DISPERSION RELATIONS IN LOOP CALCULATIONS*

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These lecture notes give a pedagogical introduction to the use of dispersion relations in loop calculations. We first derive dispersion relations which allow us to recover the real part of a physical amplitude from the knowledge of its absorptive part along the branch cut. In perturbative calculations, the latter may be constructed by means of Cutkosky's rule, which is briefly discussed. For illustration, we apply this procedure at one loop to the photon vacuum-polarization function induced by leptons as well as to the $\gamma f\bar{f}$ vertex form factor generated by the exchange of a massive vector boson between the two fermion legs. We also show how the hadronic contribution to the photon vacuum polarization may be extracted from the total cross section of hadron production in e^+e^- annihilation measured as a function of energy. Finally, we outline the application of dispersive techniques at the two-loop level, considering as an example the bosonic decay width of a high-mass Higgs boson.

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1. Introduction

Dispersion relations (DR's) provide a powerful tool for calculating higher-order radiative corrections. To evaluate the matrix element, $\mathcal{T}_{fi}$, which describes the transition from some initial state, $|i\rangle$, to some final state, $|f\rangle$, via one or more loops, one can, in principle, adopt the following two-step procedure. In the first step, one constructs $\mathrm{Im}\,\mathcal{T}_{fi}$ for arbitrary invariant mass, $s = p_i^2$, by means of Cutkosky's rule [1], which is a corollary of S-matrix unitarity. In the second step, appealing to analyticity, one derives $\mathrm{Re}\,\mathcal{T}_{fi}$ by integrating $\mathrm{Im}\,\mathcal{T}_{fi}$ over s according to a suitable DR.

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Dispersive techniques offer both technical and physical advantages. Within perturbation theory, they allow us to reduce two-loop calculations to standard one-loop problems plus phase-space and DR integrations, which can sometimes be performed analytically even if massive particles are involved [2, 3]. This procedure can also be iterated to tackle three-loop problems [4]. On the other hand, dispersive methods can often be applied where perturbation theory is unreliable. To this end, one exploits the fact that, by virtue of the optical theorem, the imaginary parts of the loop amplitudes are related to total cross sections, which may be determined experimentally as a function of s. Perhaps, the best-known example of this kind in electroweak physics is the estimation of the light-quark contributions to the photon vacuum polarization — and thus to $\alpha(M_Z^2)$ — based on experimental data of $\sigma(e^+e^- \to \text{hadrons})$ [5]. This type of analysis may be extended both to higher orders in QED [6] and to a broader class of electroweak parameters [7].

These lecture notes are organized as follows. In Section 2, we derive DR's appropriate for physical amplitudes. Cutkosky's rule is introduced in Section 3. As an elementary application, we calculate, in Section 4, the leptonic contribution to the photon vacuum polarization to lowest order in perturbation theory, and relate its hadronic contribution to the total cross section of $e^+e^- \to$ hadrons. In Section 5, we derive, from a Ward identity, a subtraction prescription for general vacuum polarizations. In Section 6, we evaluate, via a DR, the $\gamma f \bar{f}$ vertex form factor generated by the exchange of a massive vector boson between the two fermion legs. In Section 7, we outline the application of DR's at the two-loop level, considering the bosonic decay width of a high-mass Higgs boson.

2. Dispersion relations

In elementary particle physics, we often encounter form factors (*i.e.*, functions of q^2, where q is some transferred four-momentum) which are real-valued for q^2 below some threshold, M^2, and exhibit a branch cut for $q^2 > M^2$. We shall discuss various examples of form factors in Sections 4–7. In order to benefit from the powerful theorems available from the theory of complex functions, it is necessary to allow for q^2 to be complex, although this may contradict the naïve physical intuition.

