INTRODUCTION TO QUANTUM GROUPS*

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These pedagogical lectures contain some motivation for the study of quantum groups; a definition of "quasitriangular Hopf algebra" with explanations of all the concepts required to build it up; descriptions of quantised universal enveloping algebras and the quantum double; and an account of quantised function algebras and the action of quantum groups on quantum spaces.

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1. Origins and ingredients

The theory of quantum groups stands at a multiple crossroads, with paths connecting it to many widely scattered areas of mathematics and physics. Roads leading to it come from:

- Connes's non-commutative geometry [2];
- Yang's investigations on factorisable one-dimensional S-matrices [26];
- Baxter's solvable vertex models [1];
- Integrable nonlinear differential equations [6];
- Jones's construction of invariants of knots and braids [12];

On the other side, roads lead from quantum group theory to:

- Conformal field theory [8];
- Anyons (with applications to high-T_c superconductivity?) [24];
- Possible new mechanisms of symmetry breaking [25, 18];
- Group representation theory [17];
- Knots and 3-manifolds [14, 23].

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As a preparation for the study of quantum groups, I will briefly describe a few of these related topics.

Factorisable S-matrix

Consider three particles scattering successively as shown in the two diagrams below:

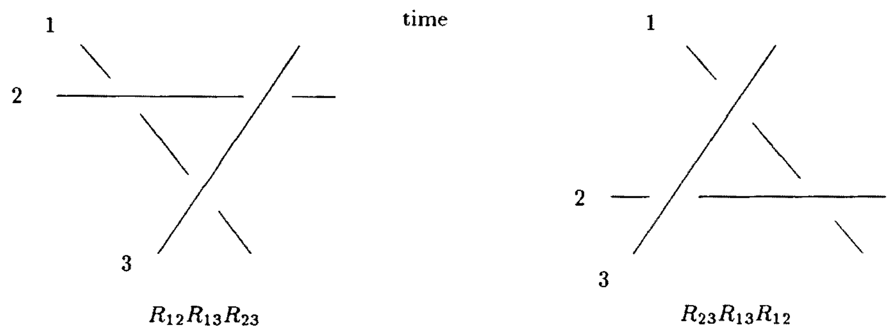

Fig. 1.

If each of the particles has the state space V, the three-particle scattering is described by an S-matrix mapping the three-particle space $V \otimes V \otimes V$ to itself, constructed out of a two-particle S-matrix $R : V \otimes V \to V \otimes V$ as shown below the diagrams. In these equations R_{ij} denotes the operator on $V \otimes V \otimes V$ which acts as R in factors i and j of the tensor product and as the identity in the third. Yang was interested in the possibility that both sequences of interactions would give the same overall three-particle S-matrix:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\,. \tag{1}$$

This is the *Yang-Baxter equation.* It is often written with R depending on a parameter, the relative velocity or rapidity of the two particles, as

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)$$

but in this lecture I will only consider the constant form (1). Another form is obtained by putting $\widehat{R} = RP$ where P is the exchange operator on $V \otimes V$; then $\widehat{R}$ satisfies

$$\widehat{R}_{12}\widehat{R}_{23}\widehat{R}_{12} = \widehat{R}_{23}\widehat{R}_{12}\widehat{R}_{23}\,. \tag{2}$$

This is the *braid relation.*

The braid group

The elements of Artin's *braid group* on n strings are the topologically distinct ways of joining two sets of n points in three-dimensional space. To be more formal, it consists of all homotopy classes of one-to-one continuous maps from $S \times I$ to $\boldsymbol{R}^3$ where S is a discrete set of n elements, I is the unit interval $[0,1]$, and the images of $S \times \{0,1\}$ are specified (say, as sets of equally spaced points on two parallel lines). The group operation, whose formal definition I leave as an exercise, is that of joining strings at their end-points as illustrated by the following diagram:

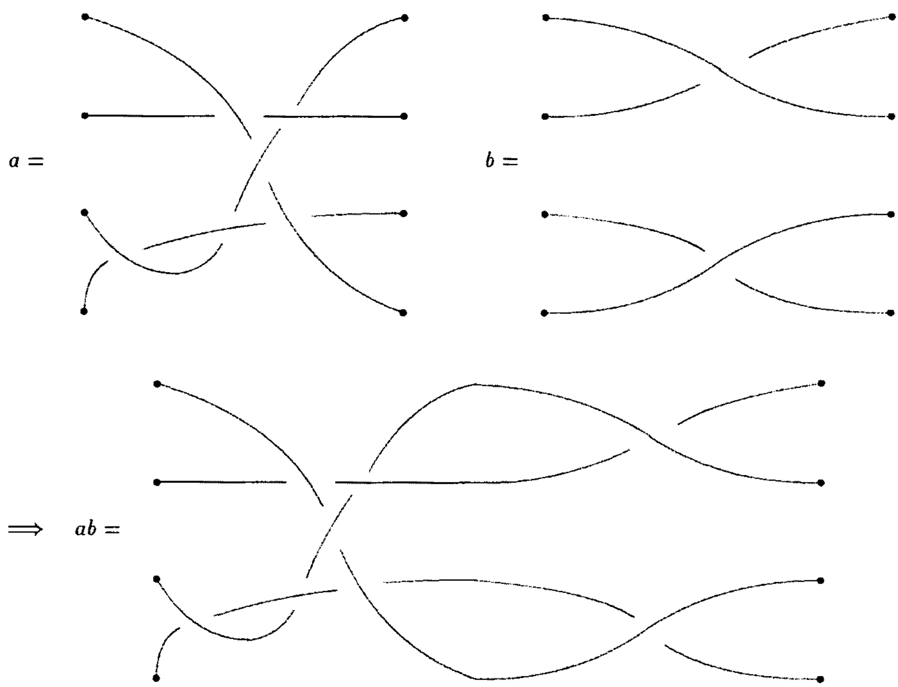

Fig. 2.

