CLASSICAL MECHANICAL SYSTEMS
BASED ON POISSON SYMMETRY*

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The existence of the theory of 'twisted cotangent bundles' (symplectic groupoids) allows to study classical mechanical systems which are generalized in the sense that their configurations form a Poisson manifold. It is natural to study from this point of view first such systems which arise in the context of some basic physical symmetry (space-time, rotations, *etc.*). We review results obtained so far in this direction.

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0. Introduction

The theory of quantum groups [1] (and Poisson groups [2–5]) has two kind of applications in physics:

1. *hidden symmetries.* The physical model is formulated in a standard way (mostly in the two-dimensional 'space-time') — the formulation does not involve the (quantum/Poisson) symmetry. This symmetry is hidden and serves as a technical tool to establish the integrability of the model.

2. *explicit symmetries.* The physical model is drastically non-standard — the configuration space is considered to be non-commutative. The quantum/Poisson symmetry controls the non-commutativity and is also used to define the dynamics, for instance the free one. These models are up to now only speculative, with no direct relation to reality.

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In the present paper we review the results of our investigations of classical mechanical models of the second type: with explicit Poisson symmetries. Our method is as follows. We pick up some typical mechanical system whose configuration space M carries a symmetry group G. We 'Poisson deform' G, *i.e.* choose a Poisson structure π on G making it a Poisson group. We choose a Poisson structure π_M on M such that the action of G on M is Poisson. The symplectic groupoid of the Poisson manifold (M, π_M) is then considered as the *phase space* (it is a natural generalization of the cotangent bundle, called also a 'twisted cotangent bundle', see [6–9] and references therein). We denote the phase space by $\mathbf{Ph}(M, \pi_M)$. The action of G is lifted to $\mathbf{Ph}(M, \pi_M)$ (with a canonical moment map J with values in G^* — the Poisson dual of G) and we can consider some dynamics compatible with the lifted action.

1. Space-time symmetry

Let M be the Minkowski space-time and let G denote the Poincaré group acting on M. $\mathbf{Ph}(M, \pi_M)$ is then the extended phase space of an elementary system (without spin; the inclusion of spin requires more subtle approach [10]) and the free dynamics is **uniquely** defined by fixing a *mass shell*, which is just the inverse image of a symplectic leaf in G^* by J. The phase trajectories are the characteristics on the shell.

1.1. Two-dimensional space-time

In [6] we have considered the following Poisson structure on the two-dimensional Minkowski space-time:

$$\{x^+, x^-\}_M = \varepsilon x^+ x^- , \tag{1}$$

where $x^\pm := x^0 \pm x^1$ are the light-cone coordinates and ε — the deformation parameter. The (left) symplectic groupoid projection of the phase trajectories turned out to be the constant-shape hyperboles

$$(x^+ - c^+)(x^- - c^-) = -\frac{1}{\varepsilon^2 m^2} \tag{2}$$

(m is the mass; the constants c^+, c^- are subject to the restriction $c^+ c^- < 0$). The result of the noncommutativity (1) of the space-time is therefore comparable to a presence of a repulsive force.

Taking into account that the Poisson structure (1) is built of commuting vector fields:

$$\pi_M = \varepsilon x^+ \frac{\partial}{\partial x^+} \wedge x^- \frac{\partial}{\partial x^-} , \tag{3}$$

we obtained in [7] a realization of $\mathbf{Ph}\,(M, \pi_M)$ in the cotangent bundle T^*M (the symplectic structure remains that of the cotangent bundle, whereas the two groupoid projections are 'deformations' of the original cotangent bundle projection, see [9]). The cotangent bundle projections of the phase trajectories have the following parametric description (p is the parameter):

$$q^+ = e^\alpha \frac{\sinh\frac{\varepsilon}{2}p}{\frac{\varepsilon}{2}m}, \qquad q^- = e^{-\alpha} \frac{\sinh\frac{\varepsilon}{2}(p-\beta)}{\frac{\varepsilon}{2}m}. \tag{4}$$

The constants α, β are in one-to-one correspondence with the 'scattering data' ($v_{\text{in}}, v_{\text{out}}$):

$$v_{\text{in}} := \lim_{p\to-\infty} \frac{q^1}{q^0} = \tan(\alpha - \frac{\varepsilon}{4}\beta), \qquad v_{\text{out}} := \lim_{p\to+\infty} \frac{q^1}{q^0} = \tan(\alpha + \frac{\varepsilon}{4}\beta) \tag{5}$$

(in [7], the factor at β was $\frac{1}{2}$; we correct that error here). See [7, 11] for the discussion of the role of different projections in the symplectic groupoid and the comparison of commuting positions q^k with non-commuting positions x^k.