Let us then consider a complex-valued function, $F(s)$, of complex argument, s, and assume that: (1) $F(s)$ is real for real $s < M^2$; (2) $F(s)$ has a branch cut for real $s > M^2$; (3) $F(s)$ is analytic for complex s (except along the branch cut). As usual, we fix the sign of the absorptive (imaginary) part of F along the branch cut by

$$F(s + i\epsilon) = \mathrm{Re}\, F(s) + i\, \mathrm{Im}\, F(s)\,, \tag{1}$$

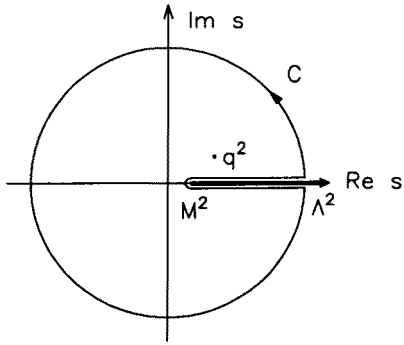

Fig. 1. Contour $\mathcal{C}$ of Eq. (3) in the complex-s plane.

where $\epsilon > 0$ is infinitesimal. By Schwartz' reflection principle, we then have

$$F(s + i\epsilon) - F(s - i\epsilon) = 2i \operatorname{Im} F(s) \,. \tag{2}$$

Since F is analytic at each point q^2 within contour $\mathcal{C}$ depicted in Fig. 1, we may apply Cauchy's theorem to find

$$\begin{aligned} F(q^2) &= \frac{1}{2\pi i} \oint\limits_{\mathcal{C}} ds \, \frac{F(s)}{s - q^2} \\ &= \frac{1}{2\pi i} \left(\int\limits_{M^2}^{\Lambda^2} ds \, \frac{F(s + i\epsilon) - F(s - i\epsilon)}{s - q^2} + \oint\limits_{|s|=\Lambda^2} ds \, \frac{F(s)}{s - q^2} \right) \\ &= \frac{1}{\pi} \int\limits_{M^2}^{\Lambda^2} ds \, \frac{\operatorname{Im} F(s)}{s - q^2 - i\epsilon} + \frac{1}{2\pi i} \oint\limits_{|s|=\Lambda^2} ds \, \frac{F(s)}{s - q^2} \,, \end{aligned} \tag{3}$$

where we have employed Eq. (2) in the last step. Suppose that we only know $\operatorname{Im} F$ along the branch cut and wish to evaluate F at some point q^2. Then, Eq. (3) is not useful for our purposes, since F also appears on the right-hand side, under the integral along the circle. Thus, our aim is to somehow get rid of the latter integral. If

$$\lim_{\Lambda^2 \to \infty} \oint\limits_{|s|=\Lambda^2} ds \, \frac{F(s)}{s - q^2} = 0 \,, \tag{4}$$

then we obtain the unsubtracted DR

$$F(q^2) = \frac{1}{\pi} \int\limits_{M^2}^{\infty} ds \, \frac{\operatorname{Im} F(s)}{s - q^2 - i\epsilon} \,. \tag{5}$$

This means that F can be reconstructed at any point q^2 from the knowledge of its absorptive part along the branch cut. In particular, the dispersive (real) part of F may be evaluated from

$$\mathrm{Re}\, F(q^2) = \frac{1}{\pi} \mathcal{P} \int\limits_{M^2}^{\infty} ds \, \frac{\mathrm{Im}\, F(s)}{s - q^2}, \tag{6}$$

where $\mathcal{P}$ denotes the principal value.

Equation (4) is not in general satisfied. It is then useful to subtract from Eq. (3) its value at some real point $q_0^2 < M^2$,

$$\begin{aligned} F(q^2) \;=\; & F(q_0^2) + \frac{q^2 - q_0^2}{\pi} \int\limits_{M^2}^{\Lambda^2} \frac{ds}{s - q_0^2} \frac{\mathrm{Im}\, F(s)}{s - q^2 - i\epsilon} \\ & + \frac{q^2 - q_0^2}{2\pi i} \oint\limits_{|s|=\Lambda^2} ds \, \frac{F(s)}{(s - q_0^2)(s - q^2)}. \end{aligned} \tag{7}$$

If the last term in Eq. (7) vanishes for $\Lambda^2 \to \infty$, then we have the once-subtracted DR

$$F(q^2) = F(q_0^2) + \frac{q^2 - q_0^2}{\pi} \int\limits_{M^2}^{\infty} \frac{ds}{s - q_0^2} \frac{\mathrm{Im}\, F(s)}{s - q^2 - i\epsilon}. \tag{8}$$

Otherwise, further subtractions will be necessary. For the use of DR's in connection with dimensional regularization, we refer to Ref. [8].