This group is generated by $\tau_1, \ldots, \tau_{n-1}$ where τ_i is the braid in which the ith and $(i+1)$th strings simply cross over. These satisfy the relations

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \tag{3}$$

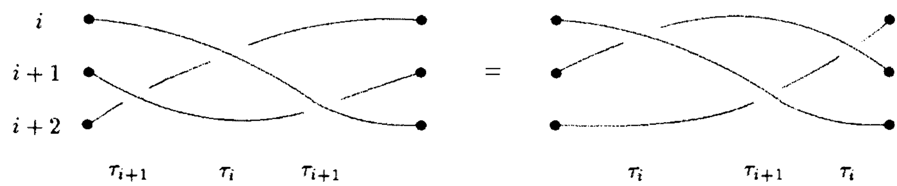

Fig. 3.

and, it can be shown, no others. The resemblance of these relations to the three-particle scattering considered by Yang is obvious. In fact, any solution $\widehat{R} : V \otimes V \to V \otimes V$ of the braid relation (2) yields representations of the braid groups for all n: on the vector space $\otimes^n V$ one takes the group generator τ_i to be represented by $\widehat{R}_{i,i+1}$. This is similar to the way in which permutations act on tensor products. The group of permutations of n letters is generated by the transpositions $\tau_1, \ldots, \tau_{n-1}$ of adjacent letters , which satisfy the relations (3) and also $\tau_i^2 = 1$; it can be represented on tensor products by taking $\widehat{R}$ to be the exchange operator, $\widehat{R}(x \otimes y) = y \otimes x$.

In quantum mechanics the action of the permutation group on a tensor product of the form $\otimes^n V$ is familiar from the theory of identical particles, leading to the classification of particles into fermions or bosons. A re-examination of this theory, however, suggests that it is only valid for particles moving in three or more dimensions; in two dimensions the permutation group should be replaced by the braid group, and instead of being restricted to the two familiar possibilities, particles can be *anyons*.

Statistics in two dimensions

Consider the quantisation of a classical system of n identical hard particles in d-dimensional space. The configuration space of the classical system of distinguishable particles is $\mathbf{R}^{dn}$; for identical particles we must factor out by the action of the permutation group, and for impenetrable particles we must first remove the diagonal subspace of n-particle configurations in which two or more particles occupy the same point (which has the advantage of making the factor space a manifold). The resulting configuration space C_n, the set of unordered subsets $\{x_1, \ldots, x_n\} \subset \mathbf{R}^n$, is not simply-connected, so that when we quantise we must allow the wave function $\psi(\{x_1, \ldots, x_n\})$ to be many-valued. It will be derived from a true one-valued function on the universal covering space of C_n, the distinct values at each point of C_n being given by a one-dimensional representation of the fundamental group of C_n.

If $d \geq 3$, the fundamental group of C_n is the permutation group S_n and its universal covering space is the distinguishable-particle configuration space $\boldsymbol{R}^{dn}$ (with the diagonal removed) which we started with. It has generators τ_i, corresponding to the interchange of particles i and $i+1$, which satisfy $\tau_i^2 = 1$. It follows that the identical-particle wave functions $\psi(\{x_1, \ldots, x_n\})$, which are many-valued functions of unordered sets of positions, can be regarded as single-valued functions of ordered sets of positions satisfying

$$\psi(x_2, x_1, \ldots) = \pm\psi(x_1, x_2, \ldots) .$$

Thus we obtain the usual conclusion that in quantum mechanics identical particles must be either bosons or fermions. On second quantisation the wave functions become commuting or anticommuting fields.

But if $d = 2$ the fundamental group of C_n is the braid group, with generators of infinite order. The basic closed loops τ_i in C_n, which describe two particles changing places, are no longer contractible when they are repeated to return the two particles to their original positions. To see this, consider the motion of the relative position vector $x_{i+1} - x_i$ as particles i and $i+1$ move around the path τ_i. This is a vector in the plane with diametrically opposite points identified and with 0 removed (since the particles cannot occupy the same position). The repeated loop τ_i^2 is a loop around the origin, which obviously cannot be shrunk to a point without crossing 0:

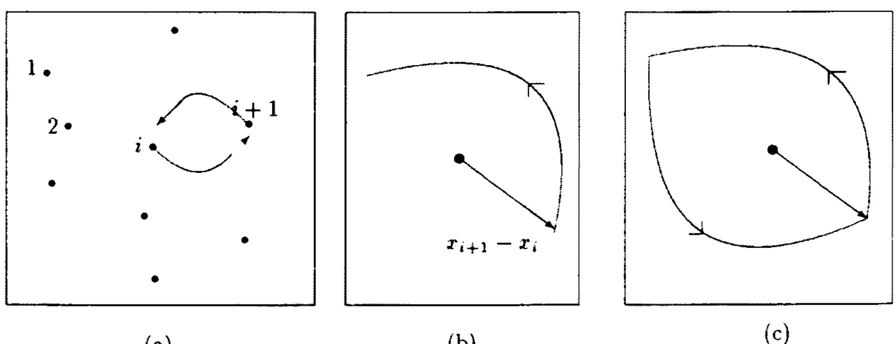

Fig. 4. (a) — the curve τ_i in C_n; (b) — the curve τ_i in terms of $x_{i+1} - x_i$; (c) — the curve τ_i^2 in terms of $x_{i+1} - x_i$.

It follows that in a one-dimensional representation the number representing τ_i is not restricted to ± 1, but could be any number q. This situation cannot be simply described in terms of wave function, but in terms of quantum fields it has the consequence [9] that in a two-dimensional quantum field theory the fields may satisfy

$$\psi(x_2)\psi(x_1) = q\psi(x_1)\psi(x_2) \tag{4}$$

for any value of q. The particles may be neither bosons nor fermions but *anyons*.

Statistics of local fields

Why should the representation of the fundamental group of the configuration space be one-dimensional? In dimensions $d \geq 3$, where the fundamental group is the permutation group, this is the same as the question why particles should not obey parastatistics. The answer is that if they did they could always be reinterpreted as fermions or bosons with an internal quantum number like colour. This is a rigorous theorem due to Doplicher, Haag and Roberts [3, 4], who showed that in space-time dimension $d+1 \geq 4$ a local relativistic quantum field theory must describe fermions or bosons with a compact internal symmetry group.

What is the corresponding statement in space-time dimension 2 or 3? It would be nice to have a statement like the Doplicher–Haag–Roberts theorem with fermions and bosons replaced by anyons, and the symmetry group replaced by a *quantum group*. The situation is not quite as simple as that [7], but this does give an intuitive idea of what sort of thing a quantum group is and what it might do.

2. Definitions and simple examples

So what is a quantum group?

— It (or maybe its dual) is a *quasitriangular Hopf algebra*.
— Oh. What's a Hopf algebra?
— It is a *bialgebra* with an antipode.
— Oh. What's a bialgebra?

