TOWARDS A MICROLOCAL SPECTRUM CONDITION*

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An attempt for the formulation of a local version of the spectrum condition on globally hyperbolic spacetime is discussed. It relies on microlocal analysis.

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The incorporation of gravity into quantum theory is an outstanding open problem of physics. A modest attempt in this direction is the formulation of quantum physics under the influence of an external classical gravitational field. But even this apparently harmless problem turns out to lead to some severe difficulties. One may hope, that the investigation of these problems provides hints for the formulation of quantum gravity. In addition, the emphasis on local concepts which is essential for the analysis of quantum fields on curved spacetime might also deepen our understanding of quantum field theory on Minkowski space.

The mathematical framework in which these problems are discussed is that of quantum field theory on a pseudoriemannian manifold with Lorentzian signature. In a first step one would like to formulate the analogue of the Wightman axioms for fields living on such a spacetime. So one looks for operator valued distributions, and wants to impose certain general requirements on them. But whereas commutativity for fields at spacelike separated points makes perfect sense on a curved background, the axioms of covariance and of positivity of the energy momentum spectrum have no obvious generalization. Covariance is not to be expected to be meaningful on a generic spacetime without nontrivial isometries. The spectrum condition, however, expresses the requirement of stability of the system, and one would like to have a local version of stability also in the generic case.

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There are different attempts to formulate a stability condition. One might for instance try a formulation in terms of the energy momentum tensor. For free fields, this idea led to the concept of a Hadamard state [1, 2] which is a quasifree state of the free Klein Gordon field on a globally hyperbolic manifold whose 2-point function has the singularity structure of Hadamard's fundamental solution of the Klein Gordon equation. It was shown by Christensen [3] that in such states the expectation value of the energy momentum tensor can be defined by a point splitting technique, after subtraction of divergent contributions which depend only on the local space time geometry and a scale parameter.

Another idea starts from the observation that, relative to a (nonunique) definition of particles, particles are created in nonstatic spacetimes. One then tries to find states with minimal particle production; these are the so called adiabatic vacua introduced by Parker [4] and carefully investigated by Lüders and Roberts [5].

These proposals are restricted to free fields. An approach which remains meaningful for interacting fields is due to Haag, Narnhofer and Stein [8, 9]. These authors look at the short distance behaviour of n-point functions. They show that provided the scaling limit exists the limit is a field theory on the tangent space. The tangent space is isomorphic to Minkowski space, and it is natural to impose as a condition of local stability that the scaling limit theory satisfies the Minkowski space spectrum condition. Hadamard states satisfy this condition, so this gives another motivation for the Hadamard condition, but for free fields the Hadamard condition is not implied by the property of local stability in the sense of Haag, Narnhofer und Stein. Recently, Buchholz and Verch [10] succeeded in giving a very general definition of the scaling limit. It would be interesting to know whether a condition of local stability in their framework leads to a stronger conclusion.

A new development was started by the work of Radzikowski [11]. In his thesis he found a characterization of Hadamard states in terms of wave front sets. This discovery makes the connection with the spectrum condition much more transparent. Furthermore, it is technically much easier to handle.

Let me recall the definition of a wave front set[1]. The wave front set of a distribution f on a manifold M is the set of all elements $(x, k), k \neq 0$ of the cotangent bundle T^*M with the following property:

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[1] Wave front sets (in the $\mathcal{C}^\infty$ sense) were introduced by Hörman der [6, 7]. The corresponding analytic wave front sets were discussed by Sato [12, 13] and, independently, by Bros and Iagolnitzer [14] (there under the name "essential support"). Here we use only the $\mathcal{C}^\infty$ version of the theory.

Let χ be a diffeomorphism from a neighbourhood U of x into the tangent space T_xM at x with $\chi(x) = 0$ and $d\chi_x = id$. Let φ be a test function with compact support and $\varphi(x) \neq 0$, and let C be a conic neighbourhood of k. Then the function

$$k' \mapsto \langle f, \varphi \exp i\langle k', \chi(\cdot)\rangle\rangle$$

is not rapidly decreasing in C.

It is sufficient to check this condition for a fixed χ.

In special cases, the wave front sets can be easily determined. For instance, the wave front set of the δ-function on $\mathbb{R}^n$ is

$$WF(\delta) = \{(0, k), k \in \mathbb{R}^n \setminus \{0\}\}$$

since for all testfunctions φ

$$\langle \delta, \varphi \exp ikx\rangle = \varphi(o)\,.$$

For the massless propagator function $D_+(x) = \frac{1}{x^2 + i\epsilon x_0}$ one finds

$$WF(D_+) = \{(x, k), x^2 = 0, k^2 = 0, k_0 > 0 \quad \text{and} \quad x = \lambda k, \lambda \geq 0\}\,,$$

and this coincides with the wave front set of the massive propagator.

