LATTICE FERMIONS*

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In this talk some recent developments in the overlap formulation of chiral gauge theories on a lattice is briefly reviewed. We argue that the overlap formalism correctly accounts for all the desirable features of the chiral Dirac determinant for slowly varying weak external gauge fields.

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1. Introduction

Weyl fermions are the building blocks of matter. The two spinor representations ψ_L and ψ_R of SO(1,3) can belong to inequivalent representations of internal symmetry group and maintain their distinct identities under local Yang–Mills and reparametrization transformations. These geometrical facts can also be incorporated in classical field theories involving such particles. Quantum mechanically, however, the situation is different. The need for ultraviolet regularization introduces new elements into our description, which modify the classical picture drastically.

Thus starting from a classical action $S = S(A, \psi_L, \psi_R)$ we need to construct a quantum effective action $\Gamma_\Lambda = \Gamma_\Lambda(A, \psi_L, \psi_R)$ which will depend on a set of regulator parameters Λ and should incorporate as much symmetries of S as possible. In an expansion of Γ_Λ in powers of the Planck's constant $\hbar$, S coincides with the $\hbar = 0$ term. It turns out that in a generic chiral gauge theory, where the left handed and right handed fermions belong to unitarily inequivalent representations of the gauge group, there is no choice of the regularization which makes Γ_Λ to respect all the symmetries of S, at least with a finite number of regularization parameters Λ. This is the origin of chiral anomalies. For the mathematical consistency of chiral gauge theories like the standard Weinberg - Salam model, it is essential that the

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local gauge symmetries are unbroken, at least in the limit of infinite cut off. In the standard electroweak model or in grand unified theories based on groups such as SU(5), where fermions belong to complex representation of the gauge group, the gauge invariance of the renormalized theory is ensured by adjusting the fermionic content such that the anomalies of the local symmetries are cancelled.

On the other hand there are global symmetries which should be anomalously broken in the standard model. According to our present understanding it is believed that nature is profiting from this symmetry violation. Indeed the non-vanishing divergence of the flavour singlet axial current

$$J_{\mu}^{5} = \bar{u}\gamma_{\mu}\gamma_{5}u - \bar{d}\gamma_{\mu}\gamma_{5}d$$

is at the basis of our theoretical description of phenomena such as $\pi^0 \to \gamma\gamma$ decay [1] and the U(1) problem [2]. This need of preserving all of the local gauge symmetries and violating, in a very specific manner, some of the global symmetries is one of the sources of difficulties in constructing a lattice regularization of chiral gauge theories.

There are of course good perturbative regularization schemes which enable us to calculate Γ_A to any desired order in perturbation theory, at least in principle. In this talk we would like to address the question of non-perturbation definition of Γ_A, *i.e.* a lattice formulation of chiral gauge theories. Such a regularization would be needed for examining the non perturbative features of the standard electroweak model.

There is a generic difficulty with the lattice transcription of fermions which is independent of the details of the lattice and the details of the coupling in the fermionic bilinear part of the action. This difficulty is known as the doubling problem and was realized rather early in the history of lattice formulation of the quarks in QCD [3]. As will be shown in the next section in vector like theories like QCD the doublers can always be removed by adding a space dependent mass term. This of course violates the global axial symmetries of QCD and thereby gives rise to the γ_5 anomaly mentioned above. Infact in the absence of a chiral symmetry breaking term there would be no axial anomaly because the lattice regularized theory , in the absence of explicit symmetry violating terms, is chirally symmetric. It has been shown by Karsten and Smit that the doublers have pairwise opposite chiral charges and therefore are responsible for the vanishing of the axial anomaly [4]. Thus the lattice regularization of the fermionic part of vector like theories poses no conceptual difficulties. This is not so for chiral gauge theories. In these theories the chiral symmetry is gauged and thus the addition of a space dependent mass term would not be compatible with the guaged chiral symmetry. Furthermore, if the lattice has any crystallographic symmetry group, it could also prevent the inclusion of a Wilson term to remove the doublers. Of course in any practical situation we work on lattices which possess some finite subgroup of the orthogonal group as its symmetry group. The fermions belong to the spinor representation of this group.

