

PRECISION TESTS OF THE STANDARD MODEL AND HIGHER ORDER EFFECTS: THE EXAMPLE OF THE EFFECTIVE MIXING ANGLE*

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(Received September 9, 1996)

The radiative corrections involved in the precise determination of the electroweak mixing angle measured at the Z^0 peak are reviewed in detail, with particular emphasis on the new calculation of two-loop heavy top effects. This example serves as a brief pedagogical introduction to the problems and techniques of higher order electroweak calculations.

PACS numbers: 12.15. Lk

The experimental verification of the Standard Model (SM) has reached in the last decade a high degree of sophistication. On the experimental side we have now a large amount of very precise data collected at high energy, and several observables are known at the permille level. On the theoretical side, a major effort has been undertaken in order to match the experimental accuracy. The one-loop corrections to all electroweak observables are by now very well established [1], and two and three-loop effects have been investigated in many cases. Because of the number of different mass scales involved, multi loop calculations in the SM are highly non-trivial, but in the last few years a surge of activity in this field has made the investigation of some higher order effects possible [2].

Although the assessment of the theoretical error is a very subjective matter, we can generally distinguish between two different kind of uncertainties: *parametric*, i.e. due to the uncertainty of the input parameters, and *intrinsic*, that is inherent to the truncation of the perturbative expansion, or to the appearance of non-perturbative effects. An important example of parametric uncertainty is the one due to the hadronic contribution to the running of the electro-magnetic coupling from low energy to the Z^0

* Presented at the XXXVI Cracow School of Theoretical Physics, Zakopane, Poland, June 1-11, 1996.

resonance. As long distance dynamics is involved in this case, one has to resort to the experimental data for $e^+e^- \rightarrow \text{hadrons}$. Through the use of dispersion relations, $\alpha(M_Z)$ is then determined with an accuracy of about 7×10^{-4} [3]. This sizable uncertainty is the main source of theoretical error for the Z^0 -peak observables (except for low-angle Bhabha scattering). On the other hand, scheme dependence, scale dependence, and explicit evaluation of higher order corrections can all give information on the intrinsic error, hence the crucial importance of higher order corrections for the tests of the SM.

In this lecture I will use a relatively simple example to illustrate some of the problems and techniques involved in the evaluation of the higher order effects relevant for the precision tests of the SM. The case of the effective Weinberg angle, $\sin^2 \theta_{\text{eff}}^{\text{lept}}$, measured at LEP and SLC, has both the relevance and the simplicity to make a detailed and self-contained discussion possible. This observable is now known with excellent accuracy (the present world average is $\sin^2 \theta_{\text{eff}}^{\text{lept}} = 0.23165 \pm 0.00024$ [4]), and its sensitivity to the Higgs mass is very high in comparison to other quantities, giving it a predominant role in the present analyses.

After introducing to the basics of the renormalization of the SM, I will review the one-loop electroweak corrections to $\sin^2 \theta_{\text{eff}}^{\text{lept}}$. I will then discuss some potentially large higher order effects: QCD corrections to hadronic loops, large mass two-loop effects (top quark and heavy Higgs case), *etc.* In the case of the two-loop heavy top corrections, I will describe in detail a new calculation of the $O(\alpha^2 m_t^2/M_W^2)$ contributions, and illustrate explicitly the heavy mass expansion method.

In the SM Lagrangian, besides the masses of the fermions and of the Higgs boson, and the CKM mixing parameters, there are only three independent parameters, which we can choose to be, for example,

$$g \quad g' \quad v, \quad \text{or} \quad g \quad M_W \quad M_Z, \quad (1)$$

where g and g' are the $SU(2)_L$ and $U(1)_Y$ couplings, and v is the vacuum expectation value of the Higgs field. On the other hand, there are presently three very obvious experimental inputs that we can use to determine these parameters: α , the fine structure constant, G_μ , the Fermi constant, and M_Z ; they are respectively known within 4×10^{-9} , 8×10^{-6} , 2×10^{-5} . What effectively enters the physics of the weak interactions is generally $\alpha(M_Z)$, the running electro-magnetic coupling at the Z^0 scale. For the reasons mentioned above, this is known only slightly better than the permille level.