Now Radzikowski [11] (with some gaps filled in by Köhler [15]) showed that the wave front set of a Hadamard bisolution of the Klein Gordon equation on a globally hyperbolic Lorentz manifold M is

$$WF(\omega_2) = \{(x, k, x', k') \in T^*M^2 \setminus \{0\}, k \in \partial V_+, (x, k) \sim (x', -k')\}\,,$$

where $(x, k) \sim (x', k')$ means that there is some lightlike geodesic γ from x to x' such that k is coparallel to the tangent vector of γ, *i.e.*, for some $\lambda \in \mathbb{R}$, $\langle k, \xi\rangle = \lambda g(\xi, \dot\gamma) \forall \xi \in T_xM$, and k' is the parallel transport of k along γ.

Moreover, he showed that any 2-point function of the free Klein Gordon field whose wave front set is given by the expression above satisfies the Hadamard condition in the formulation of Kay and Wald [2].

This result relates the Hadamard condition to the spectrum condition. In addition, it enables us to use the machinery of pseudodifferential operators and Fourier integral operators. An immediate application is the construction of new Wightman fields which are not solutions of the Klein Gordon equation. For instance, for two independent commuting free fields A and B one can define the pointwise product $C(x) = A(x)B(x)$ [15]. The n-point functions of C are just products of the n-point functions of A and B. These products of distributions exist because the sum of the wave front sets of both factors does not contain points with zero covectors. *E.g.*, the 2-point function is

$$\omega(C(x)C(y)) = \omega_2(x,y)^2\,,$$

with the wave front set contained in the set

$$\{(x, k_1 + k_2; x', k_1' + k_2') \in T^*M^2 \setminus \{0\}, k_1, k_2 \in \partial V_+, (x, k_i) \sim (x, -k_i'), i = 1, 2\}\,.$$

In particular, for coinciding points x and x', $k = k_1 + k_2$ may be an arbitrary nonzero covector in the closed forward light cone; also for conjugated points x, x', k need not to be lightlike.

A further class of Wightman fields whose existence follows from similar arguments are the Wick polynomials of a free field. Their construction was performed in a collaboration with Brunetti and Köhler [16].

Let us look at the n-point function of Wick polynomials $:\varphi^{l_1}:, \ldots, :\varphi^{l_n}:$. Formally, it is a sum of products of 2-point functions ω_2^r where r runs over the set of ordered pairs $(r_1, r_2), 1 \leq r_1 < r_2 \leq n$ and ω_2^r is the 2-point function ω_2 in the variables x_{r_1}, x_{r_2} considered as a distribution on M^n. These products are well defined, if the convex hull of the union of the wave front sets of ω_r does not contain points with vanishing covectors. But the latter property can easily be verified. Namely, let $x \in M^n$ and consider an arbitrary convex combination $k = \sum_\nu \lambda_\nu k^\nu$ with $(x, k^\nu) \in WF(\omega_2^{r^\nu})$ and $\lambda_\nu \in \mathbb{R}$, $\lambda_\nu \geq 0$, $\sum_\nu \lambda_\nu = 1$. Let $i, 1 \leq i \leq n$ be the smallest number occuring in the pairs r^ν with $\lambda_\nu > 0$. Then the i-th component k_i of k is given by $k_i = \sum_{\nu, r_1^\nu = i} \lambda_\nu k_i^\nu$ with $k_i^\nu \in \partial V_+ \setminus \{0\}$ for all ν with $\lambda_\nu > 0$. Thus k_i is a convex combination of nonzero elements of the closed forward light cone and therefore does not vanish.

The wave front set of such a n-point function is contained in the convex set just described. We obtain the following theorem [16].

Theorem: *Let φ be a free Klein Gordon field on a globally hyperbolic manifold, and let ω be a Hadamard state of φ. Then on the GNS Hilbert space of ω there exist all Wick polynomials of φ (defined by point splitting) as operator valued distributions with a dense invariant domain, and the minimal invariant domain of φ is a core for all smeared Wick polynomials.*

The definition of Wick polynomials used in the theorem depends on the choice of the Hadamard state ω, $:\varphi^l:=:\varphi^l:_\omega$. Since the 2-point functions of Hadamard states differ only by a smooth function, a change of ω to ω' formally amounts to a redefinition of Wick polynomials

$$:\varphi^n(x):_{\omega'}=:\varphi^n(x):_\omega + \sum_{l<n} c_l(x)\, :\varphi^l(x):_\omega$$

with smooth functions c_l. Moreover, since different Hadamard states induce locally unitarily equivalent states of the free field [18], the GNS Hilbert spaces $\mathcal{H}_\omega$ and $\mathcal{H}_{\omega'}$ may be identified after restriction of the free field to a bounded region, so the relation above may be understood as a relation between operators on the same Hilbert space. There is, however, the problem that the domains of definition of both sides do not necessarily coincide. One may hope that they have the same closures, but this remains to be proven.

We now want to use the information on the wave front sets of n-point functions of Wick polynomials for a guess on the wave front set of Wightman functions of interacting fields. It was again Radzikowski [11] who made the first attempt towards a formulation of a spectrum condition in terms of wave front sets. Unfortunately, his proposal is violated by the fields just described [15].

