_{l+1} \sim (-b_0/l)^k k!$, for $l = 1, 2, ...$, leading to singularities located at $t = -2l/b_0$ on the negative real axis. Near the first of the points, $l = 1$, the singularity is $(b_0 t + 2)^{-1+\gamma}$, where γ is related to the anomalous dimension of local operators of dimension 6.

(3) *Infrared renormalons* are generated by contributions behaving as $c_{l+1} \sim (...b_0/l)^k k!$, $l = 2, 3, ...$, leading to singularities located at $t = l/b_0$. Near the first of the points the singularity behaves as [20] $(b_0 t - 4)^{-1-2\lambda/b_0}$.

Brown, Yaffe and Zhai and Beneke [28–30] use the information about the structure of the first infrared renormalon. Expanding $\mathrm{Im}\Pi$ and Π in powers of the coupling constant with the expansion coefficients a_n and c_n respectively and defining their respective Borel transforms

$$A(z) = \sum_{n=1}^{\infty} \frac{n a_n}{\Gamma(n+1)} z^n , \tag{17}$$

and

$$C(z) = \tilde{c}_0 + \sum_{n=1}^{\infty} \frac{n c_n}{\Gamma(n+1)} z^n , \tag{18}$$

they obtain by comparing the expansion coefficients [28] the following relation between A and C:

$$A(z) = \sin(b_0 z) C(z) , \tag{19}$$

which turns out to be a consequence of renormalization-group invariance [30]. Defining a modified Borel transform $\mathcal{F}(z)$ by

$$\mathcal{F}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(1+\lambda z)}{\Gamma(n+1+\lambda z)} f_n z^n \tag{20}$$

(thereby accounting for the first infrared renormalon), the authors of [29] and [30] consider the case of a general beta function, which they choose in such a scheme that its inverse contains two terms:

$$1/\beta(g^2) = -1/(b_0 g^4) + \lambda/(b_0 g^2) . \tag{21}$$

They find that, for this form of $\beta(g^2)$, the relation (19) remains valid also for $\mathcal{A}(z)$ and $\mathcal{C}(z)$, the modified (according to (20)) Borel transforms of $\mathrm{Im}\Pi$ and Π respectively:

$$\mathcal{A}(z) = \sin(b_0 z) \mathcal{C}(z) . \tag{22}$$

It was already pointed out that the concept of renormalon is presently applied to concrete physical situations. This poses the topical problem to what extent renormalons are physical concepts and to what extent they are just artefacts of our, maybe inadequate, formalism.

Table 3 gives a survey of singularities in the Borel plane.

TABLE III

Singularities in the Borel plane SU(N) QCD

	$\mathrm{I}-\bar{\mathrm{I}}$ **pairs**	**UV renormalons**	**IR renormalons**
position	$t = 4, 8, 12, \ldots$	$-2l/b_0,\ l = 1, 2, \ldots$	$2l/b_0,\ l = 2, 3, \ldots$
strength of the first singularity	$\frac{11}{6}(N_f - N)$ [31]	$(b_0 t + 1)^{-1+\gamma}$	$(b_0 t - 2)^{-1-2\lambda/b_0}$ $\lambda = b_1/b_0$, [20]

5. Remarkable phenomena in the Borel plane

A look at the Table III reveals that the positions of the singularities in the Borel plane as well as their strength depend on N and N_f and will thus change as they are varied. This is relevant to our discussion because the position and strength of the singularities nearest to the origin of the Borel plane determine the large-order properties of the perturbation expansion. This phenomenon was recently discussed by Lovett-Turner and Maxwell [32]. Here I shall repeat a part of the analysis made in Ref. [32] and condense, with a few additions, a discussion I had with the second of the authors on this topic.

1. The instanton-antiinstanton pairs are covered by the infrared renormalons if $N_f = N(\mathrm{mod} 3)$. This follows from the condition $4n = 2l/b_0$, where $b_0 = (11N - 2N_f)/6$.

2. The first instanton-antiinstanton singularity ($t = 4$) disappears if $N_f \geq N$ and $N_f = N$ (mod 6).

3. Take $N = 3$ and $N_f = 15$ as a special case of item 1. Then $b_0 = \frac{1}{2}$ and the l-th infrared renormalon coincides with the l-th $\mathrm{I}-\bar{\mathrm{I}}$ pair. Since the first renormalon, $l = 1$, does not exist, the leading singularity is the first $\mathrm{I}-\bar{\mathrm{I}}$ pair.

4. If $N_f = 16$, $b_0 = \frac{1}{6}$ and the first infrared renormalon is located at $t = 24$, coinciding with the 6th $\mathrm{I}-\bar{\mathrm{I}}$ pair. The items 3 and 4 are examples of situations in which, contrary to common opinion, instantons play an important role in large-order behaviour. Analogous situations occur for different values of N; it generally holds that when the number of flavours

is sufficiently high (approaching the flavour-saturation value), instantons become more important than renormalons.

5. If, on the other hand, the number of flavours decreases, the importance of renormalons increases. For $N_f = 0$, $b_0 = \frac{11}{6}N$ and the density of renormalons is the highest; for 3 colours, there are 9 infrared renormalons below the first $I - \bar{I}$ pair, which covers with the 10th of them.

6. The case $N_f = 15$, $N = 3$ is very special also from the point of view of the strength of the infrared renormalon singularity, whose power is 44 in this case. Because of this, the nearest infrared singularity disappears, but also the nearest instanton-antiinstanton singularity disappears, because 15= 3 (mod 6), as required in the item 2 of this list.