3. Cutkosky's rule

In the previous section, we explained how to obtain the dispersive part of a form factor from its absorptive part. Here, we outline a convenient method how to evaluate the absorptive part within perturbation theory.

Decomposing the scattering matrix as $S = 1 + iT$, where T is the transition matrix, we obtain

$$-i(T - T^\dagger) = T^\dagger T \tag{9}$$

from the unitarity property $S^\dagger S = 1$. Since four-momentum is conserved in the transition from some initial state $|i\rangle$ to some final state $|f\rangle$, we may always write

$$\langle f|T|i\rangle = (2\pi)^4 \delta^{(4)}(P_f - P_i) \mathcal{T}_{fi}. \tag{10}$$

Consequently,

$$\langle f|T^\dagger|i\rangle = \langle i|T|f\rangle^* = (2\pi)^4\delta^{(4)}(P_f - P_i)\mathcal{T}_{if}^*. \tag{11}$$

Inserting a complete set of intermediate states $|n\rangle$, we find

$$\begin{aligned}\langle f|T^\dagger T|i\rangle &= \sum_n \langle f|T^\dagger|n\rangle\langle n|T|i\rangle \\ &= (2\pi)^4\delta^{(4)}(P_f - P_i)\sum_n (2\pi)^4\delta^{(4)}(P_n - P_i)\mathcal{T}_{nf}^*\mathcal{T}_{ni}. \end{aligned}\tag{12}$$

Using Eqs (10)–(12) in connection with Eq. (9) and peeling off the overall delta function, we obtain Cutkosky's rule,

$$-i(\mathcal{T}_{fi} - \mathcal{T}_{if}^*) = \sum_n (2\pi)^4\delta^{(4)}(P_n - P_i)\mathcal{T}_{nf}^*\mathcal{T}_{ni}, \tag{13}$$

where it is understood that the sum runs over all kinematically allowed intermediate states and includes phase-space integrations and spin summations. Appealing to time-reversal invariance, we may identify the left-hand side of Eq. (13) with $2\,\mathrm{Im}\,\mathcal{T}_{fi}$. In summary, we may construct the absorptive part of a loop diagram according to the following recipe: (1) cut the loop diagram in all kinematically possible ways into two pieces so that one of them is connected to $|i\rangle$ and the other one to $|f\rangle$, where cut lines correspond to real particles; (2) stitch each pair of pieces together by summing over the spins of the real particles and integrating over their phase space; (3) sum over all cuts.

4. Photon vacuum polarization

One of the most straightforward applications of Cutkosky's rule and DR's is to evaluate the one-loop photon vacuum polarization induced by fermions. To avoid possible complications due to large nonperturbative QCD corrections, we start by considering leptons. For the sake of generality, we keep the electric charge, Q, and the number of colours, N_c, arbitrary.

4.1. Leptonic contribution

Given the QED interaction Lagrangian, $\mathcal{L}_I = -eQ\bar{\psi}\not{A}\psi$, we wish to compute $\mathcal{T}_{fi}$ depicted in Fig. 2. We start from the cut amplitudes,

$$\begin{aligned} i\mathcal{T}_{ni} &= \bar{u}(k)(-ieQ)\gamma^\mu v(\bar{k})\,, \\ i\mathcal{T}_{nf} &= \bar{u}(k)(-ieQ)\gamma^\nu v(\bar{k})\,. \end{aligned}\tag{14}$$

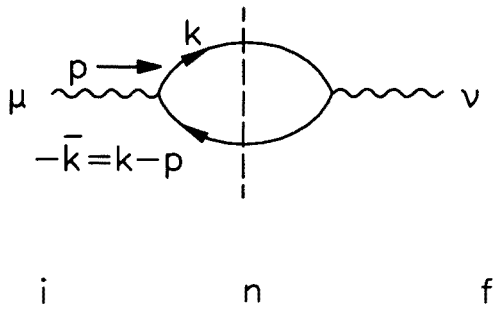

Fig. 2. Application of Cutkosky's rule to the lepton-induced photon vacuum polarization.