1.2. Four-dimensional space-time

According to [12], all Poisson structures on the Poincaré group G are of coboundary type with the r-matrix given by

$$r = a + b + c \in \bigwedge^2 \mathfrak{g} = \bigwedge^2 V \oplus (V \wedge \mathfrak{h}) \oplus \bigwedge^2 \mathfrak{h}, \tag{6}$$

(we used the semi-direct product structure of the Poincaré Lie algebra $\mathfrak{g} = V \rtimes \mathfrak{h}$ with V — translations, $\mathfrak{h}$ — Lorentz Lie algebra), where a, b and c (the components of r coming from the decomposition on the r.h.s. of (6)) are listed in the table in [12]. For each such r-matrix there is exactly one Poisson structure π_M on M such that the action is Poisson, namely $\pi_M = r_M$, where

$$r_M(x) := rx \qquad \text{for } x \in M \tag{7}$$

(see [13, 14]). The components a, b and c correspond to the constant, linear and quadratic part of π_M, respectively. Looking at the table we see that $c \neq 0$ implies that r is triangular. Therefore the construction of $\mathbf{Ph}\,(M, \pi_M)$ is relatively easy. If r is triangular we can use the methods of [15, 14]]. If not, the Poisson structure π_M is linear plus constant and $\mathbf{Ph}\,(M, \pi_M)$ is easily obtained using the cotangent bundle of an appropriate Lie group.

1.2.1. Triangular deformations and the abelian subcase

If r is triangular, one can realize $\mathbf{Ph}\,(M, \pi_M)$ as (a subset of) T^*M with the shifted Poisson structure [14]

$$\pi_{T^*M} = r_{T^*M} + \pi_0 \,, \tag{8}$$

where π_0 is the canonical Poisson structure of T^*M (the left groupoid projection is here identified with the original cotangent bundle projection; note that this projection maps π_{T^*M} on π_M). The first term after the equality sign is defined using the action of G on $T^*M = M \times V^*$ (similarly as in (7)). Since the tangent action (and its dual) involves only the action of the Lorentz part, the vertical part of (8) comes only from c: $r_{V^*} = c_{V^*}$. Hence the Poisson bracket between the momenta p_k is purely quadratic. The Lorentz square of the momenta, $p^2 := g^{ij}p_ip_j$, Poisson commutes with p_k:

$$\{p^2, p_k\} = 0 \,, \tag{9}$$

since p^2 is preserved by the action. These properties suggest that the mass shell might be defined by $p^2 = m^2$ (*cf.* [16] for a similar Ansatz on the Klein-Gordon operator). This however should be verified by a computation of the true mass shell (using the moment map). We leave it for the next future.

We have made more effective calculations in the special subcase of the triangular case: when r is **abelian** [9]. In this case we can use a little different realization of $\mathbf{Ph}\,(M, \pi_M)$ in T^*M than (8). This time $\mathbf{Ph}\,(M, \pi_M)$ is identified with T^*M with its canonical symplectic structure, but the left and right groupoid projections are given by

$$M \times V^* \ni (x, p) \mapsto (x, p)_L = \exp(-\tfrac{1}{2}rJ_0(x, p))x \in M \,, \tag{10}$$

$$M \times V^* \ni (x, p) \mapsto (x, p)_R = \exp(+\tfrac{1}{2}rJ_0(x, p))x \in M \,, \tag{11}$$

where $J_0 : T^*M \to \mathfrak{g}^*$ is the usual canonical moment map and r is treated here as a map from $\mathfrak{g}^*$ to $\mathfrak{g}$. As we know from [9], the Poisson dual of (G, π) can be naturally identified with $\mathfrak{g}^*$ as a Poisson manifold (the group structure of G^* does not however coincide with the abelian group structure of $\mathfrak{g}^*$). Let $J : \mathbf{Ph}\,(M, \pi_M) \to G^* \equiv \mathfrak{g}^*$ denote the canonical moment map for the lift of the Poisson action of G on M. Here is the main result of this subsection.

Theorem. $J = J_0$.

