THE EFFECTIVE ACTION IN QUANTUM GRAVITY*,**

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I review different methods used in description of the high-energy processes in quantum gravity. As a first I discuss the result obtained within the eikonal approximation. Next I describe the derivation of the effective action for the quantum gravity in the multi-Regge kinematics.

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In this lecture I review some results obtained in studies of quantum gravity in the multi-Regge limit. When the energy E involved in a scattering process exceeds significantly the Planck mass M_P

$$E \gg M_P \tag{1}$$

quantum effects become important. Unfortunately, the problem of construction quantum theory of gravity is unsolved till now. One of the main obstructions is that the quantization of general relativity leads to perturbatively non-renormalizable theory. A consequence of this fact is the general opinion that studies of quantum gravity which are based on perturbation theory are without predictive power.

It turns out that in the case of processes occurring in the multi-Regge kinematics (MRK) one can obtain definite predictions. In this kinematics, the produced particles which arise in the scattering of two high energy particles fly mainly in the direction of one of the incoming particles. They have

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very large and strongly ordered longitudinal momenta and small transverse momenta. The calculations of scattering amplitudes show that renormalization effects do not contribute to leading and next-to-leading terms. In particular, Lipatov has calculated the inelastic amplitudes corresponding to production of gravitons in MRK and graviton's Regge trajectory [1]. More recently Amati, Ciafaloni and Veneziano have calculated the quantum correction to the classical deflection angle of a graviton in the field of a black hole [2].

Gravity in MRK is a strongly interacting theory with the effective coupling constant sG, where G is Newton constant and s is the energy in the c.m.s. squared. For the case of the elastic processes this fact implies that the t-channel intermediate state with n gravitons leads to a contribution which behaves as s^{n+1}, *i.e.* t-channel exchanges involving more gravitons are more important. This is due to the spin of graviton $\sigma = 2$. One should confront the above result with the analogous contribution due to gluonic exchanges ($\sigma = 1$) in QCD which is of the order s (modulo $\ln s$'s), independently of how many gluons are exchanged.

As a consequence of the appearance of the big coupling constant sG, to obtain reliable results, one should sum the contributions corresponding to an arbitrary number of gravitons in the t-channel. The eikonal diagrams are those which for given power of G provide the contributions with the highest power of s.

The calculations of eikonal diagrams in the case of gravity proceed in a similar way as analogous calculations in QED (compare [4]). Kabat and Ortiz have calculated the leading terms of those diagrams in the case of scalar-scalar scattering by graviton exchanges [5]. The obtained result for the scattering amplitude M is the following

$$M(s,t=-\vec{q}_{\perp}^{\,2}) = -2is\int d^2x_{\perp} e^{-i\vec{q}_{\perp}\vec{x}_{\perp}} \left[\exp\left(i4\pi Gs \int \frac{d^2k_{\perp}}{(2\pi)^2} \frac{e^{-i\vec{k}_{\perp}\vec{x}_{\perp}}}{k_{\perp}^2+\mu^2}\right) - 1\right], \tag{2}$$

where μ is a graviton mass which serves as an infrared cut-off. Performing the integrals in Eq. (2) one gets

$$M(s,t) = \frac{8\pi Gs^2}{-t} \frac{\Gamma(1-iGs)}{\Gamma(1+iGs)} \left(\frac{4\mu^2}{-t}\right)^{-iGs} \tag{3}$$

This result can also be obtained using other methods, which do not refer to Feynman diagrams. t'Hooft has considered [6] the scattering of gravitons as a quantum mechanical problem of one particle moving in the gravitational field of the other particle. This gravitational field has the form of "shock-wave" as described by the Aichelburg-Sexl metric [7]. Amati, Ciafaloni and Veneziano [2] as well as Muzinich and Soldate [8] derived Eq. (3) by considering the low energy limit of string amplitudes. Finally, E. and H. Verlinde obtained this result by constructing the effective theory for gravity in Regge kinematics [9]. This effective theory emerges as a result of a natural separation of longitudinal and transverse degrees of freedom in the underlying kinematics.

