LINEAR BEAM SIZE EFFECT AT $\mu^+\mu^-$ COLLIDERS*

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Among various projects for future high-energy colliders a special place is occupied by the muon collider (for a review see Ref. [1]). It is interesting to note that if this machine is ever built, this will be the first place where collisions of unstable particles can be investigated. It turns out that even though the muon is only slightly unstable, this instability, as a matter of principle, brings in a new interesting effect. We name this effect a linear beam size effect because it predicts that certain scattering cross sections must be proportional to the transverse sizes of the colliding muon beams [2, 3, 4]. This paper is devoted to the detailed presentation of this effect.

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1. Introduction

It has been known since the early 60's that high-energy processes can have a t-channel singularity in the physical region [5]. Such a situation can occur when initial particles in a given reaction are unstable and the masses of the final particles are such that the real decay of the initial particles can take place.

Recently in Ref. [6] it was emphasized that this problem turns out to be of practical importance for the reaction $\mu^-\mu^+ \to e^-\bar{\nu}_e W^+$. It was noted in [6] that the standard calculation of this cross section leads to an infinite result. Indeed, if the invariant mass of the final $e\bar{\nu}_e$ system is smaller than the muon mass m, the square of the momentum transfer in the t-channel q^2 can be positive or negative, depending on the scattering angles. In a region of small q^2 defined by the inequalities $-\Lambda < q^2 < \Lambda$ and $\Lambda \ll m^2$ the main contribution to the discussed cross section is given by the diagram with the

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exchange of the muonic neutrino in the t-channel (see Fig.1). The t-channel propagator introduces a factor $1/q^2$ to the matrix element M of this process

$$M \propto \frac{1}{q^2 + i\epsilon} \tag{1}$$

which results in a power–like singularity in the standard cross section

$$d\sigma \propto \int\limits_{-\Lambda}^{\Lambda} |M|^2 \, dq^2 \propto B = \int\limits_{-\Lambda}^{\Lambda} \frac{dq^2}{|q^2 + i\epsilon|^2} = \infty \,. \tag{2}$$

It is, therefore, necessary to find a physically reliable method to make a definite prediction for the measurable number of events.

In Ref. [6] (see also [5]) it was suggested to regularize this divergence by taking into account the instability of muons in the initial state. If this is the case, then the muon mass is a complex quantity with the imaginary part proportional to the muon width Γ. It is possible then to solve the energy–momentum conservation constraints with the conclusion that the square of the momentum transfer in the t-channel acquires an imaginary part proportional to the muon width. This observation leads to the replacement

$$q^2 \to q^2 - i\gamma, \qquad \gamma \sim m\Gamma.$$

The divergent integral in (2) is therefore regularized

$$B = \int\limits_{-\Lambda}^{\Lambda} \frac{dq^2}{|q^2 + i\epsilon|^2} \to \int\limits_{-\Lambda}^{\Lambda} \frac{dq^2}{|q^2 - i\gamma|^2} \sim \frac{1}{\gamma} \sim \frac{1}{m\Gamma}. \tag{3}$$

Consequently a direct calculation of the scattering cross section appears to become possible.

We note in this regard that if such regularization is applied, one can estimate a typical time necessary for this reaction to occur. It turns out that this time is of the order of the moving muon lifetime, i.e., much longer than the time necessary for muon beams to cross each other. Therefore for realistic muon colliders the results of Ref. [6] are not applicable and a new consideration is necessary.

We have found that for muon colliders a much more important effect is connected with the finite sizes of the colliding beams. We are going to show *that accounting for the finite sizes of the colliding beams gives a finite cross section for processes like*

$$\mu^- \mu^+ \to e^- \bar{\nu}_e X \tag{4}$$

with a t-channel singularity in the physical region. This is the main result of this paper. As an example, we reconsider the reaction $\mu^-\mu^+ \to e\bar{\nu}_e W^+$ and show that the actual cross section for this process at total energies of $\sqrt{s} \sim 100$ GeV is approximately 1 fb for typical transverse beam sizes of $a \sim 10^{-3}$ cm.