A number of natural relations link the couplings and the masses of the gauge bosons occurring in the bare SM Lagrangian. Although initially defined in terms of the gauge couplings, after spontaneous symmetry breaking

the electroweak mixing angle θ_W^0 relates both bare masses and couplings among themselves:

$$\tan \theta_W^0 = \frac{g_0'}{g_0}, \quad e_0 = g_0 \sin \theta_W^0, \quad M_W^0 = \cos \theta_W^0 M_Z^0. \quad (2)$$

At any order in perturbation theory we can relate the parameters of the Lagrangian to the measured inputs, calculate them, and then make predictions for any observable. For instance, using $s \equiv \sin \theta_W$ and $c \equiv \cos \theta_W$, most relevant among such relations are

$$s^2 = \frac{\pi \alpha}{\sqrt{2} G_\mu M_W^2} \quad \rightarrow \quad s_R^2 = \frac{\pi \alpha}{\sqrt{2} G_\mu M_W^2} \frac{1}{(1 - \Delta r_W^R)} \quad (3a)$$

$$M_W^2 = c^2 M_Z^2 \quad \rightarrow \quad M_W^2 = c_R^2 M_Z^2 \rho_R. \quad (3b)$$

Here the l.h.s. corresponds to a tree level description, while on the r.h.s. I have shown that, in a given renormalization scheme "R", the quantum effects can be incorporated through the radiative corrections Δr_W^R and ρ_R . They are functions of s_R , M_W , m_t , M_H , *etc.*, and clearly depend on the renormalization scheme. In the following I will adopt $\overline{\text{MS}}$ renormalized couplings at the Z^0 scale, so that the mixing angle is defined by

$$\hat{s}^2 \equiv \sin^2 \hat{\theta}_W(M_Z)_{\overline{\text{MS}}} \equiv \frac{\hat{\alpha}(M_Z)_{\overline{\text{MS}}}}{\hat{\alpha}_2(M_Z)_{\overline{\text{MS}}}} \quad (4)$$

($\hat{\alpha}$ and $\hat{\alpha}_2$ are the $\overline{\text{MS}}$ U(1)_{e.m.} and SU(2) running couplings), while the vector boson masses are defined as the physical masses, as in Eq. (3) [5]. This choice has some advantages which will be clear later. In this framework¹, $\Delta \hat{\rho} \equiv 1 - \hat{\rho}^{-1}$ and $\Delta \hat{r}_W$ are the radiative corrections entering the l.h.s. of Eq. (3). By solving these two equations simultaneously, the mass of the W boson and the $\overline{\text{MS}}$ mixing angle can be determined.

At LEP and SLC $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ is measured from the on-resonance asymmetries. Leaving aside the photon mediated amplitudes and the QED radiative corrections, which form a finite and gauge invariant set and are routinely subtracted by the experimental groups, the left-right (LR) and forward-backward asymmetries depend only on the ratio of the vector and axial-vector couplings of the Z^0 to the leptons. In the simple case of the LR asymmetry measured at SLC,

$$A_{\text{LR}} = \frac{2 g_V^\ell g_A^\ell}{(g_V^\ell)^2 + (g_A^\ell)^2}, \quad (5)$$

¹ Alternative approaches are reviewed in [6]; the on-shell scheme was originally proposed in [7].

where g_V^ℓ and g_A^ℓ are the vector and axial-vector couplings of an on-shell Z^0 to the leptons.

At the tree level, the amplitude for the decay of a Z^0 boson on mass-shell is given by:

$$Z_\mu^0 \begin{array}{c} \nearrow \ell^- \\ \searrow \ell^+ \end{array} = -i \frac{g}{c} \langle \ell \bar{\ell} | J_Z^\mu | 0 \rangle \epsilon_\mu$$

where ϵ_μ is the polarization vector of the Z^0 , and J_Z^μ is the fermionic current coupled to the Z^0 . The latter can also be written as

$$J_Z^\mu = \frac{1}{2} J_3^\mu - s^2 J_\gamma^\mu, \quad (6)$$

with $J_\gamma^\mu = Q_q \bar{q} \gamma^\mu q$, and $J_3^\mu = I_3^q \bar{q} \gamma^\mu a_- q$ the $U(1)$ and $SU(2)_L$ currents respectively ($I_3^q = \pm 1$ and $a_\mp = \frac{1}{2}(1 \mp \gamma_5)$). We then define $1 - 4\sin^2 \theta_{\text{eff}}^{\text{lept}} \equiv g_V^\ell/g_A^\ell$, and A_{LR} directly measures $\sin^2 \theta_{\text{eff}}^{\text{lept}}$.