One possible route towards a spectrum condition is to weaken the conditions which characterize the wave front sets of Wightman functions of Wick polynomials such that the important properties of additivity (so products of independent fields exist) and of state independence (every state in the folium of ω, *i.e.* every state which can be obtained from the reference state by application of a finite sum of products of smeared field operators, has the same wave front set) are preserved. So one may assume that the wave front set of the n-point function ω_n of an interacting field is contained in a convex set constructed in complete analogy to the case of Wick polynmials with the only change that the wave front set of a Hadamard bisolution is replaced by the larger set

$$\Gamma = \{(x,k;x',k') \in T^*M^2 \setminus \{0\}, k \in \overline{V_+}, (x,k) \approx (x',-k')\},$$

where $(x,k) \approx (x',-k')$ means that there is some piecewise smooth curve γ from x to x' such that k' is the parallel transport of k along γ [16]. One then can show that on Minkowski space the usual spectrum condition implies the mentioned condition on the wave front set [16]. Actually, as was pointed out by Rainer Verch [18], in generic spacetimes where the holonomy group is the whole proper orthochronous Lorentz group, the above condition reduces to the requirement that the first nonvanishing k_i is in the closed forward light cone, and the last nonvanishing k_j is in the closed backward light cone.

One may conjecture that wave front sets of Wightman functions of noninteracting theories are significantly smaller than those of interacting theories. This might be a useful local criterion for interaction.

These ideas might be tested within perturbation theory. Perturbation theory on curved space time has been studied mainly on spaces with euclidean signature. It is likely that the degree of divergences and the structure of counter terms remains the same on manifolds with a Lorentzian metric. However, the standard formulation of perturbation theory with its emphasis on the role of the vacuum and of momentum space is not well adapted to the problem.

A candidate for a local formulation of perturbation theory is the Epstein Glaser method [19]. This method, based on ideas of Bogoliubov, starts from the observation that the time ordered products of Wick polynomials are everywhere well defined up to some submanifold of lower dimension. Renormalization is then the extension of these operator valued distributions to the whole space. The leading principle for this extension is a causality condition for a S-matrix depending on a test function which describes a space time dependent coupling.

As a little remark I want to emphasize that this approach leads to a completely local construction of interacting fields. Namely, the interacting fields in a bounded region $\mathcal{O}$ are given by Bogoliubov's formula

$$\varphi_g(x) = S(g)^{-1}\frac{\delta}{\delta h(x)}S(g,h)|_{h=0}\,,$$

where $S(g)$ is the S-matrix for a spacetime dependent coupling $g \in \mathcal{D}(M)$, $g = \text{const}$ on $\mathcal{O}$, and $S(g,h)$ includes a source term $\varphi(h)$ in the Lagrangian. Now consider (g,h) as a 2-component function $\boldsymbol{g}$ and set

$$V(\boldsymbol{g},\boldsymbol{h}) = S(\boldsymbol{g})^{-1}S(\boldsymbol{g}+\boldsymbol{h}).$$

The operators V satisfy the causality relation

$$V(\boldsymbol{g},\boldsymbol{h}_1+\boldsymbol{h}_2) = V(\boldsymbol{g},\boldsymbol{h}_2)V(\boldsymbol{g},\boldsymbol{h}_1)$$

provided the future of supp$\boldsymbol{h}_2$ does not intersect with supp$\boldsymbol{h}_1$. Let $\boldsymbol{g}'$ coincide with $\boldsymbol{g}$ on the intersection of the future and the past of $\mathcal{O}$. Then there are testfunctions $\boldsymbol{g}_+,\boldsymbol{g}_-$ with $\boldsymbol{g}' = \boldsymbol{g}+\boldsymbol{g}_+ +\boldsymbol{g}_-$ and such that the future of supp$\boldsymbol{g}_+$ does not intersect with $\mathcal{O}$ and the future of $\mathcal{O}$ does not intersect with supp$\boldsymbol{g}_+$. Let supp$\boldsymbol{h} \subset \mathcal{O}$. By the definition of V and the causality relation we get

$$\begin{aligned} V(\boldsymbol{g}',\boldsymbol{h}) &= V(\boldsymbol{g}+\boldsymbol{g}_-,\boldsymbol{g}_+)^{-1}V(\boldsymbol{g}+\boldsymbol{g}_-,\boldsymbol{g}_+ +\boldsymbol{h}) \\ &= V(\boldsymbol{g}+\boldsymbol{g}_-,\boldsymbol{h}) \\ &= V(\boldsymbol{g},\boldsymbol{g}_-)^{-1}V(\boldsymbol{g},\boldsymbol{h}+\boldsymbol{g}_-) \\ &= V(\boldsymbol{g},\boldsymbol{g}_-)^{-1}V(\boldsymbol{g},\boldsymbol{h})V(\boldsymbol{g},\boldsymbol{g}_-), \end{aligned}$$

hence choosing $\mathbf{g} = (g,0)$, $\mathbf{g}' = (g',0)$ we find

$$\varphi_{g'}(x) = V(g,g_-)^{-1}\varphi_g(x)V(g,g_-) \quad ,x \in \mathcal{O}\,.$$

So all algebraic properties of the interacting fields within $\mathcal{O}$ are independent of the choice of g outside of $\mathcal{O}$. In particular the wave front set of n-point functions in perturbation theory can be computed locally.

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