Let us mention here that the finite beam size effect at the high–energy colliders is well studied both experimentally and theoretically (for a review see Ref. [7]). For the first time this beam–size effect (BSE) was observed at the VEPP-4 collider (Novosibirsk) in 1980–81 during a study of a single bremsstrahlung in the electron–positron collisions [8]. Last year the BSE was observed at HERA in the reaction $ep \to ep\gamma$ [9]. In both cases the number of observed photons was smaller than expected according to the standard calculations. The decreased number of photons is explained by the fact that impact parameters, which give the essential contribution to the standard cross section of these reactions, are larger by 2-3 orders of magnitude compared to the transverse beam sizes. From a theoretical point of view the BSE represents a remarkable example of the situation where traditional notions of the cross section and standard rate formulas are no longer valid.

The remainder of this paper is organized as follows. In the next section, as a realistic example of a process with the t-channel singularity in the physical region, we consider reaction (4) and present its cross section. In Section 3 we prove the basic formula used in Section 2. Section 4 is devoted to the particular case — the reaction $\mu^-\mu^+ \to e\bar{\nu}_e W^+$. Further we develop a qualitative picture of the effect and present our conclusions.

2. General case

In this section we show how the cross section of the reaction $\mu^-\mu^+ \to e^-\bar{\nu}_e X$ can be calculated. We introduce the notations: $s = (p_1+p_2)^2 = 4E^2$ is the square of the total energy in the center of mass frame, m and $\Gamma = 1/\tau$ are muon mass and width respectively, $p_1^2 = p_2^2 = m^2$, p_3 is the 4-momentum of the final $e^-\bar{\nu}_e$ system, $y = p_3^2/m^2$, $q = p_1 - p_3 = (\omega, \boldsymbol{q})$ is the momentum transfer in the t-channel and $x = qp_2/p_1p_2 \approx \omega/E$.

From simple kinematics it follows that

$$q^2 = -\frac{\boldsymbol{q}_\perp{}^2}{1-x} + t_0, \quad t_0 = \frac{x(1-x-y)}{1-x}\, m^2 \,, \tag{5}$$

where $\boldsymbol{q}_\perp$ is the component of momentum $\boldsymbol{q}$ which is transverse to the momenta of initial muons. Note that

$$t_0 > 0 \quad \text{for} \quad y < 1-x \,, \tag{6}$$

and that for $q^2 = 0$

$$|\boldsymbol{q}_\perp| = q_\perp^0 = m\sqrt{x(1-x-y)}. \tag{7}$$

For $y < 1 - x$ (or $t_0 > 0$) we write the cross section in the form

$$d\sigma = d\sigma_{\text{stand}} + d\sigma_{\text{non-stand}}, \tag{8}$$

where by definition $d\sigma_{\text{stand}}$ corresponds to the region $q^2 < -m^2$ and $d\sigma_{\text{non-stand}}$ corresponds to the region

$$-m^2 < q^2 < t_0. \tag{9}$$

We will show that the main contribution to $d\sigma_{\text{non-stand}}$ is given by the region of very small values of q^2 inside the region (9):

$$-\Lambda < q^2 < \Lambda, \quad m/a \ll \Lambda \ll m^2, \tag{10}$$

where a is the typical transverse beam size. In the region (10) the main contribution comes from the diagram with the exchange of the muonic neutrino in the t-channel (Fig. 1). Since for such q^2 the exchanged neutrino is almost real, the corresponding matrix element can be considerably simplified.