7. It is worth mentioning that if another plane than the Borel one is chosen (see Table II, $\rho \neq 1$), all the singularities in the Borel plane either shrink to the origin or run away towards infinity.

Besides these general properties of the singularities in the Borel plane, considerable progress has been made in recent years in the knowledge of the singularities in the Borel plane in special physical systems and special functions. In particular, much is known in the case of heavy-light and heavy-heavy quark systems. So, in the large-N_f approximation there is a finite set of renormalon singularities [33] and a discrete infinity of the renormalon poles. Also, in the case of heavy-light quark-antiquark systems the structure of singularities in the Borel plane is known.

Effects of renormalons have also been studied in application to heavy quark physics; see [43] for renormalons in heavy-quark pole mass, [38, 39] for renormalons in inclusive heavy-quark decay rates and [38] for the case of exclusive decay rates. Connection between renormalons and power divergencies in heavy-quark physics was studied in [40].

6. Resummation of renormalon chains

The renewed interest in calculating higher-order perturbative corrections and in examining the high-order behaviour of perturbative series is intimately related to the investigation of renormalization scale and scheme dependence of a truncated series as well as to attempts to estimate its uncalculated remainder. In the past various criteria of finding a suitable renormalization prescription were proposed; they are based on an estimate of the size of the remainder. Examples are Stevenson's principle of minimal sensitivity [34], Grunberg's notion of effective charge [35], and the BLM method of scale setting [36] by Brodsky, Lepage and Mackenzie. The BLM method, which is based on an analogy with QED, was recently further developed and I shall briefly comment on it.

The BLM prescription is a method of estimating higher-order perturbative corrections of a physical quantity provided that the first approximation is known. It consists in the use of some "average virtuality" as scale in the running coupling. Instead of working with fixed scale,

$$\alpha_s(Q^2)\int d^4kF(k,Q)\,, \tag{23}$$

one averages over the logarithm of the gluon momentum:

$$\alpha_s(Q^2_{BLM})\int d^4kF(k,Q! \equiv \alpha_s(Q^2)\int d^4k\left(1-\frac{\beta_0}{4\pi}\alpha_s(Q^2)\ln\frac{-k^2}{Q^2}\right)F(k,Q). \tag{24}$$

This replacement amounts to acounting for higher-order terms in powers of $\alpha_s(Q^2)$ by making use of the renormalization-group evolution

$$\alpha_s(-k^2)=\alpha_s(Q^2)\sum_{n=1}^{\infty}\left(\frac{\beta_0\alpha_s(Q^2)}{4\pi}\right)^{n-1}(-\ln(-k^2/Q^2))^{n-1} \tag{25}$$

and retaining only the first two terms in the sum. This approach was recently generalized [37, 41, 42] by introducing the running coupling constant $\alpha_s(k^2)$ directly into the vertices of Feynman diagrams, with k being the momentum "flowing" through the line of the virtual gluon. This modification means replacement of (24) by

$$\alpha_s(Q^{*2})\int d^4kF(k,Q)=\int d^4k\alpha_s(-k^2)F(k,Q) \tag{26}$$

with $\alpha_s(x^2) = 4\pi/(\beta_0\ln\frac{x^2}{\Lambda^2_{QCD}})$. Note, however, that the beta function $\beta(\alpha_s(Q^2))$ is approximated by its first term only.

This method has been applied in phenomenology to various physical observables, like τ decay hadronic width and heavy-quark pole mass [45, 42], semileptonic B meson decay [44] and the Drell–Yan process [45]. The method makes maximal use of the information contained in one-loop perturbative corrections combined with the one-loop running of the effective coupling, thereby providing a natural extension of the BLM scale-fixing prescription.

Ellis *et al.* [46] use Padé approximants to develop another method of resumming the QCD perturbative series. The authors test their method on various known QCD results and find that it works very well.

7. Concluding remarks. Criticism

A typical feature of the present status of the QCD perturbative corrections is the trend to avoid explicit calculation of higher order corrections and, instead, to improve the result by making full use of some additional information we may dispose of. Such information may be, for instance, the renormalization group invariance (which allows one to introduce the running coupling constant instead of the fixed one), analyticity, or the structure of singularities in the Borel plane (where the regular location of singularities, to be approximated by poles, suggests the use of Padé approximants).

As already mentioned, the notion of renormalons, originally introduced and used to investigate interesting mathematical models, is now widely considered to have concrete background in physical phenomena. This universal belief meets with criticism that argues that the singularities in the Borel plane are nothing but products of special choice of the renormalization prescription [47]. Methods generalizing the scale-setting procedure developed by Brodsky, Lepage and Mackenzie meet with criticism [48] arguing that the approach is not fully independent of the choice of renormalization prescription. Further research will clarify the issue.

It seems that the present effort in further developing the idea of renormalon and the corresponding formalism will be helpful in finding a language appropriate for physical ideas in the nonperturbative sector. Generalizations of the scale-setting procedures are valuable by implementing new physical information without calculating higher-order perturbative corrections (what is not only cumbersome but also doubtful, due to divergence of the series), by using some additional, perturbatively independent information. This idea is not new, appearing in theoretical physics whenever technical difficulties force one to look for methods allowing the exploitation of all information on the system, including such that the existing formalism does not adequately take into account.

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