Although Eq. (3) was obtained by summing the leading terms arising in each order of perturbation theory, it needs to be corrected. Let us observe that the product of the last two factors in Eq. (3)

$$\frac{\Gamma(1-iGs)}{\Gamma(1+iGs)}\left(\frac{4\mu^2}{-t}\right)^{-iGs} \tag{4}$$

is a pure phase. This means that in order to obtain a meaningfull result one should correct the above calculations by taking into account non-leading terms which were previously neglected. In particular it is not enough to calculate the eikonal diagrams only up to leading accuracy but higher precision is needed. Moreover, it is necessary to consider also diagrams in which the exchanged t-channel gravitons interact each with other. In the MRK this corresponds to taking into account inelastic diagrams with the production of many gravitons in s-channel. The importance of non-leading terms for the final result is clearly shown up in calculations of quantum correction to the classical deflection angle [2]. For distances close to the Schwarzschild radius the magnitude of the quantum correction is of the order of the classical expression which requires a further improvement of the approximation.

The problem of how to go beyond the eikonal approximation in a systematic way is unsolved till now. The methods which work well within the eikonal approximation are difficult to generalize beyond it (for a discussion of these questions see Ref. [3]). The new approach to this problem was proposed by Lipatov in Ref. [3] and is based on the effective action for gravity in the MRK. The effective action involves only those degrees of freedom which are relevant for processes in the underlying kinematics. The analogous effective action to that of Ref. [3] turned out to be a useful tool also in the superstring approach to high-energy gravitational scattering [11].

In the case of QCD in MRK, the effective action was derived by Kirschner, Lipatov and the present author from the original QCD Lagrangian [12]. It turns out that using similar method one can also derive the effective action for quantum gravity in MRK [13]. The main steps of this derivation are presented in the following.

The starting point is the Einstein action. Because of the MRK it is natural to perform all derivations in the axial gauge, with the momentum of an incoming particle taken as the gauge vector ($p_B^\mu = \frac{s^{1/2}}{2}(1,0,0,-1)$).

The gauge fixing conditions are chosen as

$$g_{--} = g_{-i} = 0, \quad g_{-+} = 2e^{\psi/2}\,, \tag{5}$$

where the light-cone variables are defined as $x_\pm = x_0 \pm x_3$. The physical degrees of freedom are represented by the two independent matrix elements γ_{11}, γ_{12}, where γ_{ij} is defined by the transverse components of the metric $g_{ij}, i, j = 1, 2$

$$g_{ij} = e^{\psi} \;\; \gamma_{ij}, \qquad \det(\gamma_{ij}) = 1\,. \tag{6}$$

After solving the constraints related to the gauge choice (5) the Lagrangian will be expressed only in terms of the matrix elements γ_{ij}. We parametrize them as

$$\gamma_{ij} = (e^h)_{ij}, \qquad \mathrm{Sp}\; h = 0\,, \tag{7}$$

and we introduce the complex field h defined by the two independent elements of the matrix h_{ij} as

$$h = \frac{1}{\sqrt{2}}(h_{11} - ih_{12})\,. \tag{8}$$

Complex notations will also be used for two-dimensional transverse momentum and position vectors ($x = x^1 + ix^2, \partial = \frac{\partial}{\partial x}$) as in [12]. Expanding Einstein's Lagrangian in powers of h and keeping all terms including the quartic in h we arrive at the following starting point of our analysis