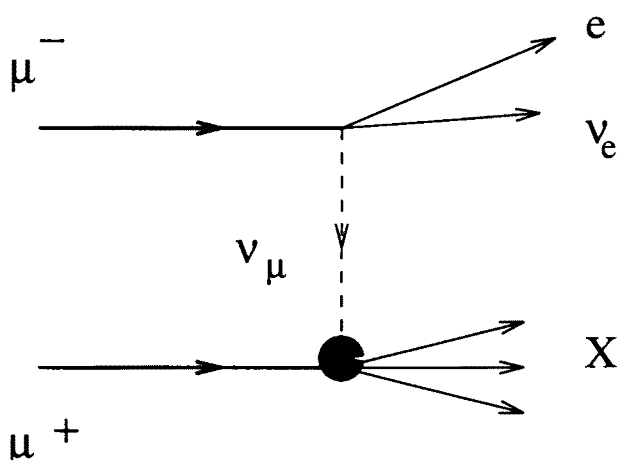

Fig. 1. The Feynman diagram for the reaction $\mu^-\mu^+ \to e\bar{\nu}_e X$ which gives the leading contribution in the region of small $|q^2|$.

As a result, in the region (10) we present the matrix element M in the form

$$M = -M_{\mu\to e\bar{\nu}_e\nu_\mu}\,\frac{1}{q^2+i\epsilon}\,M_{\nu\mu\to X}, \tag{11}$$

where $M_{\mu\to e\bar{\nu}_e\nu_\mu}$ is the matrix element for the μ^- decay and $M_{\nu\mu\to X}$ is the matrix element for the $\nu_\mu\mu^+ \to X$ process. In both of these subprocesses we take q^2 equal to zero.

Using this equation we express the non–standard contribution through the muon decay width Γ and the cross section $\sigma_{\nu\mu\to X}$ of the $\nu_\mu\mu^+ \to X$ process as

$$d\sigma_{\text{non-stand}} = \frac{1}{\pi}\, x\; m\; d\Gamma\; \frac{dq^2}{|q^2|^2}\; d\sigma_{\nu\mu\to X}\,. \tag{12}$$

For unpolarized muon beams we have

$$d\Gamma = \frac{\Gamma}{\pi}(1-y)(1+2y)dxdyd\varphi\,, \tag{13}$$

where φ is the azimuthal angle of the vector $\boldsymbol{q}$. Let us call the coefficient in front of $d\sigma_{\nu\mu\to X}$ in Eq. (12) as *the number of neutrinos*:

$$dN_\nu = \frac{1}{\pi}\, x\; m\; d\Gamma\; \frac{dq^2}{|q^2|^2} = \frac{m\Gamma}{\pi^2}\, x(1-y)(1+2y)dxdyd\varphi\; \frac{dq^2}{|q^2|^2}\,. \tag{14}$$

As it is clear from Eq. (12) the standard calculation of the cross section turns out to be impossible due to the power–like singularity, since the point $q^2 = 0$ is within the physical region for $y < 1 - x$.

The main result of our investigation of the BSE in the above process can be formulated as follows:

Accounting for the BSE results in the following treatment of the divergent integral in Eq. (12):

$$B = \int_{-\Lambda}^{\Lambda} \frac{dq^2}{|q^2|^2} \to \pi\frac{a}{q_\perp^0} = \pi\,\frac{a}{m\sqrt{x(1-x-y)}}\,. \tag{15}$$

The exact expression for the quantity a will be given below (see Eqs.(35)–(38)). We just mention here that it is proportional to the transverse sizes of the colliding beams. For identical round Gaussian beams with root-mean-square radii

$$\sigma_{ix} = \sigma_{iy} = \sigma_\perp\,, \quad i = 1,2$$

this quantity is equal to

$$a = \sqrt{\pi}\sigma_\perp\,. \tag{16}$$

The contribution (15) comes from the region (10). The contribution from the remaining part of the region (9) is smaller. Indeed, its relative value is of the order of

$$\frac{1}{B}\int_{-m^2}^{-\Lambda} \frac{dq^2}{|q^2|^2} + \frac{1}{B}\int_{\Lambda}^{t_0} \frac{dq^2}{|q^2|^2} \sim \frac{m/a}{\Lambda} \ll 1\,. \tag{17}$$

Using Eqs. (15)–(17) and integrating the number of neutrinos (14) over y (in the region $0 < y < 1 - x$) and over φ, we arrive at the following spectrum of neutrinos:

$$\frac{dN_\nu(x)}{dx} = \frac{\pi a}{2c\tau} f(x), \quad f(x) = \frac{24}{5\pi}\sqrt{x(1-x)}\left(1 + \frac{22}{9}x - \frac{16}{9}x^2\right), \quad \int_0^1 f(x)dx = 1, \tag{18}$$

where τ is the life time of the muon at rest, $c\tau = 660$ m. The plot of the function $f(x)$ is presented in Fig.2.

