SYMPLECTIC STRUCTURE FOR FIELD THEORY*

M.W. KALINOWSKI

Department of Physics, University of Łódź
Pomorska 149/153, 90-236 Łódź, Poland

W. PIECHOCKI

Soltan Institute for Nuclear Studies
Hoża 69, 00-681 Warsaw, Poland

(Received July 3, 1996)

Symplectic formalism for classical fields with intention to apply it to geometric quantization in field theory is presented.

PACS numbers: 02.40.-k, 02.30.-f, 11.10.Ef, 11.10.Lm

1. Introduction

We propose a construction of a symplectic structure for classical field theory for special type of observables. We use some generalization of differential geometry for infinite dimensional manifold of solutions to field equations. Formalism presented here seems to be useful for future geometric quantization of fields. All the results obtained here are preliminary.

2. Symplectic structure

Let us consider a classical field theory of φ^α, such that $Q \ni x \to \varphi^\alpha(x) \in \Re$, is a collection of fields on a curved spacetime Q, where α is an index labelling a certain representation of a Lorentz group.
Let us suppose that we deal with a local theory defined by the Lagrangian density $L(\varphi^\alpha, \nabla_b\varphi^\beta)$, where ∇_b is the covariant derivative with respect to the connection of the spacetime metric and where $b = 0, 1, 2, 3$.

* Presented at the II German–Polish Symposium "New Ideas in the Theory of Fundamental Interactions", Zakopane, Poland, September 1995.

From the variational principle we get the following field equations

$$\frac{\partial L}{\partial \varphi^\alpha} = \nabla_a \left(\frac{\partial L}{\partial \varphi_a^\alpha} \right), \tag{1}$$

where $\varphi_a^\alpha \equiv \nabla_a \varphi^\alpha$.

Let M be the infinite dimensional manifold of solutions of Eq. (1). We assume that an observable, $\mathcal{O}$, in our theory is of the form

$$M \ni \varphi^\alpha \to \mathcal{O}([\varphi^\alpha], [\varphi_b^\beta]) = \int_\Sigma d\sigma(x) f(\varphi^\alpha(x), \varphi_b^\beta(x)) \in \Re, \tag{2}$$

where $\Sigma \subset Q$ is a spacelike Cauchy surface, $d\sigma$ is the volume on Σ, $f \in C^\infty(Q)$.

Let us define $T_\varphi M$, the tangent space to M at φ. A tangent vector to M at φ is a solution of Eq. 1 linearized about φ (see [1]).

$$\frac{\partial^2 L}{\partial \varphi^\beta \partial \varphi^\alpha} X^\beta + \frac{\partial^2 L}{\partial \varphi_b^\beta \partial \varphi^\alpha} X_b^\beta = \nabla_a \left(\frac{\partial^2 L}{\partial \varphi^\beta \partial \varphi_a^\alpha} X^\beta + \frac{\partial^2 L}{\partial \varphi_b^\beta \partial \varphi_a^\alpha} X_b^\beta \right), \tag{3}$$

where $X_b^\beta \equiv \nabla_b X^\beta$.

Now we define an algebraic form of a vector field on an infinite dimensional manifold M (a manifold of functions). In order to do it we generalize a notion of a vector field in finite dimensional case using Gateaux derivatives in the following way.

$$\begin{aligned} X(\mathcal{O})|_\varphi &:= D\mathcal{O}|_\varphi X = \frac{d}{dt}\bigg|_{t=0} \mathcal{O}([\varphi^\alpha + tX^\alpha], [\varphi_b^\beta + tX_b^\beta]) \\ &= \int_\Sigma d\sigma \frac{d}{dt}\bigg|_{t=0} f(\varphi^\alpha + tX^\alpha, \varphi_b^\beta + tX_b^\beta) \\ &= \int_\Sigma d\sigma(x) \left[X^\alpha(x) \frac{\partial}{\partial \varphi^\alpha(x)} + X_a^\alpha(x) \frac{\partial}{\partial \varphi_a^\alpha(x)} \right] f(\varphi^\beta(x), \varphi_c^\lambda(x)), \end{aligned} \tag{4}$$

where $X(\mathcal{O})|_\varphi$ is a directional derivative of $\mathcal{O}$ along X at $\varphi \in M$ and $D\mathcal{O}|_\varphi$ is a Gateaux derivative.