$$\begin{aligned}
\mathcal{L} =& \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \mathcal{L}^{(4)} + ...,\\
\mathcal{L}^{(2)} =& -2h^*(\partial_+\partial_- - \partial\partial^*)h\,,\\
\mathcal{L}^{(3)} =& 2\alpha\{(\partial_-h^*\partial_-h)\partial^{*2}\partial_-^{-2}h\\
&+ \partial_-h^*h\partial^{*2}\partial_-^{-1}h - 2\partial_-h^*\partial^*h\partial^*\partial_-^{-1}h + \mathrm{c.c.}\},\\
\mathcal{L}^{(4)} =& 2\alpha^2\{-2|\partial_-^{-2}(\partial_-^3h^*\partial^*\partial_-^{-1}h\\
&- \partial_-^2\partial^*h^*h)|^2 + |\partial_-^{-2}(\partial_-^2h^*\partial^*h - \partial_-\partial^*h^*\partial_-h)|^2\\
&+ |\partial_-^{-1}(\partial_-h^*\partial^*h - \partial^*h^*\partial_-h)|^2 - 3|\partial_-^{-1}(\partial_-h^*\partial^*h)|^2\\
&+ 3\partial_-^{-1}(\partial_-h^*\partial_-h)\partial_-^{-1}(\partial h^*\partial^*h)\\
&+ [\partial_-^{-2}(\partial_-h^*\partial_-h) - h^*h][\partial h^*\partial^*h + \partial^*h^*\partial h\\
&- \partial\partial^*\partial_-^{-1}h^*\partial_-h - \partial_-h^*\partial\partial^*\partial_-^{-1}h]\}\,.
\end{aligned} \tag{9}$$

In writing down Eq. (9) we included a factor $(8\pi G)^{1/2}$ into the definition of h; $\alpha = (4\pi G)^{1/2}$, where G is Newton's constant. Moreover, we assume for simplicity of notation that the differential operators act only on the nearest fields.

The fields in Eg. (9) as it stays contain all modes. So our aim is to eliminate those degrees of freedom which are not present in MRK. Let us first separate the fields modes according to the MRK

$$h \to h_1 + h + h_t\,, \tag{10}$$

where h_1, h and h_t contain the modes of the following momentum ranges

$$\begin{aligned} h_1 &: |k_+k_-| \gg |\kappa|^2 \sim |q|^2, \\ h &: |k_+k_- \; - \; |\kappa|^2| \sim |q|^2, \\ h_t &: |k_+k_-| \ll |\kappa|^2 \sim |q|^2. \end{aligned} \tag{11}$$

The field h_t describes exchanged particles in t-channel whereas field h corresponds to the scattered particles. The "heavy" modes h_1 describe highly virtual particles and these have to be integrated out. This is done in the first perturbative order by means of equations of motion describing heavy modes (the saddle point method).

Consider first the kinetic term of the Lagrangian (9). With the separation (10) it decomposes as

$$\mathcal{L}^{(2)} = -2h_1^*(\partial_+\partial_- - \partial\partial^*)h_1 - 2h^*(\partial_+\partial_- - \partial\partial^*)h + 2h_t^*\partial\partial^*h_t\,. \tag{12}$$

The part of the triple interaction vertices $\mathcal{L}^{(3)}$ (9) which leads to large contributions is obtained when the inverse of ∂_- acts on the field with the smallest momentum component k_-. In order to isolate this part we separate the modes in $\mathcal{L}^{(3)}$ by making the substitution

$$h \longrightarrow \tilde{h} + \tilde{h}_t\,.$$

Here, $\tilde{h}$ denotes the field with all modes *i.e.* those of h, h_t and h_1 whereas $\tilde{h}_t$ describes the fields carrying those modes of h and h_t whose momentum component k_- is , due to the MRK, much smaller than the ones in $\tilde{h}$. Next, the interaction Lagrangian for heavy modes is obtained by keeping only terms linear in h_1 (which are contained in $\tilde{h}$). We obtain the following equation of motion for the modes $h_1^{(0)}$

$$\begin{aligned} (\partial_+\partial_- - \partial\partial^*)h_1^{(0)} = -\alpha\{&\partial_-(\partial_-\tilde{h}\tilde{\mathcal{A}}_{++}) + \tfrac{1}{2}\partial_-\tilde{h}(\partial^*\tilde{\mathcal{A}}_+) \\ &- \tfrac{1}{2}\partial_-\tilde{h}\;\; \partial\tilde{\mathcal{A}}_+^* - \partial_-(\partial^*\tilde{h}\tilde{\mathcal{A}}_+) - \partial(\partial_-\tilde{h}\tilde{\mathcal{A}}_+^*)\}\,. \end{aligned} \tag{13}$$

where

$$\begin{aligned} \tilde{\mathcal{A}}_{++} &= \partial_-^{-2}(\partial^{*2}\tilde{h}_t + \partial^2\tilde{h}_t^*), & \tilde{\mathcal{A}}'_+ &= -i\partial_-^{-1}(\partial^{*2}\tilde{h}_t - \partial^2\tilde{h}_t^*)\,, \\ \tilde{\mathcal{A}}_+ &= 2\partial_-^{-1}\partial^*\tilde{h}_t, & \tilde{\mathcal{A}}_+^* &= 2\partial_-^{-1}\partial\tilde{h}_t^*\,. \end{aligned} \tag{14}$$