Well, for a start it is an algebra; that is, it is a vector space H with a multiplication, which (because of the distributive law) can be regarded as a bilinear map from $H \times H$ to H, or a linear map $\mu : H \otimes H \to H$. This automatically gives a dual map μ^* between the dual spaces H^* and $(H \otimes H)^*$. The latter space may not (if H is infinite-dimensional) coincide with $H^* \otimes H^*$, but it always contains it. Thus with a bit of luck (and certainly if H is finite-dimensional), we will have a map $\mu^* : H^* \to H^* \otimes H^*$. A *bialgebra* is an algebra H which has a structure of this type not on the dual H^* but on H itself: it has a *coproduct* $\Delta : H \to H \otimes H$ (and therefore a multiplication $\Delta^* : H^* \otimes H^* \to H^*$ on the dual of H). This has to satisfy two axioms. First, the dual multiplication Δ^* must be associative. In terms of the coproduct Δ itself, this requirement is

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta \,.$$

Such a Δ is said to be *coassociative*. Secondly, Δ is a map between two algebras, multiplication in $H \otimes H$ being defined by

$$(a \otimes b)(c \otimes d) = ac \otimes bd\,.$$

It is therefore natural to require Δ to be an algebra homomorphism:

$$\begin{aligned} \Delta(xy) &= \Delta(x)\Delta(y) \\ &= \sum x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)} \\ \text{where} \qquad \Delta(x) &= \sum x_{(1)} \otimes x_{(2)}, \quad \Delta(y) = \sum y_{(1)} \otimes y_{(2)}\,. \end{aligned} \tag{5}$$

The notation introduced in the last line is standard for coproducts, as is the convention of omitting the summation sign. From now on we will use this notation and convention freely.

In algebraic work a bialgebra (considered as an algebra) is usually taken to have an identity. This can be regarded as a map $1 : \boldsymbol{K} \to H$, where $\boldsymbol{K}$ is the field of scalars. The dual of this is a map $1^* : H^* \to \boldsymbol{K}$. Again, in a bialgebra we require such an object in H itself, *i.e.* a map $\varepsilon : H \to \boldsymbol{K}$. This is the *counit* of H. It is required to be an algebra homomorphism, and its dual $\varepsilon : \boldsymbol{K} \to H^*$ is required to be an identity for the dual multiplication Δ^*. This second axiom is

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta(\mathrm{x}) = \mathrm{x} = (\mathrm{id} \otimes \varepsilon) \circ \Delta(\mathrm{x})\,,$$

i.e.

$$\varepsilon(x_{(1)})x_{(2)} = x = x_{(1)}\varepsilon(x_{(2)})\,.$$

A *Hopf algebra* is to a bialgebra as a group is to a semigroup. Corresponding to the inverse in a group, a Hopf algebra H has a further linear map $S : H \to H$ called the *antipode* satisfying

$$\mu \circ (\mathrm{id} \otimes \mathrm{S}) \circ \Delta(\mathrm{x}) = \varepsilon(\mathrm{x})1 = \mu \circ (\mathrm{S} \otimes \mathrm{id}) \circ \Delta(\mathrm{x})\,,$$

i.e.

$$S(x_{(1)})x_{(2)} = 1 = x_{(1)}S(x_{(2)})\,.$$

The antipode is not an inverse — you would not expect inversion to be a linear map — but there is a sense in which both S and S^* are partial inverses, S for the multiplication in H and S^* for the dual multiplication Δ^* in H^*. In what sense this is true will become clear when we consider examples. First, however, let us define the final element of structure which makes a Hopf algebra a quantum group.

A *quasitriangular* Hopf algebra is a Hopf algebra H together with an element $\mathcal{R} \in H \otimes H$ (called the *universal R-matrix* of H) which satisfies:

1.
$$\mathcal{R}\Delta(x)\mathcal{R}^{-1} = P\Delta(x)\,,$$
where P is the exchange operator on $H \otimes H$;

2.
$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}\,,$$
$$(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}\,,$$
where $\mathcal{R}_{ij}$ is the element of $H \otimes H \otimes H$ which is $\mathcal{R}$ in factors i and j with 1 in the third factor;

3.
$$(\varepsilon \otimes 1)(\mathcal{R}) = 1 = (\mathrm{id} \otimes \varepsilon)(\mathcal{R})\,,$$
$$(S \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1} = (\mathrm{id} \otimes \mathrm{S})(\mathcal{R})\,.$$

An element $\mathcal{R}$ with these properties gives rise to a solution of the Yang–Baxter equation (1) in any representation of the algebra H.

Classical Hopf algebras

As an instructive (and prototypical) example of a Hopf algebra, let G be a finite group and let $H = \boldsymbol{K}G$ be the group algebra of G over the field $\boldsymbol{K}$ (so H is the set of formal linear combinations of elements of G with coefficients in $\boldsymbol{K}$). Then the dual H^* is the set of functions on G with values in $\boldsymbol{K}$, and is also an algebra with the obvious (pointwise) multiplication of functions. From the algebra structure on H^* we get a coproduct on H which is given by the simple formula

$$\Delta(g) = g \otimes g \qquad \text{for all } g \in G\,. \tag{6}$$

(Note that this formula only applies to the subset $G \subset H$, which is a basis of the vector space H. In any bialgebra an element g whose coproduct is given by this equation is said to be *grouplike*.) It is easy to see that this satisfies the conditions for H to be a bialgebra. The identity of the function algebra H^* is the constant function on G whose value is 1; thus the counit of the bialgebra is given by

$$\varepsilon(g) = 1 \qquad \text{for all } g \in G\,. \tag{7}$$

This bialgebra is made into a Hopf algebra by defining the antipode as

$$S(g) = g^{-1} \qquad \text{for all } g \in G\,. \tag{8}$$

This is extended linearly to the rest of H, so it only coincides with the inverse on the special subset $G \subset H$.

Looking at this example from a dual standpoint, we get a Hopf algebra structure on the space $H^* = \mathcal{F}(G)$ of functions on G. The coproduct on this implied by the multiplication in H can be described by identifying $\mathcal{F}(G) \otimes \mathcal{F}(G)$ with the space of functions from $G \times G$ to $\boldsymbol{K}$; then the coproduct Δf of a function f is the function of two variables given by

$$\Delta f(g,h) = f(gh) \qquad (f, g \in G)$$

and the counit and antipode are given by

$$\varepsilon(f) = f(e), \qquad Sf(g) = g^{-1},$$

where e is the identity element of G.