It was subsequently shown that under certain assumptions chiral gauge theories cannot be defined on the lattice. This no–go theorem [5] implies that if Γ_Λ is defined on a lattice its infrared limit, which should correspond to the quantum description of the classical action S for the slowly varying fields on lattice scale, is inevitably a vector like theory. In particular, if not circumvented, the no–go theorem implies that there is no lattice formulation of the standard Weinberg–Salam theory or SU(5) GUT, even though the fermions belong to anomaly free representations of the gauge group. Our aim in this talk is to explain one possible attempt at bypassing the no–go theorem.

The formalism described in this talk has been called the overlap approach by Narayanan and Neuberger [6] and is an evolution of an idea of Kaplan [7]. According to Kaplan the 4-dimensional Euclidean space should be realised as a domain wall in a 5-dimensional space. The number of the components of the Fermi fields is the same in 4 and 5 dimensional spaces. However since the 5-dimensional fields depend on an extra coordinate, from a 4-dimensional point of view we shall have an infinite number of fermionic flavours. This is why there is a possibility for evading the no go theorem in this approach.

Recently there has been few other suggestions to solve this old problem [8, 9] which shall not be discussed here. For a summary see [10].

2. The no–go theorem

For a bilinear fermionic Euclidean action on a lattice with lattice spacing $a = 1$, the inverse propagator has the structure

$$G^{-1}(k) = \gamma^\mu C_\mu(k) \,,$$

where $k \in T^4$, with the torus T^4 denoting a Brillouin zone in the momentum space and $C_\mu(k)$ is a vector field on T^4. In order for $G^{-1}(k)$ to approach the standard Euclidean propagator in the infrared we must have $C_\mu(k) \to k_\mu$ as $k_\mu \to 0$. However, a topological theorem due to Hopf and Poincaré will then require the existence of at least one more zero of $C_\mu(k)$ on T^4. In fact this theorem states that [11]

$$\sum_{C_\mu(k)=0} \frac{\text{Det } C}{|\text{Det } C|} = 0 \,,$$

where

$$\text{Det } C = \operatorname*{Det}_{\mu,\nu}\left(\frac{\partial C_\mu(k)}{\partial k_\nu}\right).$$

Thus the zeroes of C will come in pairs. A close inspection indicates that they come in chiral pairs, *i.e.* in the infrared we will have equal number of states with $\gamma_5 = +1$ and $\gamma_5 = -1$. Furthermore, the doublers will belong to the same representation of the internal symmetry group.

For vector like theories like QCD, Wilson suggested [3] a solution to the doubling problem by adding a k-dependent mass term to $G^{-1}(k)$, *viz.*

$$G^{-1}(k) = \gamma^\mu\, C_\mu(k) + B(k)\,.$$

$B(k)$ is chosen such that as $k \to 0$, $B(k) \to k^2$, while $B(k)$ does not vanish at any other zeroes of $C_\mu(k)$. In this way the extra poles of the propagator are removed. The new term also breaks the chiral symmetry of the original problem. It has been shown that [4] this breakdown is at the origin of chiral anomaly in the divergence of the axial current in lattice regularized QCD. Since the axial symmetry is not gauged in QCD, its breakdown by the B-term does not create any problems.

Now let us consider Weyl fermions. In this case we expect the lattice G^{-1} to have the following form

$$\begin{aligned} G^{-1}(k) &= \sigma^\mu\, C_\mu(k) \\ &= iC_4(k) + \underline{\sigma} \cdot \underline{C}(k)\,, \end{aligned}$$

where σ are the Pauli matrices. The doublers will of course be still there and one may think of adding a B-term to remove them. In the continuum such a term would not be compatible with the $O(4)$ invariance of the chiral Euclidean action. On the lattice the fermions should belong to a spinor representation of a finite subgroup of O(4) which would prevent the introduction of the B-term. This argument or the more general no-go theorem of Nielsen and Ninomya imply that the 2-component Weyl fermions cannot be defined on a lattice in such a way that a chiral theory emerges in the infrared limit.