We define the dual T^*M to the TM by

$$TM \ni X \to X^*(X) := \int_\Sigma d\sigma(X_\alpha^* X^\alpha + X^{*a}{}_\alpha X_a^\alpha) \in \Re. \tag{5}$$

In this way the element of T^*M is a covector field, which generalize a notion of one-form field on a finite dimensional manifold to an infinite dimensional case.
For special type of observables considered here we can use the following representation for X and X^*

$$X = \int_\Sigma d\sigma(x) \left[X^\alpha(x) \frac{\partial}{\partial\varphi^\alpha(x)} + X_a^\alpha(x) \frac{\partial}{\partial\varphi_a^\alpha(x)} \right], \tag{6}$$

$$X^* = \int_\Sigma d\sigma(x) [X_\alpha^*(x) d\varphi^\alpha(x) + X_\alpha^{*a}(x)\, d\varphi_a^\alpha(x)], \tag{7}$$

where $\frac{\partial}{\partial\varphi^\alpha}$, $\frac{\partial}{\partial\varphi_a^\alpha}$ and $d\varphi^\alpha$, $d\varphi_a^\alpha$ are vector fields and one-forms on Σ, respectively. It is easy to check that we can get Eqs. (4) and (5) by simple manipulations modulo the following properties

$$\frac{\partial\varphi^\beta(y)}{\partial\varphi^\alpha(x)} = \delta_\alpha^\beta \delta(y,x), \quad \frac{\partial\varphi_b^\beta(y)}{\partial\varphi_a^\alpha(x)} = \delta_\alpha^\beta \delta_b^a \delta(y,x),$$
$$\frac{\partial\varphi_b^\beta(y)}{\partial\varphi^\alpha(x)} = 0 = \frac{\partial\varphi^\beta(y)}{\partial\varphi_a^\alpha(x)}; \tag{8}$$

$$\frac{\partial}{\partial\varphi^\alpha(x)} \rfloor d\varphi^\beta(y) = \delta_\alpha^\beta \delta(y,x), \quad \frac{\partial}{\partial\varphi_a^\alpha(x)} \rfloor d\varphi_b^\beta(y) = \delta_\alpha^\beta \delta_b^a \delta(y,x),$$
$$\frac{\partial}{\partial\varphi^\alpha(x)} \rfloor d\varphi_b^\beta(y) = 0 = \frac{\partial}{\partial\varphi_a^\alpha(x)} \rfloor d\varphi^\beta(y), \tag{9}$$

where $\delta(\cdot,\cdot)$ is a generalized function defined by

$$C^\infty(TM) \ni f \to F_y([f]) = \int_\Sigma d\sigma(x) f(x) \delta(x,y)$$
$$:= \begin{cases} f(y) & \text{for} \quad y \in \Sigma \\ 0 & \text{for} \quad y \notin \Sigma \end{cases} \tag{10}$$

with properties

$$\int_\Sigma d\sigma(x) \delta(x,y) = 1, \tag{11}$$

$$\int_\Sigma d\sigma(x) f(x) \delta(x,y) = \int_\Sigma d\sigma(x) f(x) \delta(y,x), \tag{12}$$

$$\int_\Sigma d\sigma(x) \delta(x,z) \delta(z,y) = \delta(x,y). \tag{13}$$

Now we pass to the symplectic strucure on M. We define the 1-form θ on M by

$$\theta|_{\varphi} := \int_{\Sigma} d\sigma n_a \frac{\partial L(\varphi, \nabla\varphi)}{\partial \varphi^{\alpha}_a} d\varphi^{\alpha}, \tag{14}$$

where n^a is the unit vector normal to Σ.

One gets

$$X \rfloor \theta = \int_{\Sigma} d\sigma n_a X^{\alpha} \frac{\partial L}{\partial \varphi^{\alpha}_a}. \tag{15}$$

It is clear that θ depends, on the choice of Σ.

Now, we define the 2-form ω on M by

$$\begin{aligned} TM \times TM \ni (X,Y) \to \omega(X,Y) &:= \delta\theta(X,Y) \\ &:= X(Y \rfloor \theta) - Y(X \rfloor \theta) - [X,Y] \rfloor \theta, \end{aligned} \tag{16}$$

where $[X,Y] := X(Y) - Y(X)$.
One can show (see Ref. [1]) that Eq. (16) leads to

$$\omega(X,Y) = \int_{\Sigma} d\sigma n_b \left[\frac{\partial^2 L}{\partial \varphi^{\alpha} \partial \varphi^{\beta}_b} (Y^{\beta} X^{\alpha} - Y^{\alpha} X^{\beta}) + \frac{\partial^2 L}{\partial \varphi^{\alpha}_a \partial \varphi^{\beta}_b} (Y^{\beta} X^{\alpha}_a - X^{\beta} Y^{\alpha}_a) \right]. \tag{17}$$