A way of extending this example from finite groups to Lie groups is to take $H = U(\boldsymbol{g})$ to be the universal enveloping algebra of a Lie algebra $\boldsymbol{g}$. Intuitively, we think of this as containing exponentials e^X of Lie algebra elements X, *i.e.* Lie group elements, and we think of Lie algebra elements as obtained by differentiating Lie group elements, so that they are (limits of) linear combinations of group elements. Formally differentiating the expression (6) for the coproduct of a group element gives the coproduct of a Lie algebra element as

$$\Delta(X) = 1 \otimes X + X \otimes 1 \qquad \text{for all } X \in \boldsymbol{g} \tag{9}$$

(conversely, exponentiating this and using the homomorphism property of the coproduct gives $\Delta(\mathrm{e}^X) = \mathrm{e}^X \otimes \mathrm{e}^X$). This only holds for the subset $\boldsymbol{g} \subset U(\boldsymbol{g})$, but since $\boldsymbol{g}$ generates $U(\boldsymbol{g})$ as an algebra the homomorphism property of Δ enables us to extend the definition to the rest of $U(\boldsymbol{g})$. (In any bialgebra an element X with coproduct given by (9) is said to be *primitive*.)

Differentiating the expressions for the counit and antipode in a group algebra give

$$\varepsilon(1) = 1, \qquad \varepsilon(X) = 0 \quad \text{for } X \in \boldsymbol{g}, \tag{10}$$

$$S(1) = 1, \qquad S(X) = -X \quad \text{for } X \in \boldsymbol{g}. \tag{11}$$

which are sufficient to define ε and S on all of $U(\boldsymbol{g})$ by the homomorphism and antihomomorphism properties (and which imply $\varepsilon(\mathrm{e}^X) = 1$ and $S(\mathrm{e}^X) = \mathrm{e}^{-X}$ as in a group algebra).

Go forth and comultiply

Comultiplication may look quite new, but in fact it codifies and extends a number of familiar ideas. For a start, the coproduct in a bialgebra H allows us to construct the tensor product of two representations of the algebra H.

If ρ_1 and ρ_2 are representations of H on vector spaces V_1 and V_2, the tensor product representation ρ_{12} acting on $V_1 \otimes V_2$ is defined by

$$\rho_{12}(x) = \rho_1(x_{(1)}) \otimes \rho_2(x_{(2)}) ,$$

i.e. from $\rho_1 : H \to \mathrm{End V_1}$ and $\rho_2 : H \to \mathrm{End V_2}$ we form

$$\rho_{12} = (\rho_1 \otimes \rho_2) \circ \Delta : H \to \mathrm{End V_1} \otimes \mathrm{End V_2} = \mathrm{End(V_1 \otimes V_2)}$$

and the homomorphism property of Δ guarantees that this is a representation of H. If H is the group algebra of a group G, with $\Delta(g) = g \otimes g$, this gives the usual rule that in the tensor product a group element acts on both vectors simultaneously,

$$\rho_{12}(g)(v_1 \otimes v_2) = \rho_1(g)v_1 \otimes \rho_2(g)v_2 ;$$

if H is the universal enveloping algebra of a Lie algebra L, with $\Delta(X) = X \otimes 1 + 1 \otimes X$, it tells us to make a Lie algebra element act on a tensor product by acting on each vector separately and adding the results:

$$\rho_{12}(X)(v_1 \otimes v_2) = \rho_1(X)v_1 \otimes v_2 + v_1 \otimes \rho_2(X)v_2 .$$

In quantum mechanics, where the Lie algebra elements can be interpreted as momenta, this coproduct can be interpreted as telling us that the total momentum of a system is the sum of the momenta of its constituents. More generally, the mathematical notion of a coproduct captures a basic idea in both classical and quantum mechanics, that of expressing a property of a collection of particles in terms of the same property of the individual particles. For non-additive properties, such as the kinetic energy of interacting particles, this raises the interesting possibility of describing interactions in terms of coproducts.

A second application of comultiplication which unifies familiar concepts in group theory and Lie algebra theory arises from considering an action of a bialgebra H on another algebra. We consider a representation of H on a vector space A which also has a multiplication, and ask how an element $x \in H$ acts on the product of two elements of A. The representation is said to consist of *generalised derivations* if

$$\rho(x)(ab) = [\rho(x_{(1)})a][\rho(x_{(2)})b] \tag{12}$$

(using the summation convention of (5)). For a grouplike element g this becomes

$$\rho(x)(ab) = [\rho(x)a][\rho(x)b] ,$$

so $\rho(x)$ is an *automorphism* of A; for a primitive (Lie algebra-like) element X it becomes

$$\rho(x)(ab) = [\rho(x)a]b + a[\rho(x)b],$$

so $\rho(x)$ is a *derivation* of A.

The notion of *invariance* in a representation is different for groups and Lie algebras, but can be formulated in a unified way using the coproduct and the counit. In a representation of a group, an invariant element is one which is kept the same by all group elements; in a representation of a Lie algebra, an invariant element is one which is annihilated by all Lie algebra elements. These two statements are both included in

$$\rho(x)v = \varepsilon(x)v. \tag{13}$$

A final example of this kind of unification of concepts uses both the coproduct and the antipode in a Hopf algebra. The *adjoint* representation of a Hopf algebra H is an action on H itself in which each $x \in H$ is represented by the operator ad$x$ given by

$$\mathrm{ad}x(y) = x_{(1)}yS(x_{(2)}). \tag{14}$$

Then

$$\mathrm{ad}x(y) = xyx^{-1} \qquad \text{if } x \text{ is grouplike},$$
$$\mathrm{ad}x(y) = xy - yx \qquad \text{if } x \text{ is primitive}.$$

The set of invariant elements of the adjoint representation is just the centre of H, *i.e.* the set $\{y : xy = yx \text{ for all } x \in H\}$. (The reader who is new to Hopf algebras should prove this as an exercise.)

Quantised enveloping algebras

All the Hopf algebras that we have seen so far have been either commutative or cocommutative (meaning that the dual multiplication Δ^* is commutative, the condition for which is that $\Delta(x)$ should be a symmetric tensor for all x). The interesting quantum groups are those which are neither commutative nor cocommutative. The word "quantum" refers to this replacement of commuting objects by non-commuting ones (though in my definition quantum includes classical). The first example was given by Jimbo [10]: it is a Hopf algebra generated by three elements J_0, J_+ and J_- with relations

$$[J_0, J_+] = J_+ \tag{15}$$
$$[J_0, J_-] = -J_- \tag{16}$$
$$[J_+, J_-] = \frac{q^{2J_0} - q^{-2J_0}}{q - q^{-1}}, \tag{17}$$

comultiplication

$$\Delta(J_0) = 1 \otimes J_0 + J_0 \otimes 1 \tag{18}$$
$$\Delta(J_\pm) = q^{J_0} \otimes J_\pm + J_\pm \otimes q^{-J_0}\,, \tag{19}$$

counit

$$\varepsilon(1) = 1, \quad \varepsilon(J_0) = \varepsilon(J_\pm) = 0\,, \tag{20}$$

and antipodes

$$S(J_0) = -J_0, \qquad S(J_\pm) = -q^{-1}J_\pm\,. \tag{21}$$

To define this algebra properly, avoiding problems with infinite series, we should use generators $K^\pm = q^{\pm}J_0$ instead of J_0 and replace (18) by

$$KJ_\pm K^{-1} = q^{\pm}J_\pm,\,, \tag{22}$$

but the above form is intuitively more illuminating since it shows the relation to the classical universal enveloping algebra of the Lie algebra $\mathbf{su(2)}$, which can be seen as the limit of the above Hopf algebra as $q \to 1$. We can put $q = e^h$ where h is Planck's constant, and regard the above algebra as a quantisation of the classical Hopf algebra $U(\mathbf{su(2)})$. It is the *quantised enveloping algebra* $U_q(\mathbf{su(2)})$.