3. The overlap proposal

The basic idea of the overlap approach is to recover the chiral theory as a limit of a vector like theory [6, 12–14], Thus, instead of starting from a 2-component theory we start from a 4-component spinor theory, in which case we can add a Wilson type B-term to remove the doubler poles. The approach involves two Hamiltonians $H_\pm$, which differ in the sign of a mass term, *viz.*

$$H_\pm = \sum_{n,m} \psi(n)^\dagger \; H_\pm(n-m) \; U(n,m) \; \psi(m) \,, \tag{3.1}$$

where the 1-body Hamiltonians $H_\pm(n-m)$ are defined in terms of their Fourier transforms

$$\sum_n H_\pm(n) \; e^{-ik\cdot n} = \gamma_5 (i \, \not{C}(k) + B(k) \pm \Lambda) \,. \tag{3.2}$$

The $U(n,m)$ in (3.1) are the link variables. For smooth background gauge fields we shall write $U(n,m)$ as

$$U(n,m) = T \exp i \int_0^1 dt (n-m) \cdot A((1-t)n + tm) \,, \tag{3.3}$$

where A is the Lie algebra valued vector potential.

Let $|A\pm\rangle$ denote the normalized states obtained by filling the negative energy states of $H_\pm(A)$. The phases of these states should be fixed by an extra condition which we take to be

$$\langle \pm | A \pm \rangle > 0 \,. \tag{3.4}$$

Define the functional $\Gamma(A)$ by

$$e^{-\Gamma(A)} = \lim_{|k/\Lambda|\to 0} \frac{\langle A+|A-\rangle}{\langle +|-\rangle} \,, \tag{3.5}$$

where k is a typical momentum in the Fourier expansion of A. The claim is that $e^{-\Gamma(A)}$ is a good candidate for the determinant of the chiral Dirac operator operator on the lattice.

4. Verification of the overlap proposal

Now we would like to explain some of the analytical tests in support of (3.5).

Firstly, if $e^{-\Gamma(A)}$ is a good definition for the determinant of the chiral Dirac operator it should be a complex functional of A with a non–gauge invariant phase, if the fermions are not in the anomaly free representations of the gauge group. Furthermore, under local gauge transformation the change in $\mathrm{Im}\,\Gamma(A)$ should be proportional to the consistent anomalies, at least for smooth external gauge fields.

In general we should try to verify that $\Gamma(A)$ incorporates all the known perturbative features of chiral gauge theories. The non perturbative effects such as those normally attributed to the instantons should also be accounted for. Here we summrize very briefly what perturbative tests have been carried out so far.

Firstly any new proposal for the effective action of chiral fermions on the lattice should reproduce the chiral anomaly correctly. The lattice regularized overlap passes this test successfully [14]. The U(1) anomaly in the 2–dimensional continuum overlap and the non Abelian anomaly in 4-dimensions had been calculated before in Ref. [6] and [13] respectively. In the next section we shall skech the manner in whcih chiral anomalies are generated in the overlap approach.

The contribution of fermions to the vacuum polarization in the standard model has also been calculated in the framework of the overlap formalism. This is also a test on the contribution of the fermions to the 1–loop β–functions of the standard model. We have shown that (3.5) also passes this test successfully [15].

On a manifold with the topology of a torus there are non contractible loops. One can have guage fields with non zero line integrals along such loops which have vanishing field strength. The effective action for a discretised torus in 2–dimensions in the background of such flat connections has also been calculated [16] and the result agrees with what was known from the string literature [17]. This calculation had also been done numerically by Narayanan, Neuberger and Vranas [18].