Summation over spins yields

$$\sum_{\text{spins}} \mathcal{T}_{nf}^{*}\mathcal{T}_{ni} = \mathrm{e}^2Q^2\,\mathrm{tr}(\bar{\not{k}} - m)\gamma^\nu(\not{k} + m)\gamma^\mu$$
$$= 4N_c\mathrm{e}^2Q^2\left(k^\mu\bar{k}^\nu + \bar{k}^\mu k^\nu - \frac{s}{2}g^{\mu\nu}\right), \tag{15}$$

where m is the fermion mass and $s = p^2 = 2(k\cdot\bar{k} + m^2)$. Defining $d\tilde{k} = (d^3k/(2\pi)^3 2k^0)$, Eq. (13) takes the from

$$2\,\mathrm{Im}\,\mathcal{T}_{fi} = \int d\tilde{k}d\tilde{\bar{k}}\,(2\pi)^4\delta^{(4)}(p-k-\bar{k})\sum_{\text{spins}}\mathcal{T}_{nf}^{*}\mathcal{T}_{ni}$$
$$= 2N_c\alpha Q^2\sqrt{1-\frac{4m^2}{s}}T^{\mu\nu}, \tag{16}$$

where $\alpha = (\mathrm{e}^2/4\pi)$ is Sommerfeld's fine-structure constant and

$$T^{\mu\nu} = \int\frac{d\Omega}{4\pi}\left(k^\mu\bar{k}^\nu + \bar{k}^\mu k^\nu - \frac{s}{2}g^{\mu\nu}\right). \tag{17}$$

Upon integration, p is the only four-momentum left, so that we can make the ansatz $T^{\mu\nu} = Ap^\mu p^\nu + Bg^{\mu\nu}$. In order to determine A and B, we form the Lorentz scalars

$$p_\mu p_\nu T^{\mu\nu} = s(sA+B) = 0$$
$$g_{\mu\nu}T^{\mu\nu} = sA + 4B = -s - 2m^2. \tag{18}$$

We so find $B = -A/s = -s - 2m^2$, so that $\operatorname{Im} \mathcal{T}_{fi} = -(sg^{\mu\nu} - p^\mu p^\nu) \operatorname{Im} \pi(s)$, where

$$\operatorname{Im} \pi(s) = \frac{N_c}{3} \alpha Q^2 \left(1 + \frac{2m^2}{s}\right) \sqrt{1 - \frac{4m^2}{s}} \tag{19}$$

is the absorptive part of the photon vacuum polarization. Notice that the dot product of $\mathcal{T}_{fi}$ with p vanishes in compliance with electromagnetic gauge invariance. Using the once-subtracted DR (8) with $q_0^2 = 0$, we obtain the renormalized photon vacuum polarization as

$$\begin{aligned} \hat{\pi}(s) &= \pi(s) - \pi(0) \\ &= \frac{s}{\pi} \int\limits_{4m^2}^{\infty} \frac{ds'}{s'} \frac{\operatorname{Im} \pi(s')}{s' - s - i\epsilon} \\ &= \frac{N_c}{3\pi} \alpha Q^2 f\left(\frac{s + i\epsilon}{4m^2}\right), \end{aligned} \tag{20}$$

where

$$f(r) = -\left(2 + \frac{1}{r}\right) \sqrt{1 - \frac{1}{r}} \operatorname{arsinh} \sqrt{-r} + \frac{5}{3} + \frac{1}{r}, \tag{21}$$

appropriate for $r < 0$. This agrees with the well-known result found in dimensional regularization [9]. Representations of f appropriate for $0 < r < 1$ and $r > 1$ emerge from Eq. (21) through analytic continuation. Specifically, we have

$$\sqrt{1 - \frac{1}{r}} \operatorname{arsinh} \sqrt{-r} = \sqrt{\frac{1}{r} - 1} \arcsin \sqrt{r} = \sqrt{1 - \frac{1}{r}} \left(\operatorname{arcosh} \sqrt{r} - i\frac{\pi}{2}\right). \tag{22}$$

We verify that, for $r > 1$, Eq. (19) is recovered from Eq. (20). The expansions of f for $|r| \ll 1$ and $r \gg 1$ read

$$\begin{aligned} f(r) &= \frac{4}{5} r + \mathcal{O}(r^2), \\ \operatorname{Re} f(r) &= -\ln(4r) + \frac{5}{3} + \mathcal{O}\left(\frac{1}{r}\right), \end{aligned} \tag{23}$$

respectively.

