EQUATIONS OF MOTION OF CHARGED TEST PARTICLES FROM FIELD EQUATIONS*

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A new method of deriving equations of motion from field equations is proposed. It is applied to classical electrodynamics. As a result, we obtain a new, perfectly gauge-invariant, second order Lagrangian for the motion of classical, charged test particles.

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1. Introduction

The motion of classical, charged test particles, in the classical Maxwell field is derived usually from the *gauge-dependent* Lagrangian function

$$L = L_{\mathrm{particle}} + L_{\mathrm{int}} = -\sqrt{1-\boldsymbol{v}^2}\ (m + eu^\mu A_\mu(t,\boldsymbol{q}))\ , \tag{1}$$

where u^μ denotes the (normalized) four-velocity vector. Since the Lorentz force $eu^\mu f_{\mu\nu}$ derived from this Lagrangian is perfectly gauge invariant, it is not clear, why we *have* to use the *gauge-dependent* interaction term $eu^\mu A_\mu$, with no direct physical interpretation. Moreover, in this approach, the equations of motion are not uniquely implied by the field equations. As an example, non-linear forces of the type $u^\mu f_{\mu\lambda} f^\lambda{}_\nu$ cannot be *a priori*

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excluded. This corresponds to the general "folklore", which says that in electrodynamics (unlike in General Relativity) equations of motion cannot be derived from field equations.

In the present paper we propose a new method of deriving equations of motion from field equations. The method is based on an analysis of the geometric structure of generators of the Poincaré group, related with any special-relativistic, lagrangian field theory. This analysis leads us to a simple theorem, which we call "variational principle for an observer" (see Section 2). Applying this observation to the specific case of classical electrodynamics, we prove that (up to gauge dependent boundary terms) the interaction Lagrangian $eu^\mu A_\mu$ is equal to the amount of the total angular-momentum acquired by the system "field + particle" due to the interaction of the external field with the particle's own Coulomb field. In this approach, equations of motion are uniquely implied by the global conservation laws.

More precisely, our approach *uniquely* leads to a manifestly gauge-invariant, second order Lagrangian $\mathcal{L}$:

$$\mathcal{L} = L_{\text{particle}} + \mathcal{L}_{\text{int}} = -\sqrt{1-\mathbf{v}^2}\ \left(m + a^\mu u^\nu M^{\text{int}}_{\mu\nu}(t, \boldsymbol{q}, \boldsymbol{v})\right) \ , \qquad (2)$$

where $a^\mu := u^\nu \nabla_\nu u^\mu$ is the particle's acceleration. The skew-symmetric tensor $M^{int}_{\mu\nu}$ is equal to the amount of the total angular momentum, which is acquired by our physical system, when the particle's own Coulomb field is added to the background (external) field.

We stress that the new Lagrangian differs from the old one by (gauge-dependent) boundary corrections only (see Section 4 for a direct proof). Therefore, both Lagrangians generate the same physical theory (although the new Lagrangian is of second differential order, it depends linearly upon the second derivatives and produces the standard, second order equations of motion).

The above result is an immediate consequence of a consistent theory of interacting particles and fields (*cf.* [1, 2]), called *Electrodynamics of Moving Particles.* All the formuale of the present paper could be derived directly from the above theory in the *test particle limit* (i.e. $m \to 0$, $e \to 0$ with the ratio e/m being fixed).

The relation between L and $\mathcal{L}$ is analogous to the one well known in General Relativity: the gauge-invariant, second order Hilbert Lagrangian for Einstein equations may be obtained starting from the first order, gauge-dependent Lagrangian and supplementing it by an appropriate boundary term.

The above theory possesses also very nice gauge-invariant Hamiltonian structure. It turns out that our approach is perfectly equivalent to that proposed by Souriau (see [5]) (although the Souriau's contact structure is different from ours, they are, however, equivalent from the physical point of view). For more details about the Hamiltonian structure see [3].