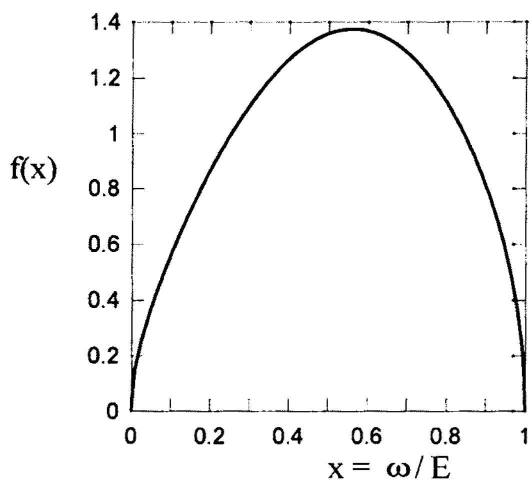

Fig. 2. The normalized spectrum of neutrinos $f(x) = (1/N_\nu)(dN_\nu/dx)$ vs. x — the fraction of the muon energy carried by neutrino (see Eq. (18)).

The total number of neutrinos is equal to

$$N_\nu = \frac{\pi}{2}\frac{a}{c\tau}. \tag{19}$$

After the spectrum of neutrinos is obtained, the non-standard cross section for the reaction $\mu^-\mu^+ \to e\bar{\nu}_e X$ is given by the equation

$$d\sigma_{\text{non-stand}} = dN_\nu(x)\, d\sigma_{\nu\mu\to X}(xs) = N_\nu\, f(x)\, dx\, d\sigma_{\nu\mu\to X}(xs)\,. \tag{20}$$

Subsequent integration over x can be performed without further difficulties. For the particular case $X = W^+$ such calculation is performed in Section 4.

3. Derivation of the basic formula

Here we are going to prove our result presented in the previous section (see Eq. (15)). To begin with, let us note that the standard notion of the cross section is an approximation itself. As is well known it corresponds to the plane wave approximation for initial and final particles. In a real experiment the particles are confined to beams of a relatively small size and it is the collision of such beams that leads to the measurable number of events.

In order to arrive at more general formulas for the scattering processes, it is necessary to describe the collisions of wave packets instead of plane waves. In view of the fact that the movement of the particles inside the beam is quasiclassical, a simple and efficient technique for taking into account the beam–size effects in the actual calculations has been developed [10] (for a review see [7]).

Below we present some results from Refs. [10], [7] which are essential for our discussion. For simplicity, we neglect the energy and angular spread of the particles in the colliding beams.

Note that in the standard approach the number of events N is the product of the cross section σ and the luminosity L:

$$dN = d\sigma\, L, \quad d\sigma \propto |M|^2, \quad L = v \int n_1(\boldsymbol{r},t)\; n_2(\boldsymbol{r},t) d^3r dt\;,, \tag{21}$$

where $v = |\boldsymbol{v}_1 - \boldsymbol{v}_2| = 2$ for the head–on collision of ultra–relativistic beams. The quantities $n_i(\boldsymbol{r},t)$ are the particle densities of the beams.