Preceeding these analytic tests there have been several numerical tests of the overlap by Narayanan, Neuberger, all of them giving substantial support to the definition (3.5) [12, 18]. These authors have also studied numerically the instanton effects on the lattice and shown that the overlap does reproduce these effects too [12]. In the continuum version of the formalism it is not so hard to study the instanton physics and the results seem to be in agreement with our expectations.

More recently Kaplan and Schmaltz showed that the phase of the determinant of the chiral Dirac operator as defined by Kaplan's domain wall formalism or by continuum overlap, in the background of topologically trivial gauge fields, coincides with the η–invariant of the Dirac operator in one higher dimension [19]. This is in agreement with an earlier result of Alvarez–Gaumé *et. al*, who have shown that the phase of the determinant of chiral Dirac operator is given by the η–invariant [20].

5. Chiral anomaly

If the fermions are not in an anomaly free representation of the gauge group, the effective action will not be gauge invariant. The non gauge invariance has a precise form given by chiral anomalies. In this section we indicate the main steps in the calculation of chiral anomalies in overlap picture.

In lattice regulated overlap the anomalies manifest themselves as a consequence of the fact that the phase convention (3.4) is not gauge invariant. Insisting on (3.4) in all gauges forces $\Gamma(A)$ to transform anomalously. To see this, consider a local gauge transformation

$$A_\mu(x) \to A_\mu^\theta(x) = e^{i\theta(x)} \left(A_\mu(x) + i\partial_\mu\right) e^{-i\theta(x)} . \tag{5.1}$$

The link variable $U(n,m)$ transform according to

$$U(n,m) \to U^\theta(n,m) = e^{i\theta(n)} \, U(n,m) \, e^{-i\theta(m)} . \tag{5.2}$$

It follows that the Hamiltonians (3.1) transform according to

$$H_\pm(A^\theta) = U_\theta \, H_\pm(A) \, U_\theta , \tag{5.3}$$

where U_θ is the unitary operator that acts on the fermions

$$U_\theta \, \psi(n) \, U_\theta^{-1} = e^{-i\theta(n)} \, \psi(n) .$$

It can be expressed as

$$U_\theta = \exp\left(i \sum_n \psi(n)^\dagger \, \theta(n) \, \psi(n) \right) . \tag{5.4}$$

Since the perturbative ground state is not degenerate we must have

$$U_\theta |A\pm\rangle = |A^\theta \pm\rangle \, e^{i \, \Phi_\pm(\theta,A)} , \tag{5.5}$$

where $\Phi_\pm$ is real. They are determined precisely from the requirement that the transformed states $|A^\theta\pm\rangle$ satisfy the same phase convention as $|A\pm\rangle$, *viz.*,

$$\langle \pm | A^\theta \pm \rangle > 0 .$$

For an infinitesimal θ one obtains

$$\Phi_+(\theta, A) = \mathrm{Re} \sum_n \frac{\langle + | \psi(n)^\dagger \, \theta(n) \, \psi(n) | A+\rangle}{\langle + | A+ \rangle} . \tag{5.6}$$

A similar expression gives $\Phi_-(\theta, A)$.

To evaluate the variation of $\Gamma(A)$ under (5.1) we simply make use of (5.5) in $\langle A^\theta + | A^\theta - \rangle$. This implies that the real part of $\Gamma(A)$ is gauge invariant and the change in its imaginary part is given by

$$g(\theta, A) \equiv i\left(\Phi_+(\theta, A) - \Phi_-(\theta, A)\right) .$$

Expand g in powers of θ and A. The first order part in θ gives the chiral anomaly. Furthermore, using the group property one can show that $g(\theta, A)$ satisfies a cocycle condition, which when expanded in powers of θ upto second order, it leads to the Wess–Zumino consistency condition.

The formalism described above has been used to evaluate chiral anomalies for the non–abelian continuum [13] as well as the lattice regularized overlap [14]. It is also powerful enough to produce gravitational anomalies in 2–dimensional quantum gravity [21].

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