2. Variational principle for an observer

Consider any relativistic-invariant, Lagrangian field theory (in this paper we will consider mainly Maxwell electrodynamics, but the construction given in the present Section may be applied to any scalar, spinor, tensor or even more general field theory). We want to describe the field evolution with respect to an observer moving along an arbitrary time-like trajectory ζ. The observer's co-moving frame is related with the laboratory one *via* a (time-dependent) boost transformation (see [2, 4] for more details). It is relatively easy to show (*cf.* [2]) that the field evolution with respect to the above *non-inertial reference frame* is a superposition of the following three transformations:

- time-translation in the direction of the local time-axis of the observer,
- boost in the direction of the acceleration a^k of the observer,
- purely spatial O(3)-rotation ω^m,

where the vector ω^m is given by

$$\omega_m = \frac{\epsilon_{mkl} v^k \dot{v}^l}{(1-\boldsymbol{v}^2)(1+\sqrt{1-\boldsymbol{v}^2})}, \tag{3}$$

and $\dot{v}^k$ is the observer's acceleration in the laboratory frame (note that the above formula is the same as the formula for an angular velocity of Thomas precession (*cf.* [4])).

It is, therefore, obvious that the field-theoretical generator of this evolution is equal to

$$H = \sqrt{1-\mathbf{v}^2}\left(\mathcal{E} + a^k R_k - \omega^m S_m\right), \tag{4}$$

where $\mathcal{E}$ is the rest-frame field energy, R_k is the rest-frame static moment and S_m is the rest-frame angular momentum. The factor $\sqrt{1-\boldsymbol{v}^2}$ in front of the generator is necessary, because the time $t = x^0$, which we used to parameterize the observer's trajectory, is not the proper time along ζ but the laboratory time. For any point $(t, \boldsymbol{q}(t)) \in \zeta$ the values of the Poincaré generators $\mathcal{E}, R_k$ and S_m are given as integrals of appropriate components of the field energy-momentum tensor over any space-like Cauchy surface Σ which intersects ζ precisely at $(t, \boldsymbol{q}(t))$ (due to Noether's theorem, the integrals are independent upon the choice of such a surface).

Given a field configuration, we are going to use the quantity H as a second order Lagrangian for the observer's trajectory. For this purpose we choose a "reference trajectory" ζ_0 and for each point $(t, \boldsymbol{q}(t)) \in \zeta_0$ we calculate the corresponding "reference values" of the generators $\mathcal{E}(t)$, $R_k(t)$, $S_m(t)$. Inserting them into H we finally consider the function obtained this way as a Lagrangian depending on a generic trajectory ζ *via* its velocity v and acceleration a.

Theorem. *Euler-Lagrange equations derived from the above Lagrangian are automatically satisfied by the trajectory* $\zeta = \zeta_0$.

This Theorem may be checked by simple inspection: the Euler-Lagrange equations derived from H are automatically satisfied due to the energy-momentum and angular-momentum conservation (it is an obvious consequence of the invariance of the theory with respect to the choice of an observer). In the next Section we show that if we add to the background (external) field a Coulomb field of a charged particle then variational principle obtained this way gives nontrivial particle's equations of motion.

3. Adding a test particle to the field

From now on we limit ourselves to the case of electrodynamics. Suppose that to a given background field $f_{\mu\nu}$ we add a test particle carrying an electric charge e. Denote by $f_{\mu\nu}^{(y,u)}$ the (boosted Coulomb) field accompanying the particle moving with constant four-velocity u, which passes through the space-time point y. Being bi-linear in fields, the energy-momentum tensor $T^{\rm total}$ of the total field

$$f_{\mu\nu}^{total} := f_{\mu\nu} + f_{\mu\nu}^{(y,u)} \tag{5}$$

may be decomposed into three terms: the energy-momentum tensor of the background field T^{field}, the Coulomb energy-momentum tensor $T^{particle}$, which is composed of terms quadratic in $f_{\mu\nu}^{(y,u)}$ and the "interaction tensor" $T^{\rm int}$, containing mixed terms:

$$T^{\rm total} = T^{\rm field} + T^{\rm particle} + T^{\rm int} . \tag{6}$$

Let us try to calculate the generator (4) for such a composed field configuration. For this purpose we have to integrate appropriate components of $T^{\rm total}$ over any Σ which passes through $y = (t, \boldsymbol{q}(t))$. Integrating $T^{\rm field}$ and $T^{\rm int}$ we obtain the corresponding generator for the background field, which we call $H^{\rm field}$, and the "interaction generator" $H^{\rm int}$. Because the left-hand side and the first two terms of the right-hand side of (6) are conserved (outside of the particle's trajectory), we conclude that also $T^{\rm int}$ is conserved.