At the **one-loop** level much of this simplicity is retained. We need to consider only the electroweak effects; the two topologies that contribute to the Z -decay amplitude at this order are depicted in Fig.1, where (a)

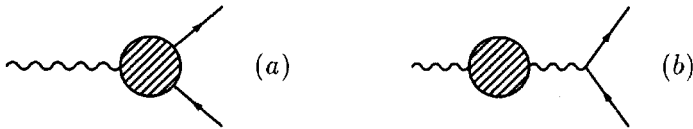


Fig. 1. One-loop Feynman diagrams contributing to Z^0 decay amplitude.

summarizes all vertex corrections and the wave function renormalization of the external fermions, and (b) represents a self-energy insertion, that either mixes the Z with the photon, or contributes to the wave function renormalization of the Z^0 , in which case it is multiplied by a factor $1/2$. Just because of their tensorial structure, the sum of these contributions and of the tree level one can be written in terms of *bare* quantities as

$$-i \frac{g_0}{c_0} A \langle \bar{\ell} \ell | \frac{1}{2} J_3^\mu - s_0^2 B J_\gamma^\mu | 0 \rangle, \quad (7)$$

where A and B are two divergent quantities, regulated in n dimensions. At this point we perform the renormalization of the couplings by a $\overline{\text{MS}}$

subtraction at the Z^0 scale. In Eq. (7) we replace s_0 by $\hat{s}(\mu) - \delta\hat{s}$, and $e_0 = g_0 s_0$ by $\hat{e}(\mu) - \delta\hat{e}$, and set the scale $\mu = M_Z$. Notice that $\delta\hat{s}$ and $\delta\hat{e}$ are pure poles in $\epsilon = (4 - n)/2$. Expanding everything up to $O(\hat{\alpha})$ we get rid of the divergences. The new amplitude is finite:

$$-i \frac{\hat{g}}{\hat{c}} \sqrt{\hat{\rho}} \langle \bar{\ell}\ell \mid \frac{1}{2} J_3^\mu - \hat{s}^2 \hat{k}(M_Z) J_\gamma^\mu \mid 0 \rangle, \quad (8)$$

and we can identify

$$\sin^2 \theta_{\text{eff}}^{\text{lept}} = \text{Re } \hat{k}(M_Z) \hat{s}^2(M_Z). \quad (9)$$

In order to actually compute $\sin^2 \theta_{\text{eff}}^{\text{lept}}$, we therefore need the form factor $\hat{k}(M_Z)$, which depends only on the J_γ^μ component of the diagrams in Fig. 1. This is given by the mixing amplitude, $-i\hat{g}\hat{s} \langle \bar{\ell}\ell \mid J_\gamma^\mu \mid 0 \rangle A_{\gamma Z}(M_Z^2)/M_Z^2$, and by the part of the vertex amplitude proportional to J_γ^μ , $-i\hat{g}/\hat{c} \langle \bar{\ell}\ell \mid J_\gamma^\mu \mid 0 \rangle \hat{s}^2 V_\gamma$. Hence

$$\hat{k}(M_Z) = 1 - \frac{\hat{c}}{\hat{s}} \frac{A_{\gamma Z}(M_Z^2)}{M_Z^2} - V_\gamma - \frac{\delta\hat{s}^2}{\hat{s}^2}. \quad (10)$$

From the ultraviolet divergent part (see for ex. the second paper in Ref. [7]) of the vector boson mass counterterms we calculate the counterterm $\delta\hat{s}^2 = -\delta\hat{c}^2 = -[\delta(M_W^2/M_Z^2)]_{UV}$,

$$\frac{\delta\hat{s}^2}{\hat{s}^2} = -\frac{\hat{g}^2}{16\pi^2} \left(\frac{19}{6} + \frac{11}{3} \hat{s}^2 \right) \frac{1}{\epsilon}. \quad (11)$$