The form in which Eq. (13) is written down emphasizes the underlying MRK: the momentum component k_- of $\tilde{A}$ is much smaller than the one of $\tilde{h}$ whereas the situation is inverse if we consider momentum components k_+. Moreover, for simplicity of notation we still use the same symbol $\tilde{h}$ even if this field does not contain heavy modes any more.

The result of integration over heavy modes $\mathcal{L}^{(1)}$ is given by the kinetic term with opposite sign for fields $h_1^{(0)}$ (13). The result takes the form

$$\begin{aligned}
\mathcal{L}^{(1)} &= \alpha^2 \tilde{T}_{--} \tilde{\mathcal{I}}_{++} , \\
\tilde{T}_{--} &= \partial_- \tilde{h}^* \partial_- \tilde{h} , \\
\tilde{\mathcal{I}}_{++} &= - (\partial_+^{-1} \tilde{A}_{++} \overleftrightarrow{\partial}_- \tilde{A}_{++}) \\
&\quad + \{\partial(\partial_+^{-1} \tilde{A}_{++} \tilde{A}_+^* - \tilde{A}_{++} \partial_+^{-1} \tilde{A}_+^*) + \partial \tilde{A}_{++} \partial_+^{-2} \partial^* \tilde{A}_{++} + \text{c.c.}\} .
\end{aligned} \tag{15}$$

Let us now observe that, although we integrated out over heavy fields propagating in the s-channel, Eq. (15) is factorized in the t-channel. The result (15) can be obtained from the new Lagrangian which contains:

a) the kinetic term for t-channel fields h_t from Eq.(12), smallskip

b) the leading triple vertex

$$\mathcal{L}^{(3+)}_{\text{leading}} = 2\alpha \tilde{T}_{--} \mathcal{A}_{++} \tag{16}$$

supplemented by Eq. (15) and Eq.(14) in which we restrict ourselves to t-channel field only,

c) the new induced vertex $\mathcal{L}^{(1)}_{\text{ind}}$ which is given by formula

$$\mathcal{L}^{(1)}_{\text{ind}} = -\alpha \partial \partial^* \mathcal{A}_{--} \tilde{\mathcal{I}}_{++} , \tag{17}$$

where we have introduced t-channel field $\mathcal{A}_{--}$ defined as

$$\mathcal{A}_{--} = \tfrac{1}{2} \partial_-^2 (\partial \partial^*)^{-2} (\partial^{*2} h_t + \partial^2 h_t^*) = \tfrac{1}{2} \partial_-^4 (\partial \partial^*)^{-2} \mathcal{A}_{++} . \tag{18}$$

The above results permit us to read off some vertices of the effective Lagrangian. Let us consider the leading vertex (16) in which we restrict the modes of both fields appearing in $\tilde{T}_{--}$ (see Eq. (15) to the modes of h fields only. This leads to the effective scattering vertex of gravitons off the $\mathcal{A}_{++}$ field

$$\begin{aligned}
\mathcal{L}^{(s+)} &= 2\alpha T_{--} \mathcal{A}_{++} , \\
T_{--} &= \partial_- h^* \partial_- h .
\end{aligned} \tag{19}$$

The vertex (16) gives also a contribution $\mathcal{L}^{(3-+)}$ to the production vertex. It is obtained by restricting the modes of one field $\tilde{h}$ to the modes of the h field and keeping in the second field $\tilde{h}$ only the modes of the h_t field. Also the induced vertex (17) contributes to graviton production. In this case one of the fields in $\tilde{\mathcal{I}}_{++}$ carries the modes h and the other the modes h_t. The sum of these both contributions leads to the following effective graviton production vertex

$$\mathcal{L}^{(-+)} = -2\alpha(\partial^{*2}\mathcal{A}_{--}\partial^2\mathcal{A}_{++} - \partial\partial^*\mathcal{A}_{--}\partial\partial^*\mathcal{A}_{++})\partial^{-2}h + \text{c.c.}. \quad (20)$$