This implies, that the above integrals are invariant with respect to changes of Σ, provided the intersection point with the trajectory does not change (see [1] for more details).

Unfortunately, the Coulomb tensor T^{particle} has an r^{-4} singularity at y and cannot be integrated. According to the renormalization procedure defined in [2] we replace its integrals by the corresponding components of the total four-momentum of the particle: $p_\lambda^{\text{particle}} = mu_\lambda$ and the total angular momentum: $M_{\mu\nu}^{\text{particle}} = 0$. The renormalized particle generator is, therefore, defined as follows:

$$H^{\text{particle}} = m\sqrt{1-\boldsymbol{v}^2}\,. \tag{7}$$

This way we obtain the total (already renormalized) generator as a sum of three terms

$$H^{\text{total}} = H^{\text{field}} + H^{\text{int}} + H^{\text{particle}}, \tag{8}$$

where the first term is quadratic and the second term is linear with respect to the background field $f_{\mu\nu}$. To calculate the "interaction generator" H^{int} let us observe that the only non-vanishing term in H^{int} comes from the static moment term R in (4), because the mixed terms in both the energy $\mathcal{E}$ and the angular momentum S vanish when integrated over any Σ (see [2] for the proof).

Finally, we have

$$H^{\text{total}} = H^{\text{field}} + \sqrt{1-\boldsymbol{v}^2}\; a^\mu u^\nu M_{\mu\nu}^{\text{int}}(t, \boldsymbol{q}, \boldsymbol{v}) + \sqrt{1-\mathbf{v}^2}\; m\,, \tag{9}$$

where the interaction term is defined as the following integral

$$M_{\mu\nu}^{\text{int}}(y) := \int\limits_{\Sigma} \left\{ (x_\mu - y_\mu) T_{\nu\lambda}^{\text{int}}(x) - (x_\nu - y_\nu) T_{\mu\lambda}^{\text{int}}(x) \right\} d\Sigma^\lambda(x)\,, \tag{10}$$

and Σ is any hypersurface which intersects the trajectory at the point $y = (t, \boldsymbol{q}(t))$. In particular, using the particle's rest-frame and integrating over the rest-frame hypersurface Σ_t we have

$$M_{k0}^{\text{int}} = \frac{e}{4\pi} \int_{\Sigma_t} \frac{x_k x^n}{r^3} D_n(x)\, d^3x\,, \tag{11}$$

where D_n is the corresponding component of the external field $f_{\mu\nu}$.

Now, let us consider the generator H^{total} as a second order Lagrangian for the particle's trajectory ζ. However, as we proved in the previous section, the background field generator H^{field} does not contribute to the particle's equations of motion. Therefore, we finally take

$$\mathcal{L} := -(H^{\text{particle}} + H^{\text{int}}) = -\sqrt{1-\boldsymbol{v}^2}\ (m + a^\mu u^\nu M^{\text{int}}_{\mu\nu}(t,\boldsymbol{q},\boldsymbol{v}))\,, \qquad (12)$$

as a particle's Lagrangian (we take the "$-$" sign to obtain correct Lagrangian for a free particle). This way the formula (2) is fully justified. The gauge-invariant Lagrangian $\mathcal{L}$ has a direct physical interpretation contrary to the standard one.

We stress that in deriving (12) we use only Maxwell equations and some (physically natural) assumptions about the particle's structure. Therefore, our approach may be treated as a natural way of deriving equations of motion from field equations.