In the t'Hooft-Feynman gauge the divergent part of the mixing self-energy is [7]

$$\frac{A_{\gamma Z}(M_Z^2)}{M_Z^2} = \frac{\hat{g}^2}{16\pi^2} \frac{\hat{s}}{\hat{c}} \frac{1}{\epsilon} \left[\frac{7}{6} + \frac{17}{3} \hat{s}^2 + 2 \frac{M_W^2}{M_Z^2} - 2\hat{c}^2 + O(\epsilon) \right]. \quad (12)$$

Similarly, the divergence of V_γ is $2\hat{g}^2\hat{c}^2/(16\pi^2) 1/\epsilon$. Using Eqs. (10-12), and the l.h.s. of Eq. (3b), we see that the form factor $\hat{k}(M_Z)$ is indeed finite through $O(\hat{\alpha})$. It is also gauge invariant and independent of the Higgs mass, as the Higgs boson does not couple to the photon, and the Higgs vertices are suppressed by the Yukawa couplings of the leptons. Explicit expressions of $\hat{k}(M_Z)$ can be found in [8,9].

Because of the top loop contribution to $A_{\gamma Z}(M_Z^2)$, however, the form factor does depend on the top mass, albeit very weakly²:

$$\hat{k}(M_Z)_{\text{top}} = -\frac{\hat{\alpha}}{6\pi\hat{s}^2} \left(1 - \frac{8}{3} \hat{s}^2 \right) \ln \frac{m_t^2}{M_Z^2} + O\left(\frac{M_Z^2}{m_t^2} \right). \quad (13)$$

² For simplicity, \hat{s}^2 is defined here without decoupling the top quark. That approach is considered in [9, 10].

Numerically, the real part of the form factor is very close to one [9]. For $m_t = 175\text{GeV}$, $M_W = 80.356\text{GeV}$, and $\hat{s}^2 = 0.2316$,

$$\sin^2 \theta_{\text{eff}}^{\text{lept}} = \hat{s}^2(M_Z^2) + 4 \times 10^{-5} \quad (14)$$

The unusual smallness of the one-loop electroweak radiative corrections ($\approx 2 \times 10^{-4}$) is due to large cancellations between bosonic and fermionic contributions, which are individually over one order of magnitude larger [9]. The imaginary part of $\hat{k}(M_Z)$, although comparatively large, gives a negligible contribution to the asymmetries [9].

Eq. (14) gives the one-loop prediction for $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ in the SM: once \hat{s}^2 is calculated from the inputs via the l.h.s. of Eqs. (3), $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ can be known with a great accuracy. Of course, \hat{s}^2 depends very sensitively on M_H and m_t , through $\Delta\hat{r}_W$ and $\hat{\rho}$, and has an uncertainty from the value of $\alpha(M_Z)$ of about 0.1%. Moreover, unknown higher order contributions could add a significant uncertainty to its determination. In that respect, however, the situation for \hat{s}^2 seems to be reasonably under control [11, 12] after the calculation of the three-loop QCD [13] and of the two-loop $O(\alpha^2 m_t^2/M_W^2)$ [14] corrections.

Here I will concentrate on the analysis of higher order effects on Eq. (14). Given the estimated intrinsic accuracy on \hat{s}^2 , we can aim to a precision of about 10^{-4} on the form factor $\hat{k}(M_Z)$. A first candidate are the **QCD corrections** to the hadronic loops. Fortunately, the scale of the process at hand is such that perturbative QCD can be reliably applied. As quark loops appear only in the amplitude of Fig. 1(b), QCD affects only the $\gamma-Z$ mixing as in the figure below, where the lowest order diagrams are displayed. Since the axial and the vector current do not mix even in presence of QCD, only the vector current correlator contributes to $A_{\gamma Z}(M_Z^2)$, reducing the two-loop calculation to the old QED one [15]. However, two-loop QCD corrections

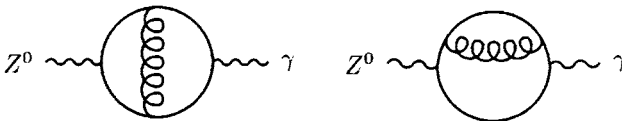


Fig. 2. Two-loop QCD corrections to the $\gamma - Z$ mixing.

to the general electroweak vector boson and scalar self-energies are known exactly [16] for any value of the quark masses and the external momentum, in terms of simple logs and dilogs. Even the three-loop massless quark contributions and the heavy top expansions of the same correlators are now available [13, 17].