Proof: We know from [9] that $\mathbf{Ph}\,(G, \pi)$ can be realized as T^*G with the left groupoid projection given by

$$T^*G \ni \alpha \mapsto \alpha_L = \exp[r(-\tfrac{1}{2}\alpha g_\alpha^{-1})]g_\alpha \exp[r(\tfrac{1}{2}g_\alpha^{-1}\alpha)] \,, \tag{12}$$

where g_α denotes the ordinary projection of α on G. Let ϕ denote the action of G on M. We shall show that $\mathbf{Ph}\,\phi = \mathbf{Ph}_0\phi$, where $\mathbf{Ph}\,\phi$ is the symplectic groupoid lift of ϕ and $\mathbf{Ph}_0\phi$ is the ordinary cotangent bundle lift of ϕ (*cf.* [9]). It is easy to see that $\mathbf{Ph}_0\phi$ is given by

$$\mathbf{Ph}_0\phi\,(\alpha,\xi) = g_\alpha\xi, \tag{13}$$

if $g_\alpha^{-1}\alpha = J_0(\xi)$ (otherwise $\mathbf{Ph}_0\phi\,(\alpha,\xi) = \varnothing$); here $\alpha \in T^*G$, $\xi \in T^*M$. The graph of $\mathbf{Ph}_0\phi$ is a lagrangian submanifold of the symplectic manifold $T^*M \times \overline{T^*G \times T^*M}$ (the bar denoting the opposite symplectic structure) which intersects the base manifold $M \times (G \times M)$ along the graph of ϕ. At the same time, it is contained in the inverse image of the graph of ϕ by the left projection in $\mathbf{Ph}\,(M,\pi_M) \times \overline{(\mathbf{Ph}\,(G,\pi) \times \mathbf{Ph}\,(M,\pi_M))}$, since

$$\begin{aligned}
\phi(\alpha_L,\xi_L) &= \exp[r(-\tfrac{1}{2}\alpha g_\alpha^{-1})]g_\alpha \exp[r(\tfrac{1}{2}g_\alpha^{-1}\alpha)] \cdot \exp(-\tfrac{1}{2}rJ_0(\xi))x_\xi \\
&= \exp[r(-\tfrac{1}{2}\alpha g_\alpha^{-1})]g_\alpha x_\xi = \exp[r(-\tfrac{1}{2}\mathrm{Ad}_{g_\alpha}g_\alpha^{-1}\alpha)]g_\alpha x_\xi \\
&= \exp[r(-\tfrac{1}{2}J_0(g_\alpha\xi)]g_\alpha x_\xi = (\phi(\alpha,\xi))_L
\end{aligned}$$

(here x_ξ is the ordinary projection of ξ) if $\mathbf{Ph}_0\phi\,(\alpha,\xi) \neq \varnothing$. It follows that $\mathbf{Ph}_0\phi$ is a solution of the problem of characteristics which is the well known procedure to lift Poisson maps [17, 18], hence $\mathbf{Ph}\,\phi = \mathbf{Ph}_0\phi$. This immediately implies the statement of the theorem. $\square$

Corollary. In the considered realization of $\mathbf{Ph}\,(M,\pi_M)$, the mass shell is 'non-deformed' and given by the standard formula: $p^2 = m^2$. The phase trajectories are usual straight line solutions. Their 'abelian' projections coincide with the usual trajectories of the free particle. Their groupoid projections are easily calculated using (10). Note that they are also straight lines. It follows from the fact that $J_0(x,p)$ is constant on the phase trajectory, hence the whole dependence of $(x,p)_L$ on the parameter comes from the second occurrence of x, where the dependence is linear.

1.2.2. Non-triangular deformations

A representative example of a non-triangular r is given by the so called 'κ-deformation' (*cf.* [19]; subcase 7 in the table, with $\beta = 0$). The phase trajectories in this case has been calculated in [11]. Using the fact that π_M is a product of two commuting vector fields:

$$\pi_M = \varepsilon\frac{\partial}{\partial x^0} \wedge \sum_{k=1}^{3} x^k \frac{\partial}{\partial x^k}, \tag{14}$$

we can, similarly as in (3), realize $\mathbf{Ph}\,(M, \pi_M)$ in the cotangent bundle and use the commuting positions. The calculations show that the intrinsic non-commuting positions are in this case 'more physical' than the commuting ones: the dependence of velocity on the momentum is monotonic in the first case and not monotonic in the second.