The transformation from plane waves to colliding wave packets results in the following changes. The squared matrix element $|M|^2$ with the initial state in the form of plane waves with momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ transforms to the product of the matrix elements M_{fi} and $M^*_{fi'}$ with different initial states:

$$d\sigma \propto |M|^2 \quad \to \quad d\sigma(\boldsymbol{\kappa}) \;\propto\; M_{fi} M^*_{fi'}\,. \tag{22}$$

Here the initial state $|i\rangle$ is a direct product of plane waves with momenta $\boldsymbol{k}_1 = \boldsymbol{p}_1 + \frac{1}{2}\boldsymbol{\kappa}$ and $\boldsymbol{k}_2 = \boldsymbol{p}_2 - \frac{1}{2}\boldsymbol{\kappa}$, while the initial state $|i'\rangle$ is a direct product of plane waves with momenta $\boldsymbol{k}'_1 = \boldsymbol{p}_1 - \frac{1}{2}\boldsymbol{\kappa}$ and $\boldsymbol{k}'_2 = \boldsymbol{p}_2 + \frac{1}{2}\boldsymbol{\kappa}$. Instead of the luminosity L the number of events begins to depend on the quantity

$$L(\boldsymbol{\varrho}) = v \int n_1(\boldsymbol{r},t)\; n_2(\boldsymbol{r}+\boldsymbol{\varrho},t)\, d^3r dt \tag{23}$$

through the following formula

$$dN = \int \frac{d^3\kappa d^3\varrho}{(2\pi)^3}\, \mathrm{e}^{i\kappa\varrho}\, d\sigma(\boldsymbol{\kappa})\, L(\boldsymbol{\varrho}). \tag{24}$$

Due to the presence of the exponent in the above equation, typical values of κ are of the order of the inverse beam sizes, *i.e.*,

$$\kappa \sim \frac{1}{a}.$$

Usually this quantity is much smaller than the typical scale for the variation of the matrix element with respect to the initial momenta. In this case we can set $\kappa = 0$ in $d\sigma(\boldsymbol{\kappa})$ which results in the standard expression for the number of events (21). If this does not work well one should analyse complete formulas which take into account the effect of the finite beam sizes.

In view of the discussion given in the previous section, this is indeed the situation which occurs in our case. Now we want to show how the finite result for the number of events can be obtained starting from the complete formula (24).

Let us first define the "observable cross section" by the relation (compare with Eq. (8))

$$d\sigma = \frac{dN}{L} = d\sigma_{\text{stand}} + d\sigma_{\text{non-stand}}, \tag{25}$$

where L is the standard luminosity. Such a definition implies that the quantity sensitive to the BSE is $d\sigma_{\text{non-stand}}$.

The study of the matrix element of the discussed process (11) suggests that the only quantity sensitive to the small variation of the initial momenta is the denominator of the neutrino propagator for small values of q^2. Consequently, the transformations (22)–(24) reduce to the modification:

$$\frac{1}{|q^2|^2} \to \frac{1}{t+i\epsilon}\,\frac{1}{t'-i\epsilon}, \tag{26}$$

where

$$q^2 = (p_1 - p_3)^2, \quad t = (k_1 - p_3)^2, \quad t' = (k_1' - p_3)^2. \tag{27}$$

Let us expand t and t' up to the terms linear in $\boldsymbol{\kappa}$. This gives

$$t = q^2 - \lambda, \quad t' = q^2 + \lambda, \tag{28}$$

where

$$\lambda = \boldsymbol{\kappa Q}, \quad \boldsymbol{Q} = -\boldsymbol{p}_3 + \frac{E_3}{E_1}\boldsymbol{p}_1. \tag{29}$$

In the center of mass frame of the colliding muons

$$\boldsymbol{Q} \approx -\boldsymbol{p}_{3\perp} = \boldsymbol{q}_\perp = |\boldsymbol{q}_\perp|\,\boldsymbol{n}, \quad \boldsymbol{n} = (\cos\varphi, \sin\varphi, 0). \tag{30}$$

As a result, the quantity B in Eq. (15) transforms to

$$B = \int \frac{d^3\kappa d^3\varrho}{(2\pi)^3}\, \mathrm{e}^{i\kappa\varrho}\, \frac{L(\varrho)}{L} \int\limits_{-\Lambda}^{\Lambda} \frac{dq^2}{(q^2 - \lambda + i\epsilon)\,(q^2 + \lambda - i\epsilon)}. \tag{31}$$

It has already been mentioned that the above equation must be understood in the sense that we isolate the part which does not possess the limit $\kappa \to 0$ to the integral B. All other quantities are smooth in the above limit and hence do not present any problem.