Unlike the case of the ρ parameter, however, QCD corrections are not important for the form factor $\hat{k}(M_Z)$. Using [16], we see that the light quark loops, neglecting all mass effects, get a correction

$$\Delta\hat{k}(M_Z)_{\text{QCD}}^{\text{light}} = \frac{\hat{\alpha}}{\pi\hat{s}^2} \left(\frac{7}{12} - \frac{11}{9}\hat{s}^2 \right) \frac{\alpha_s(M_Z)}{\pi} \left(\frac{55}{12} - 4\zeta(3) + i\pi \right), \quad (15)$$

whose real part amounts to about -3×10^{-5} . Using the expansions in the third paper of Ref. [16], we can see that the top contribution of Eq. (13) can be adjusted to include QCD corrections by the replacement $\log m_t^2/M_Z^2 \rightarrow \log m_t^2/M_Z^2 (1 + \alpha_s(m_t)/\pi) - (15/4) \alpha_s(m_t)/\pi$. Numerically, these QCD corrections contribute about 7×10^{-5} to $\hat{k}(M_Z)$. It seems that there is no reason to go beyond two-loop level in QCD.

A relatively large two-loop effect which can be easily accounted for is given by the **interference of the imaginary parts** of $A_{\gamma Z}(M_Z^2)$ and of the photon self-energy (Fig. 3(b)). Indeed, light fermion loops induce large

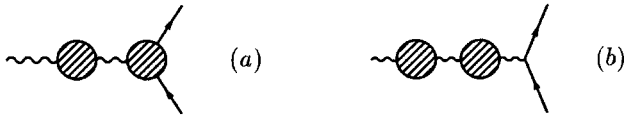


Fig. 3. Main two-loop electroweak topologies contributing to k .

imaginary parts in these two self-energies. We can resum the gauge-invariant fermionic part of the photon self-energy multiplying $A_{\gamma Z}(M_Z^2)$ in Eq. (10) by $M_Z^2/(M_Z^2 - A_{\gamma\gamma}^{(f)}(M_Z^2)|_{\overline{\text{MS}}})$ [8], where $A_{\gamma\gamma}^{(f)}(M_Z^2)|_{\overline{\text{MS}}}$ is the fermionic part of the photon two-point function renormalized in $\overline{\text{MS}}$. The interference between the two imaginary parts (the contribution of the real parts is negligible) increases the value of $\hat{k}(M_Z)$ by 1.9×10^{-4} , a quite large two-loop effect. The inclusion of this kind of contributions is necessary.

Another potential source of large higher order effects are heavy particles; it is well-known that they do not decouple in the SM, and that the corrections due to the heavy top are dominant in $\hat{\rho}$ at the one-loop level. The study of the leading effects of this origin is therefore very important, and is made possible by an expansion in powers of the heavy masses. Let us start briefly considering the possibility of **heavy Higgs** effects.

At one loop, the radiative corrections to amplitudes without Higgs bosons among the external particles depend at most logarithmically on the Higgs mass, in the limit in which this mass is heavy. At two-loop the corrections can be at most quadratic in the Higgs mass. For the ρ parameter these quadratic terms were computed long ago, and more recently a complete

analysis for electroweak observables has been performed [18], in the case where the Yukawa couplings are neglected. The result is that these corrections are completely negligible. Still, the Higgs boson could appear in the irreducible two-loop $\gamma - Z^0$ mixing in association with its Yukawa coupling with the top. In this case, however, the amplitude is also proportional to m_t^2 , and is best considered in connection with a heavy top.

Finally, we consider the two-loop effects of a **heavy top quark**. In the case of the amplitude at hand, we have seen that, at one-loop, the top contributes in a mild, logarithmic way (see Eq. (13)). At two loops, by analogy, we expect only $O(\alpha^2 m_t^4/M_W^4)$ effects, compared with the $O(\alpha^2 m_t^4/M_W^4)$ appearing in $\Delta\hat{\rho}$ [19]. Following the recent calculation of the $O(\alpha^2 m_t^4/M_W^4)$ effects on Eq. (3) [14], which turn out to be relatively important, the study of these quadratic two-loop contributions is necessary. The rest of this lecture is devoted to a detailed presentation of such calculation.