2. Free motion on Poisson SU(2)

Each Poisson-Lie structure on $G = \mathrm{SU}(2)$ is isomorphic to the one (we denote it by π) defined by the following brackets

$$
\begin{aligned}
\{\alpha, \gamma\} &= i\varepsilon\alpha\gamma \\
\{\alpha, \overline{\gamma}\} &= i\varepsilon\alpha\overline{\gamma} \\
\{\gamma, \overline{\gamma}\} &= 0 \\
\{\overline{\alpha}, \alpha\} &= 2i\varepsilon|\gamma|^2
\end{aligned}
\qquad
\begin{pmatrix} \alpha & -\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} = u \in \mathrm{SU}(2)\,,
\qquad (15)
$$

where ε is some (deformation) parameter. According to the general rule [3, 5, 20, 18], the phase space of (G, π) is given by the Manin group:

$$
\mathbf{Ph}\,(\mathrm{SU}(2), \pi) = \mathrm{SL}(2, \mathbb{C}) = \mathrm{SU}(2) \cdot \mathrm{SB}(2) = G \cdot G^*\,, \qquad (16)
$$

where $G^* = \mathrm{SB}(2)$ is the Poisson dual of (G, π) composed of upper-triangular matrices with positive diagonal elements:

$$
B = \begin{pmatrix} \rho & n \\ 0 & \rho^{-1} \end{pmatrix}, \qquad \rho > 0,\ n \in \mathbb{C}\,. \qquad (17)
$$

Each element A of $\mathrm{SL}(2, \mathbb{C})$ has a unique (Iwasawa) decomposition as a product of $u \in G$ and $B \in G^*$:

$$
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = uB = \begin{pmatrix} \alpha & -\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} \cdot \begin{pmatrix} \rho & n \\ 0 & \rho^{-1} \end{pmatrix} \qquad (18)
$$

and $A \mapsto u$ is the left groupoid projection. The symplectic structure on $\mathbf{Ph}\,(G, \pi) = \mathrm{SL}(2, \mathbb{C})$ (see the previous references for the definition), is given in terms of coordinates by the following brackets

$$
\begin{aligned}
\{a, b\} &= -i\varepsilon ab, & \{b, c\} &= 2i\varepsilon ad, \\
\{\alpha, \overline{\gamma}\} &= i\varepsilon ac, & \{b, d\} &= i\varepsilon bd, \\
\{a, d\} &= 0, & \{c, d\} &= -i\varepsilon cd,
\end{aligned}
$$

$$
\begin{aligned}
\{\overline{a}, a\} &= i\varepsilon(|a|^2 + 2|c|^2), \\
\{\overline{b}, a\} &= 2i\varepsilon c\overline{d}, \\
\{\overline{c}, a\} &= 0, \\
\{\overline{d}, a\} &= -i\varepsilon a\overline{d}.
\end{aligned}
$$

$$\{\overline{b}, b\} = i\varepsilon(|b|^2 + 2|a|^2 + 2|c|^2), \qquad \{\overline{c}, c\} = i\varepsilon|c|^2,$$
$$\{\overline{c}, b\} = -i\varepsilon b\overline{c}, \qquad\qquad\qquad\quad \{\overline{d}, c\} = 0,$$
$$\{\overline{d}, b\} = 2i\varepsilon a\overline{c}, \qquad\qquad\qquad\quad\, \{\overline{d}, d\} = i\varepsilon(|d|^2 + 2|c|^2).$$

We are now going to pick up a function on $\mathbf{Ph}\,(G, \pi)$ which plays the role of the hamiltonian of a free motion. The natural thing is to consider G-biinvariant functions on $\mathrm{SL}(2, \mathbb{C})$ (they are in one-to-one correspondence with the Casimirs of the Poisson structure on the 'momentum space' G^*). Each such a function is a function of the following basic example:

$$H = \tfrac{1}{2}\mathrm{tr}\, A^\dagger A = \tfrac{1}{2}\mathrm{tr}\, AA^\dagger \tag{19}$$

(this is also biinvariant in more general case $G = \mathrm{SU(N)}, \mathrm{SL}(N, \mathbb{C}) = \mathrm{SU(N)} \cdot \mathrm{SB(N)}$). We take (19) as the hamiltonian of the free motion. Using the brackets above, we obtain the following equations of motion:

$$\dot{a} = i\varepsilon(aH - \overline{d})$$
$$\dot{b} = i\varepsilon(bH + \overline{c})$$
$$\dot{c} = i\varepsilon(cH + \overline{b})$$
$$\dot{d} = i\varepsilon(dH - \overline{a})$$

which can be written in the following compact form

$$\dot{A} = \{H, A\} = i\varepsilon(H \cdot A + Y\overline{A}Y), \tag{20}$$

where

$$Y := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since H is a Casimir of G^*, the momentum B is constant: $\dot{B} = 0$. Using the fact that for $u \in \mathrm{SU(2)}$ we have $Y\overline{u} = uY$, we obtain from (20)

$$\dot{u}B = i\varepsilon(HuB + uY\overline{B}Y), \tag{21}$$

or,

$$u^{-1}\dot{u} = i\varepsilon(H + Y\overline{B}YB^{-1}) = \mathrm{const}. \tag{22}$$

