NONPERTURBATIVE PROPAGATORS AND QCD SUM RULES*

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QCD Sum Rules are used to study nonperturbative behaviour of quark propagators. Instead of using operator product expansion, we use Dyson-Schwinger equations and Ward-Takahashi identity. We found that good agreement with the data is obtained for logarithmically divergent effective quark mass.

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1. QCD spectral sum rules

The basic object used in the sum rules is a current with quantum numbers of the states to be investigated. In the simplest case it may be vector current build from quarks

$$J_V^\mu = \overline{q}\gamma^\mu q\,, \tag{1}$$

where appropriate summation over implicit color and flavor indices is assumed. Other simple possibilities are scalar, pseudoscalar and axial vector currents

$$J_S = \overline{q}q, \quad J_P = i\overline{q}\gamma_5 q, \quad J_A^\mu = \overline{q}\gamma_5\gamma^\mu q\,. \tag{2}$$

Correlator of such currents (we confine here to the vector case)

$$\Pi^{\mu\nu}(q^2) = i\int d^4x e^{iqx}\langle 0|T(J^\mu(x)J^\nu(0)|0\rangle \tag{3}$$

is related to its imaginary part

$$\Pi(q^2) = \frac{1}{\pi}\int\frac{\mathrm{Im}\,\Pi(s)}{s-q^2}ds\,, \tag{4}$$

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where $\Pi(q^2)$ is defined by relation

$$(q^\mu q^\nu - q^2 g^{\mu\nu})\,\Pi(q^2) = \Pi^{\mu\nu}(q^2)\,. \tag{5}$$

On the other hand, due to the optical theorem, Im $\Pi(s)$ is related to spectrum of hadrons with quantum numbers given by J^μ. This may be approximately written as

$$\mathrm{Im}\,\Pi(s) = \sum_R \frac{9\pi m_R^2}{4g_R^2}\delta(s - m_R^2) + \frac{1}{4\pi}\left(1 + \frac{\alpha_s}{\pi}\right)\theta(s - s_0)\,, \tag{6}$$

where summation is performed over resonances with the same quantum numbers as the current J^μ, m_R is mass of the resonance R, and g_R is its coupling to the current. For vector currents g_R may be related to electronic width of the resonance Γ_R

$$g_R^2 = \frac{4\pi\alpha^2 m_R}{3\Gamma_R}\,. \tag{7}$$

For large space-like s L.H.S. of (6) may be calculated perturbatively, and for small s R.H.S. of (6) is reasonably approximated by lowest resonances. The basic idea of the standard approach is to choose smallest s such that $\Pi(s)$ may be calculated perturbatively with nonperturbative SVZ–like [1] improvements. In actual calculations, instead of using directly relation (6), there are used two main methods: moments — for heavy quarks, and Borel (Laplace) transformation — for light quarks [7]. In what follows we will use moment sum rules to both cases.

Moments of $\Pi(s)$ are defined by

$$M_n = \int \frac{\mathrm{Im}\,\Pi(s)}{s^{n+1}}ds = \frac{1}{n!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2)\Big|_{q^2=0}\,. \tag{8}$$

M_n calculated using experimental data equals

$$M_n^{\mathrm{exp}} = \frac{9}{4g_{R_1}^2 m_{R_1}^{2n}}\left[1+\sum_{i>1}\frac{g_{R_i}^2}{g_{R_1}^2}\left(\frac{m_{R_1}}{m_{R_i}}\right)^{2n} + \frac{1}{n}\frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\left(\frac{m_{R_1}}{s_0}\right)^n\right], \tag{9}$$

where m_{R_1} and m_{R_1} are the parameters of the lowest lying resonance. Considering ratios of the moments

$$r_n = \frac{M_n}{M_{n-1}} \tag{10}$$

for high n we immediately obtain $r_n \approx m_{R_1}^{-2}$. Thus our aim is to calculate r_n theoretically.

2. Theoretical calculation of M_n

M_n is usually calculated semi-perturbatively using operator product expansion and assuming that only lowest twist condensates are relevant to this case. Our proposition is to use here truly nonperturbative propagators. In fact, we are not intended to calculate m_{R_1}, but rather to check experimentally favoured behaviour of the quark propagator. We start from Dyson-Schwinger equation for $\Pi^{\mu\nu}$

$$\Pi^{\mu\nu}(q^2) = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left(S(p+q)\Gamma^{\mu}(p+q,p)S(p)\Gamma_0^{\nu}\right) , \tag{11}$$

where $S(p)$ is full (nonperturbative) quark propagator, $\Gamma^{\mu}(p+q,p)$ is full, one particle irreducible quark-current vertex and Γ_0^{ν} is bare quark-current vertex (for vector current it is simply γ^{ν}).