At the two-loop level the diagrams contributing to the Z^0 decay amplitude and proportional to J_γ^μ can be grouped into three categories: (i) products of one-loop Z^0 wave function renormalization by one-loop vertices and self-energies (Fig. 3 with the first two wavy lines representing Z^0 's); (ii) products of one-loop $\gamma - Z^0$ mixing amplitudes by vertices and self-energies (Fig. 3 with the second wavy line representing a photon); (iii) two-loop irreducible vertices and $\gamma - Z^0$ self-energy, displayed in Figs 1(a), (b).

We are interested only in the two-loop contributions proportional to m_t^2 . It is easy to realize that only the last category meets this condition: we have to keep in mind that the one-loop vertices do not contain the top quark and that the $\gamma - Z^0$ self-energy depends only logarithmically on m_t (Eq. (13)). We also notice that the wave function renormalization factor of the Z^0 contains only $\log m_t$ in the heavy top limit. This can be understood on dimensional grounds, as this factor involves only the derivative of the Z^0 vacuum polarization function w.r.t. the external momentum. We therefore need to consider only the subset of two-loop irreducible vertices and $\gamma - Z^0$ self-energies which can be proportional to m_t^2 .

In addition to the two-loop diagrams, we need to include the m_t^2 part of the two-loop $\overline{\text{MS}}$ counterterm of \hat{s}^2 , and renormalize the parameters appearing in the one-loop correction Eq. (10). With our choice of $\overline{\text{MS}}$ couplings and on-shell masses, when does renormalization of the one-loop amplitude introduce m_t^2 terms? From Eq. (11) we see that $\delta\hat{s}^2$ does not contain any such term, and the same is trivially true of $\delta\hat{\alpha}$. As the one-loop amplitude is at most logarithmic in m_t , we conclude that the renormalization of the couplings does not contribute at the level of our calculation. On the other hand, the mass renormalization of the top quark and of the vector bosons does contribute $O(m_t^2)$ terms, because of the top dependence of the

on-shell mass counterterms. The Goldstone bosons and the ghost fields that appear in the one-loop loops also need mass renormalization. In the t'Hooft-Feynman gauge employed here, this is implemented assigning them the corresponding vector boson on-shell mass counterterm. In the case of the Goldstone bosons, however, a spurious quartic m_t^4 dependence must be cancelled by a tadpole contribution, as dictated by Ward identities.

At $O(\alpha^2 m_t^2/M_W^2)$, we can finally write the two-loop form factor as

$$\hat{k}^{(2,\text{top})}(M_Z) = 1 - \frac{\hat{c}}{\hat{s}} \frac{A_{\gamma Z}^{(2+\text{c.t.})}(M_Z^2)}{M_Z^2} - V_\gamma^{(2+\text{c.t.})} - \frac{\delta^{(2)}\hat{s}^2}{\hat{s}^2} + \Delta_{\text{extra}}, \quad (16)$$

where the term Δ_{extra} is generated in Eq. (12) by reexpressing the ratio $2M_W^2/M_Z^2$ in terms of \hat{c}^2 , using the l.h.s. of Eq. (3b). As $\Delta\hat{\rho}$ is $O(\hat{\alpha}m_t^2/M_W^2)$, this introduces a crucial term for the finiteness of our calculation. In n dimensions it is necessary to keep $O(\epsilon)$ terms in $\Delta\hat{\rho}$, so that a finite part is generated: $\Delta_{\text{extra}} = -N_c \hat{g}^4/(16\pi^2)^2 m_t^2/M_Z^2 1/(2\epsilon) + \text{finite part}$, where $N_c = 3$ is the number of colors. The two-loop counterterm $\delta^{(2)}\hat{s}^2$ is a crucial component of the calculation in [14], can be checked with [20], and reads

$$\frac{\delta^{(2)}\hat{s}^2}{\hat{s}^2} = -\frac{N_c \hat{g}^4}{(16\pi^2)^2} \frac{m_t^2}{\hat{c}^2 M_Z^2} \frac{1}{\epsilon} \left(\frac{1}{8} - \frac{13}{36} \hat{s}^2 \right). \quad (17)$$

