

PERTURBATION THEORY AND ITS LIMITATIONS IN THE HIGGS SECTOR OF THE SM *

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This lecture reviews various Higgs-sector amplitudes which have been calculated to two loops in the Higgs quartic coupling. After explaining the framework of these calculations, the perturbative behaviour of these amplitudes is discussed, and perturbative upper bounds on the Higgs boson mass are given.

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1. Introduction

The simplest model of breaking spontaneously the electroweak gauge symmetry $SU(2)_L \times U(1)_Y$ is the standard Higgs model. It consists of the spin-zero Higgs boson and three massless Goldstone bosons, the latter ultimately being absorbed by the weak gauge bosons. The standard Higgs model is regarded to be an effective theory, only valid up to a cutoff energy Λ . The maximal allowed value of Λ depends on the Higgs value of the Higgs mass M_H , and it is connected to the renormalization group (RG) behaviour of the Higgs sector.

In the absence of a more complete theory, it is important to understand the perturbative limitations of the standard Higgs model: if M_H is too large, perturbation theory ceases to be a useful tool for calculating physical observables of the Higgs sector, such as cross sections and Higgs decay width. Here we review the derivation of upper perturbative bound on M_H .

In Sect. 2 we briefly present the framework of the Higgs sector, introducing the Lagrangian and the Higgs running coupling. Sect. 3 lists the various criteria which can be used to judge breakdown of perturbation theory, and the results for perturbative M_H upper bounds are derived. A short discussion and comparison with lattice results are given in Sect. 4.

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2. Basics

The Lagrangian of the standard Higgs sector is

$$\mathcal{L}_H = \frac{1}{2} (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - \frac{1}{4} \lambda (\Phi^\dagger \Phi)^2 + \frac{1}{2} \mu^2 \Phi^\dagger \Phi, \quad (1)$$

where

$$\Phi = \begin{pmatrix} w_1 + iw_2 \\ h + iz \end{pmatrix} = \begin{pmatrix} \sqrt{2} w^+ \\ h + iz \end{pmatrix}. \quad (2)$$

The doublet Φ has a nonzero expectation value in the physical vacuum,

$$\langle \Omega | \Phi^\dagger \Phi | \Omega \rangle = v^2. \quad (3)$$

To facilitate perturbative calculations, the field h is expanded around the physical vacuum, absorbing the vacuum expectation value by the shift $h \rightarrow H + v$. Hence the field H has zero vacuum expectation value. Rewriting Eq. (1), the Higgs Lagrangian has the form

$$\mathcal{L}_H = \frac{1}{2} \partial_\mu \mathbf{w} \cdot \partial^\mu \mathbf{w} + \frac{1}{2} \partial_\mu H \partial^\mu H - \frac{1}{2} M_H^2 H^2 + \mathcal{L}_{3pt} + \mathcal{L}_{4pt}, \quad (4)$$

with the three-point and four-point interactions of the fields given by

$$\mathcal{L}_{3pt} = -\lambda v (\mathbf{w}^2 H + H^3), \quad (5)$$

$$\mathcal{L}_{4pt} = -\frac{1}{4} \lambda (\mathbf{w}^4 + 2\mathbf{w}^2 H^2 + H^4). \quad (6)$$

Here \mathbf{w} is the $\text{SO}(3)$ vector of Goldstone scalars, (w_1, w_2, w_3) , with $w_3 = z$. The tadpole term and an additive constant have been dropped. Looking at Eq. (4), the w^\pm and z bosons are massless, in agreement with the Goldstone theorem [1]. The Higgs mass M_H is and the Higgs quartic coupling λ are related by

$$\lambda = M_H^2 / 2v^2 = G_F M_H^2 / \sqrt{2}, \quad (7)$$

where G_F is the Fermi constant, and v is the physical vacuum expectation value, $v = 2^{-1/4} G_F^{-1/2} = 246 \text{ GeV}$. The Lagrangian \mathcal{L}_H corresponds to taking the limit of vanishing gauge and Yukawa couplings of the full standard model Lagrangian, and it is the starting point for carrying out calculations using the equivalence theorem [2, 3]. The equivalence theorem can also be used to calculate radiative corrections without having to use the full SM Lagrangian [4], but the use of proper renormalization conditions is crucial [5].