This construction was generalised from $\mathbf{su(2)}$ to any semisimple Lie algebra $\boldsymbol{g}$ by Jimbo [11] and Drinfel'd [5]. Classically, a Lie algebra of rank r can be generated by $3r$ elements: the fundamental roots $h_1, \ldots, h_r$, which form a basis of the Cartan subalgebra; positive root elements $e_1, \ldots, e_r$ associated with these roots; and the negative root elements $f_1, \ldots, f_r$ associated with $-h_1, \ldots, -h_r$. The elements associated with the other positive roots of the Cartan subalgebra can be obtained by forming repeated Lie brackets of the e_i, and those associated with the negative roots can be obtained from repeated Lie brackets of the f_i. Classically, as was discovered by Serre, these are subject only to the relations

$$[e_i, [e_i, \ldots, [e_i, e_j] \ldots] = 0 = [f_i, [f_i, \ldots, [f_i, f_j] \ldots]\,, \tag{23}$$

where the number of brackets is the number of times the root h_i can be added to h_j while remaining in the root system of L. This number is $1 - a_{ij}$ where

$$a_{ij} = 2\frac{\langle h_i, h_j\rangle}{\langle h_i, h_i\rangle}$$

is a negative integer, an element of the Cartan matrix of $\boldsymbol{g}$. The angle brackets denote the Killing form in the Cartan subalgebra H.

The quantised enveloping algebra of $\boldsymbol{g}$ is also generated by elements e_i, f_i, h_i $(i = 1, \ldots, r)$; each triple (e_i, f_i, h_i) generates a copy of $U_{q_i}(\mathbf{su(2)})$ with parameter $q_i = q^{1/2}\langle h_i, h_i\rangle$. The full set of relations is

$$[h_i, e_j] = \langle h_i, h_j\rangle e_j\,,$$
$$[h_i, f_j] = -\langle h_i, h_j\rangle f_j\,,$$
$$[e_i, f_j] = \frac{q_i^{2h_i} - q_i^{-2h_i}}{q_i - q_i^{-1}}\delta_{ij}\,,$$
$$[e_i, [e_i, \ldots, [e_i, e_j]_{q_i^{-n+1}}]_{q_i^{-n+3}} \cdots]_{q_i^{n-1}} = 0\,, \tag{24}$$

where

$$[x, y]_q = qxy - q^{-1}yx$$

and $n = 1 - a_{ij}$ as in the classical Serre relations. The coproducts, counit and antipode are as in $U_{q_i}(\mathbf{su(2)})$. For this to be a valid definition, it must be checked that the definitions (18), (20) and (21) are consistent with the quantised Serre relations (24) — a decidedly non-trivial requirement (for an illuminating treatment, see [13]). Then the quantised Serre relations can be written in terms of the Hopf-algebra adjoint representation as

$$(\mathrm{ade_i})^n \mathrm{e_j} = 0\,. \tag{25}$$

The quantum double

The final ingredient needed to make the quantised enveloping algebra $U_q(L)$ into a quantum group is a universal R-matrix. Drinfel'd showed that this could be obtained as an application of the following general construction. Let H and H^* be dual Hopf algebras, so that there is a non-degenerate bilinear function: $H \times H^* \to \boldsymbol{K}$, which we denote by angle brackets, satisfying

$$\langle \xi\eta, x\rangle = \langle \xi \otimes \eta, \Delta(x)\rangle\,, \tag{26}$$

$$\text{and} \qquad \langle \Delta(\xi), x \otimes y\rangle = \langle \xi, xy\rangle \qquad (x, y \in H, \quad \xi, \eta \in H^*) \tag{27}$$

(using Δ to denote the coproduct in both H and H^*). The *quantum double* is $D = H^* \otimes H$ with multiplication

$$(\xi \otimes x)(\eta \otimes y) = \langle \eta_{(1)}, Sx_{(1)}\rangle\, \eta_{(2)}\xi \otimes x_{(2)}y\, \langle \eta_{(3)}, x_{(3)}\rangle \tag{28}$$

and comultiplication

$$\Delta(\xi \otimes x) = (\xi_{(1)} \otimes x_{(1)}) \otimes (\xi_{(2)} \otimes x_{(2)})\,, \tag{29}$$

counit given by multiplying the counits in H and H^*, and antipode given by the tensor product of the antipodes in H and H^*. Drinfel'd showed that this quantum double was quasitriangular, the universal R-matrix $\mathcal{R}$ being simply the identity operator on H regarded as an element of $H^* \otimes H$. More explicitly, if $\{b_i\}$ is a basis of H and $\{\beta^i\}$ is the dual basis of H^* under the pairing $\langle , \rangle$, then

$$\mathcal{R} = \sum_i \beta^i \otimes b_i \,.$$

In the application of Drinfel'd's construction to $U_q(\boldsymbol{g})$, the Hopf algebra H is taken to be the subalgebra of $U_q(\boldsymbol{g})$ generated by $h_1, \ldots, h_r, e_1, \ldots, e_r$, and the dual H^* is taken to be the subalgebra generated by $h_1, \ldots, h_r, f_1, \ldots, f_r$. The resulting quantum double is bigger than $U_q(\boldsymbol{g})$ because of the duplication of $h_1, \ldots, h_r$, so one has to take a quotient of it; and it is hard work to construct the dual bases $\{\beta^i, b_i\}$, but the result is an elegant explicit formula for $\mathcal{R}$ [16, 15].

3. Quantum spaces and quantum symmetry

In the final section of this talk I will take a dual look at quantum groups, considering a quantisation not of the classical group algebra but of the classical algebra of functions on a group. This will give us a way of talking about the action of a quantum group on a quantum space.