One can also find Eq. (17) by the formula

$$\begin{aligned} \omega = \delta\theta &:= \int_{\Sigma} d\sigma n_a d\left(\frac{\partial L}{\partial \varphi^{\alpha}_a} d\varphi^{\alpha} \right) \\ &= \int_{\Sigma} d\sigma n_b \left(\frac{\partial^2 L}{\partial \varphi^{\alpha} \partial \varphi^{\beta}_b} d\varphi^{\alpha} \wedge d\varphi^{\beta} + \frac{\partial^2 L}{\partial \varphi^{\alpha}_a \partial \varphi^{\beta}_b} d\varphi^{\alpha}_a \wedge d\varphi^{\beta} \right) \end{aligned} \tag{18}$$

and our expressions, Eq. (6), for X and Y.
One can prove that ω is independent on the choice of Σ. The 2-form ω is closed on M and if det $[\frac{\partial^2 L}{\partial \varphi^{\alpha}_a \partial \varphi^{\beta}_b}] \neq 0$, then (M, ω) is a sympletic manifold (because ω is weakly nondegenerate).
One can show that the field equations, Eq. (1), are equivalent to the set of equations

$$X \rfloor w + \delta H = 0\,,$$

where

$$X = \int_\Sigma d\sigma \left(n^a \varphi^\alpha_a \frac{\partial}{\partial \varphi^\alpha} + n^b \varphi^\alpha_{ab} \frac{\partial}{\partial \varphi^\alpha_a} \right),$$

$$H = \int_\Sigma d\sigma \left(\varphi^\alpha_a \frac{\partial L}{\partial \varphi^\alpha_a} - L \right) := \int_\Sigma d\sigma h(\varphi^\alpha, \varphi^\beta_b),$$

$$\delta H := \int_\Sigma d\sigma dh, \qquad \varphi^\alpha_{ab} \equiv \nabla_b \nabla_a \varphi^\alpha. \tag{19}$$

In order to be closer to the canonical formalisms of classical mechanics and classical field theory (see [2]), we introduce new type of variables

$$(\varphi^\alpha, \varphi^\beta_b) \to (\varphi^\alpha, \pi^b_\beta) := \left(\varphi^\alpha, \frac{\partial L}{\partial \varphi^\beta_b} \right). \tag{20}$$

The above formula means that we deal with a map from the infinite dimensional manifold M to the infinite dimensional manifold N, where N is a manifold of functions φ^α and π^b_β. We assume that this map is smooth and invertible.
Now we have

$$\theta = \int_\Sigma d\sigma n_a \pi^a_\alpha d\varphi^\alpha, \tag{21}$$

$$\omega = \delta\theta = \int_\Sigma d\sigma n_a d\pi^a_\alpha \wedge d\varphi^\alpha. \tag{22}$$

One can show that in the new variables Eq. 1 reads

$$\frac{\partial h}{\partial \varphi^\alpha} = -\nabla_a \pi^a_\alpha,$$

$$\frac{\partial h}{\partial \pi^a_\alpha} = \nabla_a \varphi^\alpha, \tag{23}$$

which are equivalent to the equations

$$X \rfloor \omega + \delta H = 0,$$

where

$$X = \int_\Sigma d\sigma \left(X^\alpha \frac{\partial}{\partial \varphi^\alpha} + X^a_\alpha \frac{\partial}{\partial \pi^a_\alpha} \right), \tag{24}$$

$$X^\alpha = n^a \varphi^\alpha_a,$$

$$n_a X^a_\alpha = \nabla_a \pi^a_\alpha.$$

3. Conclusion

It seems that the symplectic formalism for local classical field theory would be constructed according to the general lines of this paper and would be a good starting point for a future program of geometric quantization of fields analogous to the finite dimensional case, *i.e.* classical mechanics [1, 3, 4].

One of the authors (W.P.) would like to thank the organizers of this symposium for a pleasant and stimulating atmosphere.

REFERENCES

[1] N.M.J. Woodhouse, *Geometric Quantization,* Clarendon Press, Oxford 1992 (second edition).

[2] J. Rzewuski, *Field Theory Part I, Classical Field Theory,* PWN Warszawa 1958.

[3] D.J. Simms, N.M.J. Woodhouse, *Lecture on Geometric Quantization, Lecture Notes in Physics 53*, Springer Verlag, Berlin, Heidelberg 1977; J. Śniatycki, *Geometric Quantization and Quantum Mechanics,* Springer Verlag, Berlin, Heidelberg 1980; G.M. Tuynman, Geometric Quantization, Proc. Seminar 1983–85, Mathematical structures in field theories, Vol. 1, Centre for Mathematics and Computer Science.

[4] M. Gadella, *Fortschr. Phys.* **43**, 229 (1995).