The standard Higgs model Lagrangian contains only one degree of freedom. Once M_H is determined experimentally, the coupling λ is fixed according to Eq. (7). This will be one of the many predictions to be tested experimentally when a Higgs particle has been discovered.

Calculating physical observables with energy scales larger than M_H , renormalization group methods suggest the use of the running coupling $\lambda(\mu)$. The evolution of $\lambda(\mu)$ as a function of μ is given by the renormalization group equation

$$\frac{d\lambda(\mu)}{d\ln\mu} = \beta_\lambda(\lambda(\mu), g_t(\mu), \dots), \quad (8)$$

with the initial condition at scale $\mu = M_H$ imposed by Eq. (7). At one loop, the beta function is

$$\beta_\lambda = 24\lambda^2 + 12\lambda g_t^2 - 6g_t^4 + \dots, \quad (9)$$

and we find $\beta_\lambda > 0$ for $M_H > 208$ GeV, assuming $m_t = 175$ GeV. For even larger values of M_H , the beta function is dominated by the contributions from the Higgs coupling $\lambda \propto M_H^2$, and we can neglect the top Yukawa coupling, g_t , as well as other contributions. In this limit, the three-loop beta function for the Higgs coupling is [6, 7]:

$$\beta(\lambda) = 24 \frac{\lambda^2}{16\pi^2} \left[1 - 13 \frac{\lambda}{16\pi^2} + 176.6 \left(\frac{\lambda}{16\pi^2} \right)^2 \right]. \quad (10)$$

Neglecting the appropriate powers of λ , the above equations determine the n -loop running coupling for $n \leq 3$. Explicitly, the one-loop heavy-Higgs running coupling is

$$\lambda(\mu) = \frac{\lambda(M_H)}{1 - 12 \frac{\lambda(M_H)}{16\pi^2} \ln \left(\frac{\mu^2}{M_H^2} \right)}. \quad (11)$$

3. Limitations of perturbation theory

There are two scenarios in which perturbation theory can break down. First, the Higgs coupling can be nonperturbative for large values of M_H ; see Eq. (7). Secondly, the Higgs *running* coupling can be nonperturbative if $\beta_\lambda > 0$ and μ is large. Increasing the scale μ in Eq. (11), $\lambda(\mu)$ eventually becomes infinite, indicating the one-loop Landau pole.¹ Perturbation theory ceases to be reliable long before reaching the location of this pole.

To judge for which values of the (running) Higgs coupling the interactions become strong, one calculates perturbative amplitudes, assuming

¹ In the case of $\beta_\lambda < 0$, increasing μ eventually leads to a negative Higgs running coupling. This observation is related to the problem of vacuum stability. See [8] for a review.

small perturbative couplings. Increasing $\lambda \propto M_H^2$, the perturbative expansion will eventually break down, giving upper perturbative bounds on the (running) Higgs coupling.

One has to distinguish two different kinds of physical observables. Low-energy quantities such as Higgs decay amplitudes do not require RG methods, hence no running coupling is involved. Therefore they immediately yield an upper perturbative bound on M_H via Eq. (7). On the other hand, cross sections with large center-of-mass energy, $\sqrt{s} \gg M_H$, do require the use of the running coupling. Hence the onset of nonperturbative effects in such processes is a function of both M_H and $\mu = O(\sqrt{s})$.

There are several criteria we can use to judge the convergence of perturbation theory:

- The size of radiative corrections should be small, with higher order contributions being suppressed.
- The renormalization scheme dependence (*e.g.*, $\overline{\text{MS}}$ vs. OMS) of physical amplitudes should diminish when including higher order corrections.
- The dependence of an amplitude on the scale μ should decrease with increasing order in perturbation theory.
- Scattering amplitudes should not violate perturbative unitarity by a large amount.

Exact amplitudes, of course, have *no* scheme dependence, *no* scale dependence, and feature *no* unitarity violations. Converging perturbative amplitudes should therefore approach this limit reasonably fast. Yet there are also limitations to nonperturbative calculations in the SM Higgs sector. Since the heavy-Higgs running coupling has a positive beta function even in the nonperturbative approach [6, 9, 10], there remains a cutoff scale Λ beyond which the theory fails. In return, this cutoff scale bounds the possible Higgs mass range from above. We will return to this aspect in Sect. 4.