We now begin the study of the irreducible diagrams with the vertices. The presence of the top quark is only induced by loop insertions on the internal vector boson propagators. The relevant diagrams are shown in Fig. 4(a), (b). Scalar loops are generally suppressed by powers of the lepton

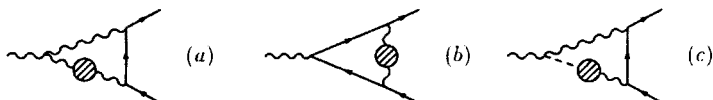


Fig. 4. Two-loop vertex diagrams containing the top.

mass, but at two-loop level the mixing $\phi^+ - W^+$ takes place through the top in the diagram of Fig. 4(c). For what concerns the diagrams of Fig. 4(a), (b), we observe that the leading m_t^2 contribution is cancelled by the mass renormalization of the corresponding vector boson. As the overall divergence of such graphs is logarithmic, and the quadratic subdivergence of the top loop is the same as in the counterterm $\delta M_W^2 = A_{WW}(M_W^2) \approx A_{WW}(0)$, the use of on-shell masses in the one-loop vertices removes all $O(\alpha^2 m_t^2/M_W^2)$ vertex contribution at two-loop level. This shows a clear advantage of the renormalization procedure adopted. The case of Fig. 4(c) is obviously very

different and requires an explicit two-loop calculation, which I describe as an example of the heavy mass expansion method.

A few kinematical simplifications allow us to write the amplitude in Fig. 4(c) as

$$I = -i \frac{\hat{g}^5 \hat{s}^2 m_t^2}{2 \hat{c}} \langle \ell \bar{\ell} | J_3^\mu | 0 \rangle \int \frac{d^n k d^n p}{(2\pi)^{2n} \mu^{-4\epsilon}} \frac{k \cdot p}{k^2 p^2 P_W^2 P K_t P Q_W}, \quad (18)$$

where I used the abbreviations $P_W = p^2 - M_W^2$, $P K_t = (p - k)^2 - m_t^2$, and $P Q_W = (p - q)^2 - M_W^2$. First we notice that I is in fact a self-energy integral, because all the dependence on the external lepton momenta has dropped out. It can therefore be treated as a two-loop two-point function.

In general, a two-loop two-point function cannot be expressed in terms of polylogarithms or other known functions (of course in special cases, like for QCD corrections considered above, the two-loop integrals can be expressed in a compact way). The choice is then between numerical evaluation, which can be now very efficient [22], and approximate analytic evaluation. In particular, we are now interested in a heavy mass expansion [21] that enables us to expand the integral in Eq. (18) in powers of m_t , keeping only the leading m_t^2 term. The simple method here illustrated is suggestive of the strategy adopted in [14].

The integral I has a threshold at $q^2 = M_W^2$, and is therefore not analytic at $q^2 = M_W^2$. We cannot use a simple Taylor expansion in the external momentum in the approximation $q^2 \ll m_t^2$, but we have to resort to an asymptotic expansion. The identity

$$\frac{1}{(p - k)^2 - m_t^2} \rightarrow \frac{1}{k^2 - m_t^2} + \frac{2 p \cdot k - p^2}{(k^2 - m_t^2) [(p - k)^2 - m_t^2]} \quad (19)$$

helps us separate different regions of the domain of integration. Using Eq. (19) in I , we obtain a disconnected two-loop integral (product of two one-loop integrals), plus a two-loop integral with improved ultraviolet convergence in the k integration and improved infrared convergence in the p integration. Applying Eq. (19) twice to I , we obtain, up to disconnected integrals proportional to $k \cdot p$ that vanish by symmetry,

$$\int \frac{d^n k d^n p}{(2\pi)^{2n} \mu^{-4\epsilon}} \left[\frac{2(k \cdot p)^2}{k^2 p^2 K_t^2 P_W P Q_W} - \frac{k \cdot p (2 k \cdot p - p^2)^2}{k^2 K_t^2 p^2 P_W^2 P K_t P Q_W} \right] \quad (20)$$

with $K_t = k^2 - m_t^2$. The first integral can be easily reduced to the product of two well-known one-loop integrals, while the second is $O(1/m_t^2)$, as can be checked counting the degrees of infrared and ultraviolet divergence,