The quantised function algebra $F_{p,q}(M(2))$

We start with an elementary example of an algebra, from which, by considering the dual, we will construct another example of a classical bialgebra. Let $M(2)$ be the algebra of 2×2 matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The dual of this as a vector space is the four-dimensional vector space $H = M(2)^*$ spanned by the functions a, b, c, d which assign to each matrix M its top left, top right, bottom left and bottom right entries. The multiplication on $M(2)$ gives rise to a comultiplication on $M(2)^*$ given by

$$\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{30}$$

$$i.e. \qquad \Delta(a) = a \otimes a + b \otimes c \qquad \text{etc.} \tag{31}$$

which we can write as

$$\Delta(a) = a_1 a_2 + b_1 c_2 \qquad \text{where} \qquad a_1 = a \otimes 1, \quad a_2 = 1 \otimes a, \quad \text{etc.} \tag{32}$$

Thus the dual of matrix multiplication is the formula for matrix multiplication. We will call this *matrix comultiplication.* The matrix counit is defined similarly by the formula for the identity matrix:

$$\begin{pmatrix} \varepsilon(a) & \varepsilon(b) \\ \varepsilon(c) & \varepsilon(d) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The four-dimensional vector space $M(2)^*$ generates a commutative algebra $\mathcal{F}(M(2)) = \boldsymbol{K}[a,b,c,d]$, the algebra of polynomial functions on $M(2)$. The comultiplication (30) can be extended to this infinite-dimensional algebra by the homomorphism property, *i.e.* by defining

$$\Delta(a^k b^l c^m d^n) = \Delta(a)^k \Delta(b)^l \Delta(c)^m \Delta(d)^n.$$

The matrix counit can be extended similarly. This makes $\mathcal{F}(M(2))$ into a bialgebra which is commutative but not cocommutative (since matrix multiplication is not commutative). If we adjoin an inverse of the determinant element $ad - bc$, we can define an antipode by the formula for the inverse of a 2×2 matrix,

$$\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

making $\mathcal{F}(M(2))$ into a Hopf algebra.

We now quantise this classical Hopf algebra. Let p and q be any two scalars, and let $\mathcal{F}_{p,q}(\mathrm{GL}(2))$ be the algebra generated by four elements a, b, c, d with relations

$$\begin{aligned} ba &= pab, & db &= qbd, \\ ca &= qac, & dc &= pcd, \end{aligned} \tag{33}$$

$$\begin{aligned} pcb &= qbc, \\ da - ad &= (q - p^{-1})bc. \end{aligned}$$

This becomes a bialgebra with the same matrix comultiplication and counit as in the classical case. If we introduce an inverse for the *quantum determinant*

$$D = ad - p^{-1}bc = da - q^{-1}cb, \tag{34}$$

then $\mathcal{F}_{p,q}(\mathrm{GL}(2))$ becomes a Hopf algebra with the antipode

$$\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = D^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} = \begin{pmatrix} d & -pb \\ -p^{-1}c & a \end{pmatrix} D^{-1}. \tag{35}$$

Properties of $\mathcal{F}_{p,q}(GL(2))$

The simple-looking statement that the above multiplication, comultiplication, counit and antipode make $\mathcal{F} = \mathcal{F}_{p,q}(GL(2))$ into a Hopf algebra includes a number of non-trivial (indeed, rather unlikely) propositions:

1. The definition (30) of the comultiplication can be consistently extended from the generators a, b, c, d to any polynomial function of them so as to give a homomorphism: $\mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$.

The potential inconsistency is that applying Δ to the two sides of (33), assuming the homomorphism property, might give different results. The fact that it does not is the fact that $\Delta(a), \Delta(b), \Delta(c), \Delta(d)$ satisfy the same relations (30) as a, b, c, d. In view of (32), it is tempting to state this like the closure property of a group:

> If the matrices A_1 and A_2 both have elements satisfying the relations of $\mathcal{F}_{p,q}(GL(2))$, and if the elements of A_1 commute with those of A_2, then the elements of A_1A_2 satisfy the relations.

2. The antipode S defined on the generators by (35) extends to an antihomomorphism: $\mathcal{F} \to \mathcal{F}$.

A major part in the proof of this property is played by the following properties of the quantum homomorphism D:

3. The quantum determinant D is normal in $\mathcal{F}_{p,q}(GL(2))$, *i.e.*

$$Df = \sigma(f)D \qquad \text{for all } f \in \mathcal{F}_{p,q}(GL(2)) , \tag{36}$$

where σ is an automorphism of $\mathcal{F}_{p,q}(GL(2))$. Explicitly,

$$Da = aD, \qquad Db = qp^{-1}bD , \tag{37}$$
$$Dc = pq^{-1}cD, \qquad Dd = dD . \tag{38}$$

(Note that if $p = q$ the determinant D commutes with every element of the algebra and can therefore be put equal to 1. This gives the quantum special linear group $\mathcal{F}_q(\mathrm{SL}(2))$.)

4. D is multiplicative in the sense that

$$\Delta(D) = D \otimes D \tag{39}$$

which is a bialgebra version of the multiplicative property of the classical determinant, viz. $\det(A_1A_2) = \det A_1 \det A_2$.

Finally, we note a property of $\mathcal{F}_{p,q}(GL(2))$ which says that it is the same size as the classical (commutative) algebra $K[a, b, c, d]$:

5. The ordered monomials $a^k b^l c^m d^n$ (where k, l, m, n are non-negative integers) form a basis of $\mathcal{F}_{p,q}(GL(2))$.

This is the *Poincaré–Birkhoff-Witt* theorem for $\mathcal{F}_{p,q}(GL(2))$.

These four properties are so unlikely that we are led to look for an explanation of them in terms of some special features of the original relations (33).

Why does $\mathcal{F}_{p,q}(GL(2))$ work?

At one level, an explanation of the properties of the relations (33) can be given by noting that they can be put into a particular form, namely

$$\widehat{R}_{ij}^{kl} t_k^m t_l^n = t_i^k t_j^l \widehat{R}_{kl}^{mn} \tag{40}$$

or, in the same notation as for the Yang–Baxter equation (1),

$$\widehat{R}_{12} T_1 T_2 = T_1 T_2 \widehat{R}_{12} \,, \tag{41}$$

where $T = (t_i^j)$ is the matrix of generators of the algebra and R is a matrix of scalars which satisfies the braid relation:

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \widehat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & qp^{-1} & q - p^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{42}$$

the rows and columns of $\widehat{R}$ being ordered as (ij) =(11), (12), (21), (22).