From Eq. (23), we conclude that heavy fermions, with mass $m \gg \sqrt{|s|}/2$, decouple from QED [10], while light fermions, with $m \ll \sqrt{|s|}/2$, generate large logarithmic corrections. The latter point creates a principal problem for the estimation of the hadronic contribution to $\hat{\pi}$. The evaluation of Eq. (20) using the poorly known light-quark masses, m_q, would suffer from large uncertainties proportional to $\delta m_q / m_q$. In addition, there would be

large nonperturbative QCD corrections in connection with the subtraction term $\pi(0)$. In the next section, we discuss an elegant way to circumvent this problem.

4.2. Hadronic contribution

Let us consider the creation of a quark pair by e^+e^- annihilation via a virtual photon, $e^-(l)e^+(\bar{l}) \to \gamma^*(p) \to q(k)\bar{q}(\bar{k})$. The corresponding T-matrix element reads

$$i\mathcal{T} = \bar{v}(\bar{l})(-ieQ_e)\gamma^\mu u(l)\frac{-ig_{\mu\nu}}{s+i\epsilon}i(\mathcal{T}_{ni})^\nu, \tag{24}$$

where $s = p^2$ and $i(\mathcal{T}_{ni})^\mu$ is given in Eq. (14). Taking into account the $e^\pm$ spin average and the flux factor, we evaluate the total cross section as

$$\begin{aligned}\sigma(s) &= \frac{1}{4}\frac{1}{2s}\int d^3\tilde{k}d^3\tilde{\bar{k}}\,(2\pi)^4\delta^{(4)}(l+\bar{l}-k-\bar{k})\sum_{\text{spins}}|\mathcal{T}|^2\\ &= \frac{e^2Q_e^2}{8s^3}\,\mathrm{tr}\,\not{l}\gamma_\nu\,\not{\bar{l}}\gamma_\mu\int d^3\tilde{k}d^3\tilde{\bar{k}}\,(2\pi)^4\delta^{(4)}(p-k-\bar{k})\sum_{\text{spins}}(\mathcal{T}_{ni}^*)^\nu(\mathcal{T}_{ni})^\mu\\ &= \frac{e^2Q_e^2}{8s^3}4\left(l_\mu\bar{l}_\nu+\bar{l}_\mu l_\nu-\frac{s}{2}g_{\mu\nu}\right)(-2)(sg^{\mu\nu}-p^\mu p^\nu)\,\mathrm{Im}\,\pi(s)\\ &= \frac{e^2Q_e^2}{s}\,\mathrm{Im}\,\pi(s),\end{aligned} \tag{25}$$

where we have exploited Eq. (16). Substituting Eq. (25) into Eq. (20), we obtain

$$\hat{\pi}(s) = \frac{s}{4\pi^2\alpha Q_e^2}\int\limits_{4m^2}^{\infty} ds'\frac{\sigma(s')}{s'-s-i\epsilon}. \tag{26}$$

Equation (26) allows us to estimate the hadronic contribution to $\hat{\pi}$ from the total cross section of hadron production by e^+e^- annihilation measured as a function of s. A recent analysis [5] has yielded $-\hat{\pi}(M_Z^2)|_{\text{hadrons}} = 0.0280 \pm 0.0007$.

5. Subtraction prescription for general vacuum polarizations

Vacuum polarizations have mass dimension two, so that the unsubtracted DR (5) in general leads to ultraviolet divergences quadratic in the cutoff Λ, which violate the Ward identities of the theory and are not removed by renormalization. It is therefore necessary to use a subtraction.

As we have seen in Section 3, in the case of QED, the naïve subtraction of Eq. (7) with $q_0^2 = 0$ leads to the correct physical result. In the presence of unconserved currents, as in the Standard Model, the situation is more complicated [2]. In the following, we discuss a suitable subtraction prescription [11].