and can be dropped. Finally, we are able to write the vertex contribution proportional to J_γ^μ as

$$V_\gamma^{(2+\text{c.t.})} = -\frac{N_c \hat{g}^4 \hat{s}^2}{(16\pi^2)^2} \frac{m_t^2}{M_Z^2} \left[\frac{C0[M_Z^2, M_W, M_W]}{\epsilon} + \text{finite} + O\left(\frac{1}{m_t^2}\right) \right], \quad (21)$$

where $C0[q^2, m_1, m_2]$ is a convergent one-loop integral. The method followed in this example can be used in a similar way in the case of the irreducible $\gamma - Z^0$ self-energies. Schematically, we can summarize the effect of the heavy top expansion as in the figure below, where the one loop diagrams are evaluated

exactly at the appropriate momentum $q^2 = M_Z^2$. The two-loop vacuum ($q^2 = 0$) integrals can always be reduced to a closed analytic expression containing dilogarithms following the strategy proposed, for ex., in [23]. For what concerns the counterterm pieces, the procedure consists simply of expanding all masses in the one-loop $A_{\gamma Z}(M_Z^2)$ according to $m \rightarrow m - \delta m$, keeping all $O(\epsilon)$ terms. The combination of irreducible and counterterm pieces is

$$A_{\gamma Z}^{(2+\text{c.t.})}(M_Z^2) = \frac{N_c \hat{g}^4 \hat{s} m_t^2}{(16\pi^2)^2 \hat{c}^3} \left[\left(\hat{s}^2 \hat{c}^2 C0[M_Z^2, M_W, M_W] - \frac{3}{8} + \frac{5}{36} \hat{s}^2 \right) + \text{finite} \right] \quad (22)$$

Putting together the various pieces of Eq. (16) the cancellation of divergences can be verified. In the notation of [14], the final result for $\text{Re } \hat{k}^{(2,\text{top})}(M_Z)$, in units $N_c(\hat{\alpha}/(4\pi\hat{s}^2))^2 m_t^2/M_W^2$, reads

$$\begin{aligned} \text{Re } \Delta \hat{k}^{(2,\text{top})} = & \frac{-211 + 24 ht + 462 \hat{s}^2 - 64 ht \hat{s}^2}{432} \\ & + \left(\frac{3}{8} - \frac{\hat{s}^2}{3} \right) B0[M_Z^2, M_W^2, M_W^2] + \frac{(ht - 4) \sqrt{ht} (8 \hat{s}^2 - 3) g(ht)}{108} \\ & - \frac{(6 + 27 ht - 10 ht^2 + ht^3) (3 - 8 \hat{s}^2) \ln ht}{108 (ht - 4)} \\ & - \frac{\hat{c}^2}{6} \ln \hat{c}^2 + \left(\frac{5}{36} + \frac{7}{18} \hat{s}^2 \right) \ln zt + \frac{(ht - 1) (8 \hat{s}^2 - 3)}{18 (4 - ht) ht} \phi\left(\frac{ht}{4}\right). \quad (23) \end{aligned}$$

Numerically, this correction is negligible. It grows with M_H , reaching about 2.5×10^{-5} for $M_H = 700 \text{ GeV}$, well below our precision goal. However, it

must be emphasized that this result is strictly related to the renormalized parameters that have been chosen here, and to the way the one-loop result has been expressed. When all the two-loop effects considered here are included, and using the same inputs, Eq. (14) is modified into

$$\sin^2 \theta_{\text{eff}}^{\text{lept}} = \hat{s}^2(M_Z^2) + 1.0 \times 10^{-4}. \quad (24)$$

In summary, I have reviewed the radiative corrections entering the relation between the effective leptonic sine and the $\overline{\text{MS}}$ parameter $\sin^2 \hat{\theta}_W(M_Z^2)$. The main higher order effects have been considered, and a new calculation of the $O(\alpha^2 m_t^2/M_W^2)$ contributions has been illustrated in detail. All these corrections are very small and we can conclude that in the case at hand higher order effects are unlikely to be a major source of uncertainty.

I am grateful to Marek Jeżabek and to the organizers of the School for the kind invitation, the hospitality, and the excellent organization. The results of the two-loop heavy top calculation have been obtained in collaboration with G. Degrassi.

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