If we confine to electromagnetic current (vector) case, then $\Gamma^{\mu}(p+q,p)$ should satisfy Ward-Takahashi identity

$$q_{\mu}\Gamma^{\mu}(p+q,p) = S^{-1}(p+q) - S^{-1}(p) . \tag{12}$$

One may divide $\Gamma^{\mu}(p+q,p)$ into two parts

$$\Gamma^{\mu}(p+q,p) = \Gamma_L^{\mu}(p+q,p) + \Gamma_T^{\mu}(p+q,p) , \tag{13}$$

where $\Gamma_L^{\mu}(p+q,p)$ (longitudinal part) alone satisfies (12) and

$$q_{\mu}\Gamma_T^{\mu}(p+q,p) = 0, \qquad \Gamma_T^{\mu}(p,p) = 0 . \tag{14}$$

From (14) it follows that for small q^2

$$\Gamma^{\mu}(p,p) \approx \Gamma_L^{\mu}(p,p) . \tag{15}$$

Thus for any given S we may calculate $\Pi(q^2 = 0)$ (unfortunately it is divergent), and (with accuracy decreasing with n) moments M_n.

The simplest $\Gamma_L^{\mu}(k,p)$ (12) (gauge technique [2]) is

$$S(k)\Gamma_{L,gt}^{\mu}(k,p)S(p) = \int \rho(\sigma) \frac{1}{\gamma\cdot p - m + i\epsilon}\gamma^{\mu}\frac{1}{\gamma\cdot p - m + i\epsilon} d\sigma , \tag{16}$$

where it is assumed spectral representation for $S(p)$

$$S(p) = \int \frac{\rho(\sigma)d\sigma}{\gamma\cdot p - m + i\epsilon} . \tag{17}$$

Eq. (16) may be written also in another form, without using spectral representation

$$S(k)\Gamma^{\mu}_{L,gt}(k,p)S(p) = \frac{k^2A(k^2)-p^2A(p^2)}{k^2-p^2}\gamma^{\mu} + \frac{A(k^2)-A(p^2)}{k^2-p^2}\gamma\cdot k\,\gamma^{\mu}\,\gamma\cdot p - \frac{B(k^2)-B(p^2)}{k^2-p^2}(\gamma\cdot k\gamma^{\mu}+\gamma^{\mu}\gamma\cdot p)\,, \quad (18)$$

where $S(p) = A(p^2)\gamma\cdot p + B(p^2)$.

Another Γ_L was proposed by Ball and Chiu [3]

$$\Gamma^{\mu}_{L,BC}(k,p) = \frac{\gamma^{\mu}}{2}\left(\frac{1}{Z(k^2)}+\frac{1}{Z(p^2)}\right) - \frac{(p+k)^{\mu}}{k^2-p^2}\left(\frac{\Sigma(k^2)}{Z(k^2)}-\frac{\Sigma(p^2)}{Z(p^2)}\right) + \frac{1}{2}\frac{(\gamma\cdot k+\gamma\cdot\ p)(k+p)^{\mu}}{k^2-p^2}\left(\frac{1}{Z(k^2)}-\frac{1}{Z(p^2)}\right), \quad (19)$$

where

$$S(p) = \frac{Z(p^2}{\gamma\cdot p-\Sigma(p^2)}\,. \quad (20)$$

If one consider perturbative S then both those Γ^{μ} have improper large momentum behaviour. To remedy this situation there were proposed appropriate transversal parts of Γ. For $\Gamma^{\mu}_{L,gt}$ [4]:

$$S(k)\Gamma^{\mu}_{T,gt}S(p) = \left(Z(k^2)-Z(p^2)\right)\frac{\gamma^{\mu}(k^2-p^2)-(k+p)^{\mu}(\gamma\cdot k-\gamma\cdot p)}{D(k,p)}\,, \quad (21)$$

where simplest $D(k,p) = \left(k^2+p^2\right)^2$.

For $\Gamma^{\mu}_{L,BC}$ (19) appropriate form of transverse vertex was proposed by Curtis and Pennington [5]

$$\Gamma^{\mu}_{T,BC} = \frac{1}{2}\left(\frac{1}{Z(k^2)}-\frac{1}{Z(p^2)}\right)\frac{\gamma^{\mu}(k^2-p^2)-(k+p)^{\mu}(\gamma\cdot k-\gamma\cdot p)}{d(k,p)} \quad (22)$$

with $d(k,p) = \frac{(k^2-p^2)^2+(M(k^2)+M(k^2))^2}{k^2+p^2}$.

3. Heavy quark (charm) mesons

For vector charmed mesons we take

$$J^\mu = \bar{c}\gamma^\mu c\,. \tag{23}$$

In this case usual expression for M_n, including one loop corrections and gluon condensate, is

$$M_n = \frac{1}{m_c^{2n}} \cdot w_n \cdot (1 + \alpha_s a_n + b_n \Phi)\,, \tag{24}$$

where m_c is c quark mass,

$$w_n = \frac{3}{4\pi^2}\frac{2^n(n+1)(n-1)!}{(2n+3)!!2^{2n}}\,, \quad \Phi = \frac{4\pi^2}{9}\left\langle 0\left|\frac{\alpha_s}{\pi}G_{\mu\nu}G^{\mu\nu}\right|0\right\rangle \Big/ (4m_c^2)^2\,,$$

and coefficients a_n and b_n may be found in [6].