It follows that the trajectories in $\mathbf{Ph}\,(G, \pi)$ projected on G coincide with the configurational trajectories of the usual free motion. Of course, the phase description is different. The right hand side of (22) may be considered as a (deformed) Legendre transformation (transforming momentum B into velocity $u^{-1}\dot{u}$).

In order to make a closer comparison with the usual free motion on SU(2), let us note that the Poisson structure on the 'momentum space' SB(2, $\mathbb{C}$),

$$\{\zeta, w\} = -iw, \qquad \{\overline{w}, w\} = i\frac{\sinh 2\varepsilon\zeta}{\varepsilon}, \qquad \text{where } \rho \equiv e^{\varepsilon\zeta}, \; n \equiv 2\varepsilon w \quad (23)$$

(calculated by the projection $A \mapsto B$) is isomorphic to the usual linear Poisson structure

$$\{x - iy, x + iy\} = 2iz, \qquad \{z, x + iy\} = i(x + iy) \qquad (24)$$

on $su(2)^*$ (see [21] for a similar fact concerning SU(N)) by the following transformation:

$$\zeta = z, \quad w = \frac{1}{\varepsilon}\sqrt{\frac{\sinh^2 \varepsilon r - \sinh^2 \varepsilon z}{r^2 - z^2}}(x + iy) \quad (r^2 := x^2 + y^2 + z^2). \quad (25)$$

Recall that such an isomorphism defines a symplectomorphism of $\mathbf{Ph}\,(G, \pi)$ with T^*G, transforming the symplectic groupoid over G^* into the usual symplectic group algebra (T^*G as the symplectic groupoid over $\mathfrak{g}^*$). The simplest Casimir of (23),

$$R^2 := |w|^2 + \left(\frac{\sinh \varepsilon\zeta}{\varepsilon}\right)^2 = \frac{H - 1}{2\varepsilon^2}, \qquad (26)$$

is related to r^2 (the natural Casimir of (24)) as follows:

$$\varepsilon R = \sinh \varepsilon r. \qquad (27)$$

It follows that the relation between our hamiltonian H and the usual hamiltonian $h \equiv \frac{1}{2}r^2$ is given by

$$H = \cosh 2\varepsilon r = \cosh(2\varepsilon\sqrt{2h}), . \qquad (28)$$

Let us make only two remarks at this point:

(i) if we took $H' := \frac{1}{2}(\frac{1}{2\varepsilon}\operatorname{arcosh} H)^2$ instead of H, we would obtain a hamiltonian system on $SL(2, \mathbb{C})$ which is symplectomorphic with the usual one of the free motion,

(ii) we conjecture that the quantum version of our model (which likely exists) should have the quantized version $\hat{H}$ of H with the spectrum equal

$$\operatorname{Sp} \hat{H} = \cosh(2\varepsilon\sqrt{2\operatorname{Sp}\hat{h}}), \text{ where } \hat{h} \text{ is the quantum hamiltonian of the}$$

free motion on SU(2) (the laplacian). If one prefers to work with a hamiltonian which tends to h as $\varepsilon \to 0$, it is reasonable to replace H by

$$H'' = \frac{1}{2}R^2 = \frac{H-1}{4\varepsilon^2} = \frac{1}{2}\left(\frac{\sinh\varepsilon r}{\varepsilon}\right)^2 = \frac{1}{2}\left(\frac{\sinh\varepsilon\sqrt{2h}}{\varepsilon}\right)^2 \qquad (29)$$

(see also [22] for an independent treatment of this example).

3. Rotational symmetry

We consider the rotation group $G := \mathrm{SO}(n)$ acting in the fundamental representation: $M \equiv V = \mathbb{R}^n$. It is not difficult to prove [14] that for any classical r-matrix on $\mathfrak{g}$, the bivector field $\pi_M := r_M$ (*cf.* (7)) is Poisson (and the action of G on M is Poisson). In [14] we have constructed also the (essential part of the structure of the) symplectic groupoid of (M, π_M) for the standard r-matrix given by

$$r = i\varepsilon \sum_{\alpha>0} \frac{X_\alpha \wedge X_{-\alpha}}{\langle X_\alpha, X_{-\alpha}\rangle}, \qquad (30)$$

where X_α are root vectors relative to a Cartan subalgebra, $\alpha > 0$ are 'positive' roots and $\langle\cdot,\cdot\rangle$ is the Killing form.