Starting from the general interaction Lagrangian $\mathcal{L}_I = -B_\mu J^\mu + \text{h.c.}$, where B^μ is some vector field and J^μ is the associated current, it is straightforward to derive the vacuum-polarization tensor, which, by convention, differs from the T-matrix element of $B^\mu(p) \to B^\nu(p)$ by a minus sign, as

$$\Pi^{\mu\nu}(p) = -i \int d^4x \, \mathrm{e}^{ip\cdot x} \langle 0|T J^\mu(x) J^{\nu\dagger}(0)|0\rangle, \tag{27}$$

where T denotes the time-ordered product. By Lorentz covariance, $\Pi^{\mu\nu}$ has the decomposition

$$\Pi^{\mu\nu}(p) = \Pi(s) g^{\mu\nu} + \lambda(s) p^\mu p^\nu, \tag{28}$$

where $s = p^2$. Integrating by parts, we obtain from Eq. (27)

$$\begin{aligned} p_\mu \Pi^{\mu\nu}(p) &= \int d^4x \, \mathrm{e}^{ip\cdot x} \langle 0|T \partial_\mu J^\mu(x) J^{\nu\dagger}(0)|0\rangle \\ &= p^\nu \Delta(s), \end{aligned} \tag{29}$$

where the last step follows from Lorentz covariance. On the other hand, from Eq. (28) we get $p_\mu \Pi^{\mu\nu}(p) = p^\nu [\Pi(s) + s\lambda(s)]$. Consequently, we have

$$\Pi(s) = \Delta(s) - s\lambda(s). \tag{30}$$

By its definition (27), Π has mass dimension two, so that its evaluation from $\operatorname{Im}\Pi$ via the unsubtracted DR (5) would be quadratically divergent. However, Eq. (30) relates Π to quantities for which unsubtracted DR's are only logarithmically divergent. This is obvious for λ, which is dimensionless. In the case of Δ, this may be understood by observing that J^μ is softly broken by mass terms and that one power of the external four-momentum is extracted in Eq. (29). Writing unsubtracted DR's for Δ and λ, we find

$$\begin{aligned} \Pi(s) &= \frac{1}{\pi} \int_{M^2}^{\Lambda^2} \frac{ds'}{s' - s - i\epsilon} \left[\operatorname{Im}\Delta(s') - s \operatorname{Im}\lambda(s')\right] \\ &= \frac{1}{\pi} \int_{M^2}^{\Lambda^2} ds' \left[\frac{\operatorname{Im}\Pi(s')}{s' - s - i\epsilon} + \operatorname{Im}\lambda(s')\right], \end{aligned} \tag{31}$$

where M^2 is the lowest threshold. Detailed inspection at $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha\alpha_s)$ [11] reveals that the logarithmic divergences of Eq. (31) exhibit a similar structure as the poles in $\varepsilon = 2 - n/2$ found in n-dimensional regularization, so that both methods lead to the same physical results.

6. Vertex correction

Cutkosky's rule (13) is also applicable if the initial and final states are different. As an example, we now consider the interaction between a massless-fermion current and a neutral vector boson, B, with mass M characterized by the Lagrangian $\mathcal{L}_I = -g\bar{\psi}\not{B}\psi$, where g is the coupling constant, and evaluate the $Bf\bar{f}$ vertex form factor induced by the exchange of B between the two fermion legs as depicted in Fig. 3.

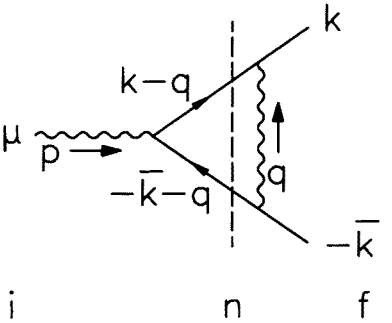

Fig. 3. Application of Cutkosky's rule to the $Bf\bar{f}$ vertex form factor induced by the exchange of B between the two fermion legs.

The cut amplitudes read

$$\begin{aligned} i\mathcal{T}_{ni} &= \bar{u}(k-q)(-ig)\gamma^\mu v(\bar{k}+q), \\ i\mathcal{T}_{nf} &= -\bar{u}(k-q)(-ig)\gamma^\mu u(k)\,\bar{v}(\bar{k})(-ig)\gamma^\nu v(\bar{k}+q)\frac{-ig_{\mu\nu}}{q^2-M^2+i\epsilon}. \end{aligned} \tag{32}$$