To perform the q^2-integration in (31) we note that in the region (10)

$$\lambda = \boldsymbol{\kappa n}\, \sqrt{(q_\perp^0)^2 - (1 - x)q^2} \approx \lambda_0 = \boldsymbol{\kappa n}\; q_\perp^0. \tag{32}$$

This approximation is justified because

$$\frac{|q^2|}{m^2} \lesssim \frac{\Lambda}{m^2} \ll 1.$$

Now we extend the region of integration over q^2 up to $\pm\infty$. The error introduced by this procedure is of the order of $(m/a)/\Lambda \ll 1$ (cf. Eq. (17)). Now the integral over q^2 along the real axis can be replaced by the integral over a contour C which goes around the upper half plane of the complex variable q^2:

$$\int\limits_{-\infty}^{\infty} \frac{dq^2}{D} = \int\limits_{C} \frac{dq^2}{D} = \frac{\pi i}{-\lambda_0 + i\epsilon}. \tag{33}$$

Here

$$D = (q^2 - \lambda_0 + i\epsilon)\;(q^2 + \lambda_0 - i\epsilon).$$

Further integrations are simply performed with the help of the exponential representation

$$\frac{i}{-\lambda_0 + i\epsilon} = \int\limits_0^{\infty} \mathrm{e}^{i\alpha(-\lambda_0 + i\epsilon)}\, d\alpha. \tag{34}$$

Subsequent integrations over $\boldsymbol{\kappa}$ and $\boldsymbol{\varrho}$ become trivial and we obtain

$$B = \pi \int\limits_0^{\infty} \frac{L(\alpha\, \boldsymbol{n}\, q_\perp^0)}{L}\, d\alpha.$$

After redefinition $\alpha q_\perp^0 = \varrho$ we get for the factor B

$$B = \pi \, \frac{a}{q_\perp^0}, \quad a = \int\limits_0^\infty \frac{L(\varrho \boldsymbol{n})}{L} \, d\varrho, \quad \boldsymbol{n} = \frac{\boldsymbol{Q}}{|\boldsymbol{Q}|} \approx \frac{\boldsymbol{q}_\perp^0}{q_\perp^0}. \tag{35}$$

This completes the proof of Eq. (15).

At high energy colliders the distribution of particles inside the beams can be often considered as Gaussian. In this case $L(\varrho \boldsymbol{n})$ equals

$$L(\varrho \boldsymbol{n}) = L \exp\Big\{ - \varrho^2 \Big(\frac{\cos^2 \varphi}{2a_x^2} + \frac{\sin^2 \varphi}{2a_y^2} \Big) \Big\}, \quad \boldsymbol{n} = (\cos\varphi, \sin\varphi, 0)\,, \tag{36}$$

where

$$a_x^2 = \sigma_{1x}^2 + \sigma_{2x}^2, \quad a_y^2 = \sigma_{1y}^2 + \sigma_{2y}^2.$$

This results in the following expression for a:

$$a = \sqrt{\frac{\pi}{2}} \; \frac{a_x \; a_y}{\sqrt{a_y^2 \cos^2 \varphi + a_x^2 \sin^2 \varphi}}. \tag{37}$$

For circular (but not identical) beams with the root-mean-square radii

$$\sigma_{1x} = \sigma_{1y} = \sigma_{1\perp}, \quad \sigma_{2x} = \sigma_{2y} = \sigma_{2\perp}$$

we have

$$a = \sqrt{\frac{\pi}{2}} \, \sqrt{\sigma_{1\perp}^2 + \sigma_{2\perp}^2}. \tag{38}$$

It is interesting to note that the quantity a and the non–standard cross section are determined by the size of the largest beam[1]. For circular and identical beams with $\sigma_{1\perp} = \sigma_{2\perp} = \sigma_\perp$ the result Eq. (16) can be obtained.