3.1. Perturbative bounds from Higgs decays

A heavy Higgs particle ($M_H > 2m_t$) decays mostly into pairs of (longitudinally polarized) gauge bosons. The branching ratio for decays into a pair of top quarks is about 10%. Neglecting the subleading corrections due to gauge and Yukawa couplings, the two-loop $O(\lambda^2)$ corrections to these decay channels have been calculated. Using the OMS renormalization scheme, the bosonic decay channel receives corrections[11, 12]

$$\Gamma(H \rightarrow ZZ, W^+W^-) \propto \lambda(M_H) \left(1 + 2.8 \frac{\lambda(M_H)}{16\pi^2} + 62.1 \frac{\lambda^2(M_H)}{(16\pi^2)^2} \right), \quad (12)$$

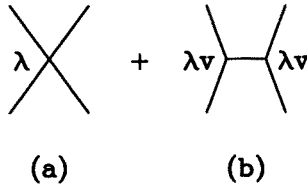


Fig. 1. The two Feynman graphs contributing to the tree-level $O(\lambda)$ term of the $W_L^+ W_L^- \rightarrow Z_L Z_L$ amplitude.

and the two-loop result for the fermionic decay width is [13, 14]

$$\Gamma(H \rightarrow t\bar{t}) \propto g_t^2 \left(1 + 2.1 \frac{\lambda(M_H)}{16\pi^2} - 32.7 \frac{\lambda^2(M_H)}{(16\pi^2)^2} \right). \quad (13)$$

Comparing the magnitude of the one-loop and two-loop corrections, perturbation theory seems to work up to values of $\lambda(M_H) \approx 7$, that is, Higgs masses of about 1 TeV. However, conversion of the results into the $\overline{\text{MS}}$ scheme as well as an analysis of the scale dependence using RG methods reveals that higher-order terms may spoil perturbation theory already for values of $\lambda(M_H) \approx 4$, or equivalently, $M_H \approx 700$ GeV [7]. The principles of such analyses will be discussed in the following section in connection with cross sections rather than decay widths.

3.2. Perturbative bounds from scattering processes

A typical scattering process related to the SM Higgs sector is $W^+ W^- \rightarrow ZZ$. This process is of interest for future colliders such as the LHC or linear e^+e^- and $\mu^+\mu^-$ colliders. At tree level, the amplitude consists of a gauge contribution $O(g^2)$ and a Higgs sector contribution $O(\lambda)$. The $O(\lambda)$ term is connected to the longitudinally polarized component of the gauge bosons. It can be calculated using \mathcal{L}_H , receiving contributions from both three-point and four-point interactions, see Fig. 1.

At high center-of-mass energy, $\sqrt{s} \gg M_H$, the three-point interactions contributing to the amplitude A are suppressed by powers of M_H^2/s :

$$A(W_L^+ W_L^- \rightarrow Z_L Z_L) = -2\lambda - \frac{4\lambda^2 v^2}{s - M_H^2} + O(\lambda^2) + O(\lambda^4 v^4), \quad (14)$$

$$A(W_L^+ W_L^- \rightarrow Z_L Z_L) = -2\lambda \left(1 + \frac{M_H^2}{s - M_H^2} \right) + O(\lambda^2) + O(\lambda^2 M_H^2), \quad (15)$$

$$A(W_L^+ W_L^- \rightarrow Z_L Z_L) \xrightarrow{s \gg M_H^2} -2\lambda + O(\lambda^2) + O\left(\frac{M_H^2}{s}\right). \quad (16)$$

This is also true when including higher order corrections, as can be shown using dimensional arguments. Hence the entire high-energy s -dependence originates from scattering graphs related to \mathcal{L}_{4pt} , the part of the Higgs Lagrangian which is identical to a massless ϕ^4 theory with $N=4$. The resulting high-energy scattering graphs up to two loops are shown in Fig. 2. Calculating these leading corrections up to two loops, the renormalized OMS amplitude is [15, 16]

$$\sigma(s) = \frac{1}{8\pi s} [\lambda(\mu)]^2 \left[1 + \left(24 \ln \frac{s}{\mu^2} - 42.65 \right) \frac{\lambda(\mu)}{16\pi^2} + \left(432 \ln^2 \frac{s}{\mu^2} - 1823.3 \ln \frac{s}{\mu^2} + 2457.9 \right) \frac{\lambda^2(\mu)}{(16\pi^2)^2} + \mathcal{O}(\lambda^3(\mu)) \right], \quad (17)$$