Notice the extra minus sign of $\mathcal{T}_{nf}$. The spin summation yields

$$\sum_{\text{spins}} \mathcal{T}_{nf}^*\mathcal{T}_{ni} = \frac{g^3}{q^2-M^2}\bar{u}(k)\Gamma^\mu v(\bar{k}), \tag{33}$$

where $\Gamma^\mu = \gamma^\nu(\not{k}-\not{q})\gamma^\mu(\not{\bar{k}}+\not{q})\gamma_\nu$. Anticipating the multiplication with the $Bf\bar{f}$ tree-level amplitude, we can bring Γ^μ into the form $\Gamma^\mu = N\gamma^\mu$, where

$$\begin{aligned} N &= -\frac{1}{4s}\,\mathrm{tr}\,\not{k}\Gamma^\mu\,\not{\bar{k}}\gamma_\mu \\ &= \frac{8}{s}k\cdot(\bar{k}+q)\,\bar{k}\cdot(k-q) \end{aligned} \tag{34}$$

and $s = p^2$. The integration over the phase space of the cut particles, with four-momenta $k_+ = \bar{k}+q$ and $k_- = k-q$, is most conveniently carried out in

the centre-of-mass frame using the variables $p = k_+ + k_-$ and $r = k_+ - k_-$. Then, we have $q^2 = -s(1+z)/2$ and $N = s(1-z)^2/2$, where z is the cosine of the angle between $\boldsymbol{r}$ and $\boldsymbol{k}$. Cutkosky's rule (13) now takes the form

$$\begin{aligned} \operatorname{Im} \mathcal{T}_{fi} &= \frac{1}{2} \int d\tilde{k}_+ d\tilde{k}_- \, (2\pi)^4 \delta^{(4)}(p - k_+ - k_-) \sum_{\text{spins}} \mathcal{T}^*_{nf} \mathcal{T}_{ni} \\ &= -g \bar{u}(k) \gamma^\mu v(\bar{k}) \frac{g^2}{16\pi^2} \operatorname{Im} F(s), \end{aligned} \tag{35}$$

where

$$\begin{aligned} \operatorname{Im} F(s) &= \frac{\pi}{2} \int_{-1}^{1} dz \, \frac{(1-z)^2}{1+z+2/x} \\ &= \pi \left[2 \left(1 + \frac{1}{x} \right)^2 \ln(1+x) - 3 - \frac{2}{x} \right] \end{aligned} \tag{36}$$

and $x = s/M^2$. Finally, we obtain the renormalized vertex function through the once-subtracted DR (8) as

$$\begin{aligned} \hat{F}(s) &= F(s) - F(0) \\ &= \frac{s}{\pi} \int_0^\infty \frac{ds'}{s'} \, \frac{\operatorname{Im} F(s')}{s' - s - i\epsilon} \\ &= 2 \left(1 + \frac{1}{x} \right)^2 [\operatorname{Li}_2(1+x) - \zeta(2)] + \left(3 + \frac{2}{x} \right) \ln(-x) - \frac{7}{2} - \frac{2}{x}, \end{aligned} \tag{37}$$

where $\operatorname{Li}_2(z) = -\int_0^1 dt \, \ln(1 - zt)/t$ is the dilogarithm. This agrees with the corresponding result found in dimensional regularization [12]. Notice that, in Eq. (37), $x = (s + i\epsilon)/M^2$ comes with an infinitesimal imaginary part. The dispersive and absorptive parts of F for $x > 0$ are conveniently separated by using the relations

$$\begin{aligned} \ln(-x) &= \ln x - i\pi, \\ \operatorname{Li}_2(1+x) - \zeta(2) &= -\operatorname{Li}_2(-x) - \ln(-x) \ln(1+x). \end{aligned} \tag{38}$$

We so recover Eq. (36), which serves as a useful check.

7. Two-loop application

In this final section, we illustrate the usefulness of DR's beyond one loop, considering the massive scalar two-loop three-point diagram depicted in Fig. 4. The idea is to write the one-loop two-point subdiagram contained in that diagram as a once-subtracted DR and to interchange the loop and DR integrations, so as to reduce the two-loop problem at hand to a one-loop one with a subsequent DR integration [13].