Taking $\Gamma^\mu = \Gamma^\mu_{L,gt}$ (see (18)) and using Dyson–Schwinger equation (11) we obtain

$$M_n = -\frac{1}{(n-1)!}\frac{d^{n-1}A(q^2)}{(dq^2)^{n-1}} \cdot w_n\,, \tag{25}$$

where $S(p) = A(p^2)\gamma \cdot p + B(p^2)$.

For $\Gamma^\mu = \Gamma^\mu_{L,BC}$ there is no simple expression for M_n and moments were calculated numerically.

Results for heavy quarks

- For $\Gamma^\mu = \Gamma^\mu_{L,gt}$ simplest quark propagator giving good agreement (within 1% for r_n, $n \leq 8$) with data is

$$S(p) = \frac{1}{\gamma \cdot p - m(p^2)}\,, \tag{26}$$

  where $m(p^2) = m_0\left[1 - \delta\ln(1 - \frac{p^2}{m_0^2})\right]$ with $m_0 = 1.28\text{GeV}$, $\delta = 0.06$.
- The case $\Gamma^\mu = \Gamma^\mu_{L,CP}$ agrees with the case $\Gamma^\mu = \Gamma^\mu_{L,gt}$ within 3%.
- Taking $\Gamma^\mu = \Gamma^\mu_{L,BC} + \Gamma^\mu_{T,BC}$ gives another 3%.
- For $\Gamma^\mu = \Gamma^\mu_{L,gt} + \Gamma^\mu_{T,gt}$ moments M_n are divergent.

4. Light quark mesons

For light quarks we consider a current

$$J^\mu = \tfrac{1}{2}(\overline{u}\gamma^\mu u - \overline{d}\gamma^\mu d) \tag{27}$$

corresponding to ρ meson family. For null quark mass perturbative expressions for M_n are divergent even in the lowest order, as may be easily seen from (24). But, on the other hand, nothing special happens for the same moments calculated from experimental data using (9)!

We assume that null current quark mass means that

$$S(p) = A(p^2)\gamma\cdot p + 0 \tag{28}$$

as it should be in the chiral limit.

Taking $\Gamma^\mu = \Gamma^\mu_{L,gt}$ we again obtain for M_n expression (25).

In this case Ball and Chiu vertex (19) takes simpler form

$$\begin{aligned}\Gamma^\mu_{L,BC}(k,p) =& \frac{\gamma^\mu}{2}\left(\frac{1}{Z(k^2)}+\frac{1}{Z(p^2)}\right)\\ &+\frac{1}{2}\frac{(\gamma\cdot k+\gamma\cdot p)(k+p)^\mu}{k^2-p^2}\left(\frac{1}{Z(k^2)}-\frac{1}{Z(p^2)}\right).\end{aligned} \tag{29}$$

Expression for M_n is in this case much simpler and very close to (25)

$$M_n = -\frac{1}{(n-1)!}\frac{d^{n-1}A(q^2)}{(dq^2)^{n-1}}w'_n\,, \tag{30}$$

where coefficients w'_n were calculated numerically and differ only slightly from w_n.

Results for light quarks

- For $\Gamma^\mu = \Gamma^\mu_{L,gt}$ good agreement with the data (within 1.5% for $n \le 6$) is for
$$S(p) = \frac{\gamma\cdot p}{p^2 - m(p^2)}\,, \tag{31}$$
where $m(p^2) = m_0\left[1-\frac{\delta}{2}\ln(1-\frac{p^2}{m_0^2})\right]$ with $m_0 = 0.29\text{GeV}$, $\delta = 0.05$.
- The case $\Gamma^\mu = \Gamma^\mu_{L,CP}$ agrees with the case $\Gamma^\mu = \Gamma^\mu_{L,gt}$ within 2.5%.
- $\Gamma^\mu = \Gamma^\mu_{L,BC} + \Gamma^\mu_{T,BC}$ leads to divergent M_n.
- For $\Gamma^\mu = \Gamma^\mu_{L,gt} + \Gamma^\mu_{T,gt}$ moments M_n are divergent.

Divergence of M_n, coming from considered Γ_T^μ, does not mean that our approach is inaccurate, but rather that those Γ_T^μ are wrong — we remind: experimental data does not lead to divergences in M_n.

5. Discussion and conclusions

At first view it seems that presented approach to bound states cannot be correct. As one can learn from Bethe–Salpeter equation for positronium, bound states come from infinite ladder of photon propagators, and in fact all the information about these states is contained in the electron–photon vertex. But usual QCD sum rules approach with perturbatively calculated coefficients in Wilson expansion and phenomenologically treated condensates also does not contain infinite ladder of propagators. Since many people claims that this approach works good, there is no reason to reject the method based on Dyson–Schwinger equation and Ward–Takahashi identity.

Our results for heavy quark system are close to everyone's expectations: c quark propagator does not differs much from the usual non confined propagator. We found that logaritmically divergent mass leads to good agreement with the data, but since we expect agreement only with first few moments and modification to the constant mass is rather small, other parametrisations are also possible.

The situation is quite different for light quarks. From the above analysis it seems clear that the light quark propagator does not contain pole for $p^2 = 0$. For small momenta chiral symmetric part of the propagator behaves like usual propagator with mass close to the nucleon mass divided by 3.

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