where the explicit μ dependence has been kept using the results in [7]. An anomalous dimension prefactor has been neglected since it is close to unity for the values of \sqrt{s} and M_H considered here. Using the three-loop running coupling in connection with the above two-loop result, we obtain the RG improved next-to-next-to-leading-log (NNLL) cross section. The NNLL result contains a complete summation of the leading logarithms (LL), $\lambda^{k+1} \ln^k(s/\mu^2)$, the next-to-leading logarithms, $\lambda^{k+2} \ln^k(s/\mu^2)$, and the NNLLs, $\lambda^{k+3} \ln^k(s/\mu^2)$, $1 \leq k \leq \infty$. Just using the one-loop cross section together with the two-loop running coupling yields the NLL cross section, and the LL cross section corresponds to the tree-level result with $\lambda(\mu)$ given by Eq. (11). The product $s\sigma$ depends only on the running cou-

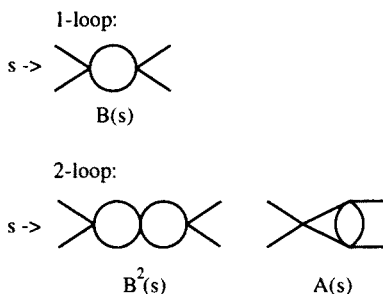


Fig. 2. The dominant s -channel high-energy topologies up to two loops. The t - and u -channel can be obtained using crossing relations.

pling and the ratio μ/\sqrt{s} , making the above cross section a useful quantity in determining perturbative upper bounds on $\lambda(\mu)$. Taking $\mu = \sqrt{s}$, all RG

logarithms are completely resummed into $\lambda(\sqrt{s})$ and we obtain

$$s\sigma \propto \lambda^2(\sqrt{s}) \left(1 - 42.65 \frac{\lambda(\sqrt{s})}{16\pi^2} + 2457.9 \frac{\lambda^2(\sqrt{s})}{(16\pi^2)^2} \right). \quad (18)$$

It is striking that the coefficients appearing in the cross section are much larger than the corresponding coefficients of the decay widths, Eqs. (12) and (13). Peculiarly, a value $\lambda(\sqrt{s}) > 2.7$ leads to a *negative* NLL cross section, indicating the complete failure of perturbation theory for such a coupling.² Using the various criteria listed earlier, the upper perturbative bound is about $\lambda(\sqrt{s}) \approx 2.2$ [7, 16–18]. This upper bound on a perturbative Higgs running coupling represents a strong restriction on the Higgs mass, stronger than the bound obtained in the case of the decay widths, $\lambda(M_H) \approx 4$. Choosing, for example, \sqrt{s} to be about 2 TeV, the Higgs mass has to be less than about 400 GeV to allow for a perturbative calculation of the cross section. Recent results [19], however, show that this bound can be relaxed; see the following section.

3.3. Improving perturbation theory

The perturbative behaviour of the scattering cross sections can be improved by performing a partial summation of higher-order contributions [19]. Realizing that chains of bubble diagrams, $B(s)$, and diagrams with bubble substructures (see Fig. 2 for examples up to two loops) are the dominant contribution to the s -dependent part of the scattering amplitude, a class of nonlogarithmic terms accompanying the logarithmic s -dependent terms can be partially summed. This summation can be formulated in terms of a modified scale entering the running coupling. Instead of using the standard choice $\mu = \sqrt{s}$, the improved choice is [19]

$$\mu = \frac{\sqrt{s}}{e} \approx \frac{\sqrt{s}}{2.7}. \quad (19)$$

The vast improvement due to this choice can be seen when plotting scale dependence of the cross section. In Fig. 3 the quantity $s\sigma$ is given as a function of μ/\sqrt{s} , fixing the running coupling at $\mu = \sqrt{s}/e$ to be 1.5 (which corresponds to $\lambda(\sqrt{s}) = 1.9$).