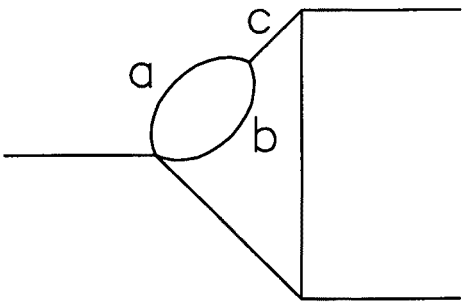

Fig. 4. Massive scalar two-loop three-point diagram, which contributes to the Higgs-boson decay to a pair of intermediate bosons. a, b, and c denote the squared masses of the respective loop particles.

In dimensional regularization, the scalar one-loop two-point integral is given by

$$\begin{aligned} \mathcal{B}(s,a,b) &= \left(\frac{\mu^2 e^\gamma}{4\pi}\right)^\varepsilon \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2-a+i\epsilon)\left[(q+p)^2-b+i\epsilon\right]} \\ &= \frac{i}{(4\pi)^2} e^{\varepsilon\gamma} \Gamma(\varepsilon) \int_0^1 \frac{dx}{X^\varepsilon} \\ &= \frac{i}{(4\pi)^2} e^{\varepsilon\gamma} \Gamma(1+\varepsilon) \left[\frac{1}{\varepsilon} + f(s) + \mathcal{O}(\varepsilon)\right], \end{aligned} \tag{39}$$

where γ is the Euler–Mascheroni constant, Γ is the gamma function, $n = 4-2\varepsilon$ is the dimensionality of space time, μ is the 't Hooft mass scale introduced to keep the coupling constants dimensionless, $s = p^2$, $X = [(1-x)a + xb - x(1-x)s - i\epsilon]/\mu^2$, and $f(s) = -\int_0^1 dx \ln X$. The peculiar form of the prefactor in Eq. (39) is to suppress the appearance of the

familiar terms proportional to $\gamma - \ln(4\pi)$ in the expressions. The absorptive part of f is

$$\begin{aligned} \operatorname{Im} f(s) &= \pi \int_0^1 dx\, \theta\Big(x(1-x)s - (1-x)a - xb\Big) \\ &= \pi \frac{\sqrt{\lambda(s,a,b)}}{s} \theta\left(s - \left(\sqrt{a}+\sqrt{b}\right)^2\right), \end{aligned} \tag{40}$$

where $\lambda(s,a,b) = s^2 + a^2 + b^2 - 2(sa + ab + bs)$ is the Källén function. Using Eq. (5), we then find

$$\begin{aligned} &\mathcal{B}(s,a,b) - \mathcal{B}(c,a,b) = \frac{i}{(4\pi)^2} \mathrm{e}^{\varepsilon\gamma} \Gamma(1+\varepsilon) \left[f(s) - f(c) + \mathcal{O}(\varepsilon)\right] \\ &= \frac{i}{(4\pi)^2} \mathrm{e}^{\varepsilon\gamma} \Gamma(1+\varepsilon) \left[\frac{1}{\pi} \int_0^\infty d\sigma\, \operatorname{Im} f(\sigma) \left(\frac{1}{\sigma - s - i\epsilon} - \frac{1}{\sigma - c - i\epsilon}\right) + \mathcal{O}(\varepsilon)\right]. \end{aligned} \tag{41}$$

Consequently, the subdiagram in Fig. 4 consisting of the bubble and the adjacent propagator with four-momentum q and squared mass c may be written as [13]

$$\begin{aligned} &\frac{\mathcal{B}(q^2,a,b)}{q^2 - c + i\epsilon} = \frac{\mathcal{B}(c,a,b)}{q^2 - c + i\epsilon} \\ &- \frac{i}{(4\pi)^2} \mathrm{e}^{\varepsilon\gamma} \Gamma(1+\varepsilon) \int_{\left(\sqrt{a}+\sqrt{b}\right)^2}^\infty \frac{d\sigma}{\sigma - c - i\epsilon} \frac{\sqrt{\lambda(\sigma,a,b)}}{\sigma} \frac{1}{q^2 - \sigma + i\epsilon} \\ &+ \mathcal{O}(\varepsilon). \end{aligned} \tag{42}$$

If Eq. (42) is inserted in the expression for the one-loop seed diagram in Fig. 4, the first term turns into a product of two one-loop diagrams, which contains all divergences. In the second term, we may interchange the DR and loop integrations and are left with a finite dispersion integral, which may be solved analytically.

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