In this notation the matrix comultiplication is given by

$$\Delta(T) = T \otimes T \,, \qquad i.e. \qquad \Delta(t_i^j) = t_i^k \otimes t_k^j \,. \tag{43}$$

The "closure" property of $\mathcal{F}_{p,q}(GL(2))$ follows very easily:

$$\begin{aligned} \widehat{R}_{12}\Delta(T_1)\Delta(T_2) &= \widehat{R}_{12}(T_1 \otimes T_1)(T_2 \otimes T_2) \\ &= \widehat{R}_{12}(T_1 T_2 \otimes T_1 T_2) \\ &= T_1 T_2 \otimes \widehat{R}_{12} T_1 T_2 \\ &= (T_1 T_2 \otimes T_1 T_2)\widehat{R}_{12} \\ &= \Delta(T_1)\Delta(T_2)\widehat{R}_{12} \,. \end{aligned}$$

There is a connection between the relations (41) and the fifth property of $\mathcal{F}_{p,q}(GL(2))$, the independence of the ordered monomials in a, b, c, d, though the connection is not as direct as is often supposed. The significance of this property is that no new relations between products of three generators are introduced by the associative law. One might expect that there would be such extra relations, because there are two ways of reducing a product such as *cba* to a multiple of *abc* using the relations (33), and it is not obvious that these will coincide. If they did not, we would have an extra and undesirable relation $abc = 0$; but we find that they do. The relevance of the braid relation to this happy accident can be seen by doing an analogous calculation in matrix form, using (41):

$$\begin{aligned} T_1(T_2T_3) &= T_1(\widehat{R}_{23}^{-1}T_2T_3\widehat{R}_{23} \\ &= \widehat{R}_{23}^{-1}\widehat{R}_{12}^{-1}T_1T_2\widehat{R}_{12}T_3\widehat{R}_{23} \\ &= \widehat{R}_{23}^{-1}\widehat{R}_{12}^{-1}\widehat{R}_{23}^{-1}T_1T_2T_3\widehat{R}_{23}\widehat{R}_{12}\widehat{R}_{23}\,, \end{aligned}$$

whereas

$$\begin{aligned} (T_1T_2)T_3 &= \widehat{R}_{12}^{-1}T_1T_2\widehat{R}_{12}T_3 \\ &= \widehat{R}_{12}^{-1}T_1\widehat{R}_{23}^{-1}T_2T_3\widehat{R}_{23}\widehat{R}_{12} \\ &= \widehat{R}_{12}^{-1}\widehat{R}_{23}^{-1}\widehat{R}_{12}^{-1}T_1T_2T_3\widehat{R}_{12}\widehat{R}_{23}\widehat{R}_{12} \end{aligned}$$

so, if the braid relation (2) is satisfied the two calculations give the same result. This suggests that the braid relation is a sufficient condition for independence of monomials. It is not completely clear, however, that this sequence of manipulations is the same as that involved in reordering a simple monomial like *cba*; and it is certainly not true that the braid relation by itself is sufficient to guarantee the PBW theorem for an algebra defined by the RTT relations (41), because not every R-matrix will give *enough* relations between a, b, c and d to allow one to reduce all products of them to ordered form. To get the precise relation between the PBW theorem and the braid relation [20, 19] one must consider the origin of the algebra generated by matrix elements in more elementary algebras generated by the coordinates on which the matrix acts.

Action and coaction

The bialgebra $\mathcal{F}_{p,q}(GL(2))$ is generated by matrix elements; what maps are these the matrix elements of? Whatever these maps are, their composition will be described not by the multiplication of matrices but by the matrix comultiplication (30). To understand this, we first look at the classical situation.

Consider a representation ρ of an algebra E on a vector space V. This can be regarded as a map $\rho : E \otimes V \to V$ satisfying

$$\rho(ab \otimes v) = \rho(a \otimes \rho(b \otimes v)) \qquad (a, b \in E, \quad v \in V)\,,$$

or

$$\rho \circ (\mu \otimes \mathrm{id}) = \rho \circ (\mathrm{id} \otimes \rho) : \mathrm{E} \otimes \mathrm{E} \otimes \mathrm{V} \to \mathrm{V}\,,$$

where $\mu : E \otimes E \to E$ is the multiplication in E. Now if E is a finite-dimensional algebra, its dual E^* is a *coalgebra*, *i.e.* it has a comultiplication $\Delta = \mu^* : E^* \to E^* \otimes E^*$. The dual of the representation $\rho : E \otimes V \to V$ is a map $\delta = \rho^* : V^* \to E^* \otimes V^*$ satisfying

$$(\Delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta\,. \tag{44}$$

A map satisfying (44) is a *corepresentation* or *coaction* of the coalgebra E^* on the vector space V^*. The simplest example of such a coaction is the case where $E = \mathrm{EndV}$ is the algebra of all linear maps of V to itself, and the representation ρ is the defining one:

$$\rho(T \otimes v) = Tv\,.$$

Then V^* has a basis of coordinates x_i with respect to a basis of V, E^* has a basis of matrix elements t_i^j with respect to the same basis, and the coaction of E^* on V^* is given by

$$\delta(x_i) = t_i^j \otimes x_j\,. \tag{45}$$

Thus the coaction of E^* is given by the formula for the action of E.

The quantum plane

In the classical case we can always extend the coaction (45) to the algebra of all polynomial functions on V, *i.e.* the commutative algebra A with generators x_i, to get an algebra homomorphism $\delta : A \to F \otimes A$ where F is the algebra of polynomial functions on E — the commutative algebra generated by the matrix elements t_i^j. If the algebras A and F are not commutative, however, extending the map (45) is more problematical; there has to be some consistency between the two algebras.

The *algebra of functions on the quantum plane* (or just *quantum plane* for short) is the algebra A_q generated by two elements x, y which satisfy

$$yx = qxy\,, \tag{46}$$

where q is an element of the ground field K. (Compare the anyon field equation (4).)It is a remarkable fact that this coordinate algebra is consistent with the matrix-element algebra $F_{p,q} = \mathcal{F}_{p,q}(GL(2))$, in the sense that there is an algebra homomorphism $\delta : A_q \to F_{p,q} \otimes A_q$ given by

$$\begin{pmatrix} \delta(x) \\ \delta(y) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}\,. \tag{47}$$

In other words, if x and y satisfy $yx = qxy$ and a, b, c, d satisfy the relations (33) while commuting with x and y, then $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ satisfy

$vu = quv$. Thus $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be regarded as the matrix of a transformation of the quantum plane which preserves the relation (46). In the same sense, the matrix coaction

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{48}$$

also preserves the relations

$$\xi^2 = \eta^2 = 0, \qquad \eta\xi = -p^{-1}\xi\eta \tag{49}$$

(the *quantum superplane*).