Taking $\mu = \sqrt{s}$, the cross section receives large radiative corrections and has a strong scale dependence. In particular, the scale dependence remains large even when performing a NNLL resummation. In contrast, the

² The value $\lambda(\sqrt{s}) = 2.7$ corresponds to, for example, $M_H = 500$ GeV and $\sqrt{s} = 1.2$ TeV.

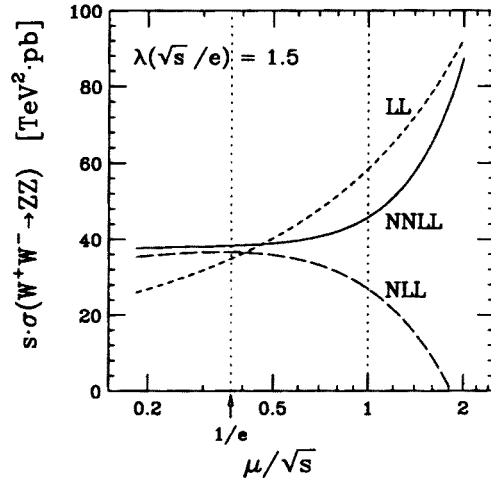


Fig. 3. The scale dependence of the high-energy cross section $W_L^+ W_L^- \rightarrow Z_L Z_L$. The running coupling is fixed to be $\lambda(\sqrt{s}/e) = 1.5$, corresponding to $\lambda(\sqrt{s}) = 1.9$.

improved scale $\mu = \sqrt{s}/e$ leads to much smaller corrections, and the scale dependence almost vanishes when going beyond the LL approximation. Of course, increasing the value of the running coupling, perturbation theory also breaks down for the summed cross section. This happens for $\lambda(\sqrt{s}/e)$ of about 4.0, where perturbative unitarity is violated [19]. Such a running couplings corresponds to, for example, $M_H \approx 700$ GeV and \sqrt{s} close to 2 TeV.

Observing the Higgs sector, $O(\lambda)$, contribution to high-energy $W^+ W^-$ scattering and verifying its theoretical prediction is experimentally challenging. For small values of λ , the electroweak $O(g^2)$ terms are dominant. Larger values of λ increase the $O(\lambda)$ part of the cross section (Fig. 4), but accumulation of a sufficient amount of data and elimination of background make this a difficult task.

4. Discussion

The perturbative upper bounds on the Higgs (running) coupling indicate that perturbation theory is reliable for Higgs masses up to 700 GeV if the process considered has a center-of-mass energy of less than 2 TeV. Going to higher cms energy, the value of M_H has to be reduced to satisfy $\lambda(\sqrt{s}/e) < 4.0$. Of course, these bounds are process dependent. However, all $2 \rightarrow 2$ high-energy scattering processes involving W_L^+ , W_L^- , Z_L and H particles are

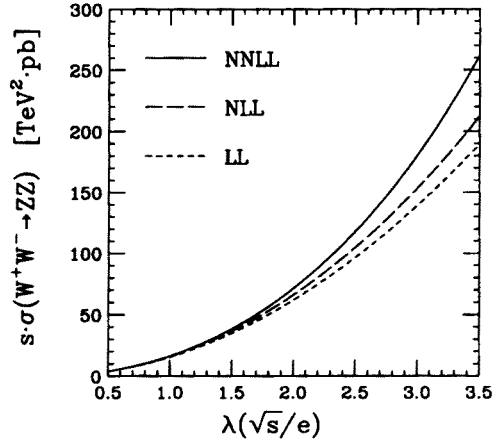


Fig. 4. The scaled cross section of $W^+W^- \rightarrow ZZ$ for $0.5 < \lambda < 3.5$ in the high-energy approximation, fixing $\mu = \sqrt{s}/e$. Gauge and Yukawa coupling contributions are neglected.

related by the underlying $SO(4)$ symmetry of \mathcal{L}_{4pt} and therefore have similar amplitudes. In particular the unitarity argument, first used by [2, 20], involves the different channels and supports the above bound.

The perturbative upper bounds can be compared with results obtained from lattice calculations [6, 9, 10]. Such results depend somewhat on the lattice action and regularization used. Interestingly, they obtain similar bounds on the Higgs mass, excluding the existence of a heavier standard model Higgs boson. Together with the results reviewed here, this excludes the possibility of a strongly interacting Higgs boson [19].

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