Due to the complicated structure of the phase space in this case, simple examples of hamiltonian systems having this symmetry, like for example the deformed harmonic oscillator, are still under investigation.

REFERENCES

[1] V. G. Drinfeld, *Quantum groups*, Proc. ICM, Berkeley, 1986, vol.1, p. 789.

[2] V. G. Drinfeld, *Sov. Math. Dokl.* **27**, 68 (1983).

[3] M. A. Semenov-Tian-Shansky, *Publ. Res. Inst. Math. Sci.*, Kyoto University **21**, 1237 (1985).

[4] J.-H. Lu, A. Weinstein, *J. Diff. Geom.* **31**, 501 (1990).

[5] J.-H. Lu, Multiplicative and affine Poisson structures on Lie groups, Ph.D. Thesis, University of California, Berkeley 1990.

[6] S. Zakrzewski, Poisson space-time symmetry and corresponding elementary systems, in: "Quantum Symmetries", Proceedings of the II International Wigner Symposium, Goslar 1991, H.D. Doebner and V.K. Dobrev (Eds.), pp. 111–123.

[7] S. Zakrzewski, Poisson Poincaré particle and canonical variables, in: "Generalized Symmetries", Proceedings of the International Symposium on Mathematical Physics, Clausthal, July 27–29, 1993, H.-D. Doebner, V.K. Dobrev and A.G. Ushveridze (Eds.), 1994, pp. 165–171.

[8] S. Zakrzewski, Symplectic models of groups with noncommutative spaces, in: Proceedings of the 12th Winter School on Geometry and Topology, Srni, 11–18 January, 1992, Supplemento ai rendiconti del Circolo Matematico di Palermo, serie II, No. 32 (1993), 185–194.

[9] S. Zakrzewski, *Contemporary Mathematics* **179**, 271 (1994).

[10] S. Zakrzewski, Extended phase space for a spinning particle, preprint, Warsaw 1994.

[11] S. Zakrzewski, On the classical κ-particle, in: "Quantum Groups, Formalism and Applications", Proceedings of the XXX Winter School on Theoretical Physics 14–26 February 1994, Karpacz, J. Lukierski, Z. Popowicz, J. Sobczyk (eds.), Polish Scientific Publishers PWN, Warsaw 1995, pp. 573–577. Also: hep-th/9412098.

[12] S. Zakrzewski, Poisson Poincaré groups, in: "Quantum Groups, Formalism and Applications", Proceedings of the XXX Winter School on Theoretical Physics 14–26 February 1994, Karpacz, J. Lukierski, Z. Popowicz, J. Sobczyk (eds.), Polish Scientific Publishers PWN, Warsaw 1995, pp. 433–439. Also: hep-th/9412099.

[13] S. Zakrzewski, Poisson homogeneous spaces, in: "Quantum Groups, Formalism and Applications", Proceedings of the XXX Winter School on Theoretical Physics 14–26 February 1994, Karpacz, J. Lukierski, Z. Popowicz, J. Sobczyk (eds.), Polish Scientific Publishers PWN, Warsaw 1995, pp. 629–639. Also: hep-th/9412101.

[14] S. Zakrzewski, Phase spaces associated with standard r-matrices, in preparation.

[15] P. Xu, *J. Funct. Anal.* **112**, 218 (1993).

[16] P. Podleś, Klein-Gordon and Dirac Equations on Quantum Minkowski Spaces, preprint, Berkeley 1995.

[17] A. Coste, P. Dazord, A. Weinstein, Groupoïdes symplectiques, Publications du Départment de Mathématiques, Université Claude Bernard Lyon I (1987).

[18] S. Zakrzewski, *Comm. Math. Phys.* **134**, 347 (1990).

[19] S. Zakrzewski, *J. Phys. A: Math. Gen.* **27**, 2075 (1994).

[20] J.-H. Lu, A. Weinstein, *C. R. Acad. Sci. Paris* Sér. I Math. **309**, 951 (1989).

[21] V. Ginzburg, A. Weinstein, *J. Amer. Math. Soc.* **5**, 445 (1992).

[22] G. Marmo, A. Simoni, A. Stern, Poisson Lie group symmetries for the isotropic rotator, hep-th/9310145.