Let us note that the production vertex (20) contains the non-local expression $\partial^{-2}h$. As our aim is to have an effective Lagrangian which is local we introduce a new s-channel field ϕ defined as

$$\phi = -\frac{1}{\partial^{-2}}h \quad (21)$$

(compare [12]). In this way we obtain the following kinetic term for the s-channel fields

$$\mathcal{L}_s^{(2)} = -2\phi^*(\partial_+\partial_- - \partial\partial^*)\partial^2\partial^{*2}\phi\,, \quad (22)$$

the scattering vertex off $\mathcal{A}_{++}$ fields

$$\mathcal{L}^{(s+)} = 2\alpha\partial_-\partial^{*2}\phi^*\partial_-\partial^2\phi\mathcal{A}_{++}\,, \quad (23)$$

and the production vertex

$$\mathcal{L}^{(-+)} = 2\alpha(\partial^{*2}\mathcal{A}_{--}\partial^2\mathcal{A}_{++} - \partial\partial^*\mathcal{A}_{--}\partial\partial^*\mathcal{A}_{++})\phi + \text{c.c.}. \quad (24)$$

In order to complete the derivation of the effective Lagrangian we have to determine the vertex $\mathcal{L}^{(3-)}$ describing graviton scattering off the field $\mathcal{A}_{--}$. The parity invariance of the theory implies that $\mathcal{L}^{(3-)}$ is obtained from $\mathcal{L}^{(3+)}$ (Eq. (23)) by the simultaneous exchange of $+ \leftrightarrow -$ and $\phi \leftrightarrow \phi^*$

$$\mathcal{L}^{(s-)} = 2\alpha\mathcal{A}_{--}\partial_+\partial^{*2}\phi\partial_+\partial^2\phi^*\,. \quad (25)$$

However, this way to derive $\mathcal{L}^{(s-)}$ is not satisfactory. The reason is that our derivation of the effective Lagrangian is performed in the peculiar axial gauge (5) which breaks the $+ \leftrightarrow -$ symmetry of the whole procedure. Of course the final result does not depend on this choice of gauge but expressions in the intermediate steps are gauge dependent. Therefore, to check the consistency of the method it is necessary to reproduce the vertex $\mathcal{L}^{(s-)}$ by careful collecting all ingredients which contribute to it. It turnes out that the scattering vertex $\mathcal{L}^{(s+)}$ (23) is obtained in a rather straightforward way whereas many terms contribute to the vertex $\mathcal{L}^{(s-)}$. This part of our derivation is rather technical and tedious so I skip the details (see Ref. [13]). I want only to mention that the consistency of the calculations requires that the t-channel fields $\mathcal{A}_{++}$ and $\mathcal{A}_{--}$ have to be treated as independent fields, despite the fact that they were originally defined by Eqs. (14) and (18) in terms of the same field h_t. This requirement leads to the kinetic term $\mathcal{L}_t^{(2)}$ for those fields

$$\mathcal{L}_t^{(2)} = 2\mathcal{A}_{++}\partial\partial^*\mathcal{A}_{--} \tag{26}$$

which differs by a factor 2 from the expression obtained by formal substituting definitions (14) and (18) into Eq. (12). In such a way we have arrived to the effective Lagrangian $\mathcal{L}^{(\mathrm{eff})}$ for gravity in the MRK which is given by the sum of the terms

$$\mathcal{L}^{(eff)} = \mathcal{L}_s^{(2)} + \mathcal{L}_t^{(2)} + \mathcal{L}^{(3+)} + \mathcal{L}^{(3-)} + \mathcal{L}^{(-+)}\,, \tag{27}$$

where the elements of the sum are given by Eqs. (22), (26), (23), (25), and (24).

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