Conversely, it can be shown that if $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$ preserves the relation $yx = qxy$ and $\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ preserves the relations $\xi^2 = \eta^2 = 0, \xi\eta = -p^{-1}\eta\xi$, then a, b, c, d satisfy the relations of $\mathcal{F}_{p,q}(M(2))$. Thus the invariance of the quantum plane and the quantum superplane can be seen as the origin of the relations of this quantum group, in somewhat the same way as the property of preserving a Euclidean inner product is the origin of the equation satisfied by an orthogonal matrix. This is the basis of a general definition of quantum groups in their dual (function-algebra) form.

Quantum groups as invariance groups

We consider the general situation of a finite-dimensional space V and the algebra $E = \mathrm{End}V$ of linear endomorphisms of V, together with their dual spaces V^* (spanned by coordinates x_i on V) and E^* (spanned by matrix elements t_i^j). Then we have the natural action: $E \otimes V \to V$ and its dual, the natural coaction $\delta_0 : V^* \to E^* \otimes V^*$ given by $x_i \mapsto t_i^j \otimes x_j$. A *coordinate algebra* on V is any algebra generated by V^*, *i.e.* a quotient of the tensor algebra $T(V^*)$. A coordinate algebra is *quadratic* if its relations are quadratic, *i.e.* if it is the quotient of $T(V^*)$ by an ideal generated by a subspace $S \subset V^* \otimes V^*$ (the *relation subspace* of the quadratic algebra). A coordinate algebra $\mathcal{M}$ on V is *compatible* with a coordinate algebra $\mathcal{C}$ on $\mathrm{End}V$ if the natural coaction δ_0 extends to a homomorphism $\delta : \mathcal{C} \to \mathcal{M} \otimes \mathcal{C}$, *i.e.* if x_i satisfy the relations of $\mathcal{C}$ and t_i^j satisfy the relations of $\mathcal{M}$, then $a_i^j \otimes x_j$ satisfy the relations of $\mathcal{C}$.

The basic theorem on such coordinate algebras is

Theorem (Takeuchi) [22, 20] Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ be coordinate algebras on the vector space V. Then

1. There is a unique coordinate algebra $\mathcal{M}$ on End$V$ which is compatible with $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ and which is universal in the sense that any other algebra with this property is a quotient of $\mathcal{M}$ (*i.e.* the relations of $\mathcal{M}$ are necessary for compatibility).
2. $\mathcal{M}$ is a bialgebra with matrix comultiplication and counit.
3. If $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ are quadratic algebras, so is $\mathcal{M}$. Let $S_i \subset V^* \otimes V^*$ be the relation subspace of $\mathcal{C}_i$; then the relation subspace of $\mathcal{M}$ is

$$\Sigma = S_1 \otimes S_1^\perp + \cdots + S_r \otimes S_r^\perp \tag{50}$$
$$\subset (V^* \otimes V^*) \otimes (V \otimes V) \cong (\mathrm{EndV})^* \otimes (\mathrm{EndV})^* . \tag{51}$$

4. If $V^* \otimes V^* = S_1 \oplus \cdots \oplus S_r$, then $\mathcal{M}$ is defined by an $\widehat{R}$-matrix $\widehat{R} = \lambda_1 \Pi_1 + \cdots + \lambda_r \Pi_r$ where the λ_i are distinct scalars, Π_i is the projection onto S_i, in the sense that the relation subspace of $\mathcal{M}$ is

$$\Sigma = \{[R, X] : X \in \mathrm{End}(\mathrm{V} \otimes \mathrm{V})\} .$$

This means that the relations in $\mathcal{M}$ are given by (41).

The quantum orthogonal group

I will finish by describing the quantum orthogonal group in the manner of Takeuchi's theorem. There is a similar description of the quantum symplectic group [21].

The q-deformation of SO(N) is the algebra $\mathcal{M}$ generated by matrix elements which preserves each of the following three algebras generated by coordinates x_m, x_{-m} ($m = 1, \ldots, l$ where l is the integer part of $N/2$):

1. 1.A deformation of the algebra of functions of N commuting coordinates:

$$x_n x_m = q x_m x_n \qquad (m < n,\ m \neq -n) ,$$

$$\frac{1}{[k]} \sum_{m=1}^{k} \left(q^{k-m} x_m x_{-m} - q^{-(k-m)} x_{-m} x_m \right) - (q^{1/2} - q^{-1/2}) x_0^2 = 0 ,$$

where

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}} \qquad (k = 1, \ldots, l)$$

(the last term in the second equation being absent if N is even). The corresponding relation subspace is a deformation of the antisymmetric subspace of $V^* \otimes V^*$.

2. A deformation of the exterior algebra on N anticommuting coordinates, but with one relation missing, the relation subspace being a deformation of the symmetric traceless subspace of $V^* \otimes V^*$:

$$x_m^2 = 0 \quad (m \neq 0)\,,$$

$$x_n x_m = -q x_m x_n \quad (m < n,\ m \neq -n)\,,$$

$$\begin{aligned}&\frac{1}{[k]}\sum_{m=1}^{k}\left(q^{k-m+1}x_m x_{-m} + q^{-(k-m+1)}x_{-m}x_m\right)\\ &= \text{constant } (k = 1, \ldots, l)\\ &= q^{1/2} + q^{-1/2} \quad \text{if } N \text{ is odd}\,.\end{aligned}$$

3. An algebra with a single relation, a deformation of the equation of a sphere:

$$\sum_{m=1}^{l}\left(q^{m+1/2}x_m x_{-m} + q^{m-1/2}x_{-m}x_m\right) + x_0^2 = 1\,, \qquad (52)$$

the term x_0^2 being absent if N is even.

Because the third of these algebras is not quadratic, the relation (52) not being homogeneous, this does not fit into the R-matrix framework provided by the last two parts of Takeuchi's theorem. To make it do so, we can replace the q-sphere relation (52) by a homogeneous relation with 0 on the right-hand side. This yields an algebra generated by matrix elements with RTT relations, which is a deformation of the algebra of functions on the classical similarity group — the orthogonal group together with multiples of the identity. To obtain the q-orthogonal group, these relations must be supplemented by a further relation putting equal to 1 a certain central quadratic element whose classical limit is $\mathrm{tr}(T^{\mathrm{T}}T)/N$ (just as one obtains $\mathcal{F}_q(SL(N))$ from $\mathcal{F}_{q,q}(GL(N))$ by putting the quantum determinant equal to 1).

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