The same method may be used to derive particle's equations of motion from field equations in other field theories. We are going to apply this method to derive the equations of motion for particles interacting with the classical Yang–Mills gauge field and finally to describe the motion of particles in General Relativity.

4. Equivalence between the two variational principles

The easiest way to prove the equivalence consists in rewriting the field $f_{\mu\nu}$ in terms of the electric and the magnetic induction, using an arbitrary (curvilinear) coordinate system. We have:

$$\dot{A}_k - \partial_k A_0 := f_{0k} = -ND_k + \epsilon_{mkl}N^m B^l\,, \qquad (13)$$

where N is the *lapse function* and N^m is the *shift vector.* Let us use coordinates (t, x^k) defined, for a given trajectory ζ, by the tetrad $(\boldsymbol{e}_\mu)$:

$$y(t,x) := (t, \mathbf{q}(t)) + x^k \mathbf{e}_k(t)\,. \qquad (14)$$

It is easy to check, that we obtain this way the following transformation between our curvilinear coordinates (t, x^k) and the laboratory (Lorentzian) coordinates $y^\mu := (y^0, y^k)$ (*cf.* [2]).

$$y^0(t,x^l) := t + \frac{1}{\sqrt{1-\mathbf{v}^2(t)}}\ x^l v_l(t)\,,$$
$$y^k(t,x^l) := q^k(t) + \left(\delta^k_l + \varphi(\mathbf{v}^2)v^k v_l\right) x^l\,. \qquad (15)$$

The above formula may be used as a starting point of the entire proof. Calculating the components of the flat Minkowskian metric in our new coordinates we easily get $g_{kl} = \delta_{kl}$ for the space-space components, whereas the lapse and the shift are given by:

$$N = \frac{1}{\sqrt{-g^{00}}} = \sqrt{1 - \boldsymbol{v}^2}\ (1 + a_i x^i)\,,$$
$$N_m = g_{0m} = \sqrt{1 - \boldsymbol{v}^2}\ \epsilon_{mkl}\omega^k x^l\,. \tag{16}$$

The transformation (15) is *not* invertible (there are points where the lapse vanishes) but it does not produce any difficulty for what follows.

We multiply (13) by $\frac{x^k}{r^3}$ and integrate over dx^3 outside of the sphere $S(r_0)$. Observe, that $\frac{x^k}{r^3}\partial_k A_0 = \partial_k \frac{x^k A_0}{r^3}$. Moreover, we have:

$$\frac{x^k}{r^3}\epsilon_{mkl} N^m B^l = -\sqrt{1 - \boldsymbol{v}^2}\ \partial_l \left(B^l \frac{\omega_k x^k}{r} \right)\,. \tag{17}$$

Hence, after integration, we obtain in the limit $r_0 \to 0$:

$$\int\limits_{\Sigma_t} \frac{x^k}{r^3}\dot{A}_k\, d^3x + 4\pi A_0(0) = \int\limits_{\Sigma_t} \frac{x^k}{r^3} N D_k\, d^3x \tag{18}$$

(A_0 is the only surface term which survives, due to the standard asymptotic behaviour of the field). Observe that the constant part of the lapse function (16) does not produce any contribution to the above formula, because the flux of the field D_k through any sphere $S(r)$ vanishes due to the Gauss law. Hence, we may replace "N" by "$\sqrt{1 - \boldsymbol{v}^2}\ a_i x^i$" under the integral. Observe moreover, that A_0 in our particular coordinate system is equal to $\sqrt{1 - \boldsymbol{v}^2}\ u^\mu A_\mu$ in any coordinate system. Hence, dividing (18) by 4π, integrating it over a time interval $[t_1, t_2]$ and using (11) we finally obtain

$$\int\limits_{t_1}^{t_2} \mathcal{L} = \int\limits_{t_1}^{t_2} L + \frac{e}{4\pi}\left(\int\limits_{\Sigma_{t_2}} \frac{x^k}{r^3} A_k\, d^3x - \int\limits_{\Sigma_{t_1}} \frac{x^k}{r^3} A_k\, d^3x \right)\,. \tag{19}$$

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