4. The cross section for the reaction $\mu^- \mu^+ \to e \bar{\nu}_e W^+$

As an example consider $X = W^+$. A detailed discussion of this case has been presented in [2]. Here we present a short summary of the numerical results.

Using Eq. (20) we obtain:

$$\sigma_{\text{non-stand}}(\mu^- \mu^+ \to e \bar{\nu}_e W^+) = \frac{\pi a}{2 c \tau} \, \sigma_0 \; x_0 \; f(x_0), \quad x_0 = \frac{M^2}{s}, \tag{39}$$

[1] This feature can be readily understood in the frame of the qualitative picture described below.

$$\sigma_0 = \frac{12\pi^2}{M^2}\frac{\Gamma(W \to \mu\nu)}{M} = 20 \text{ nb},$$

where $\Gamma(W \to \mu\bar{\nu}_\mu) = 0.22$ GeV is the partial W decay width and $M = 80.2$ GeV is the W boson mass.

For numerical estimates we take $a = \sqrt{\pi}\sigma_\perp$ (which corresponds to the case of circular identical Gaussian beams) with (see Ref.[1])

$$\sigma_\perp = 10^{-3} \text{ cm}.$$

This non-standard cross section reaches a maximum of 0.76 fb at $\sqrt{s} =$ 93 GeV. For larger energies this cross section decreases as $s^{-3/2}$. It is interesting to note that the modest value of the non-standard cross section 0.76 fb is the result of the product of the very small number of neutrinos $N_\nu = 4.2 \cdot 10^{-8}$ (see Eq. (19) and the huge value of the cross section for the $\nu_\mu \mu^+ \to W^+$ transition averaged over the neutrino spectrum

$$\langle \sigma_{\nu\mu\to W}\rangle = \sigma_0\, x_0\, f(x_0) = 1.8 \cdot 10^7 \text{ fb}, \qquad x_0 = 0.74.$$

First, let us compare this non-standard piece with the standard contribution to the same cross section. Remember that by the "standard" contribution we mean the cross section of the same reaction calculated according to the standard rules excluding the region of the final phase space where $q^2 \;>\; -m^2$. This contribution was calculated [12] with the help of the CompHEP package [13]. The comparison of both contributions is shown in Fig.3. It is seen that the non-standard contribution dominates up to energies $\sqrt{s} \approx 105$ GeV.

Second, we compare our non-standard cross section with the cross section for single W boson production in the reaction $\mu^-\mu^+ \to \mu^-\bar{\nu}_\mu W^+$. The latter is a completely standard process since it has no t-channel singularity. A reasonable estimate of its cross section can be quickly obtained with the help of the equivalent photon approximation. It gives a cross section of ≈ 1 fb at $\sqrt{s} \approx 95$ GeV (where it almost coincides with our non-standard cross section). At higher energies the process $\mu^-\mu^+ \to \mu^-\bar{\nu}_\mu W^+$ dominates as compared with the process $\mu^-\mu^+ \to e\bar{\nu}_e W^+$.

5. Qualitative picture

In this section we develop a qualitative but precise picture of the phenomena discussed above. Our aim is to demonstrate the physics responsible for this effect.

To begin we note again, that the diagram shown in Fig.1 can be viewed as a sequence of two processes:

$$\mu^- \to e\bar{\nu}_e\nu_\mu \tag{40}$$

and

$$\nu_\mu \mu^+ \to W^+ \tag{41}$$

both of which can occur for *real* muonic neutrino. Therefore, there exists a region of the final phase space corresponding to the reaction (1), where the denominator of the propagator of the muonic neutrino q^2 can be equal to zero – in other words the *virtual* neutrino can go *on its mass shell.*

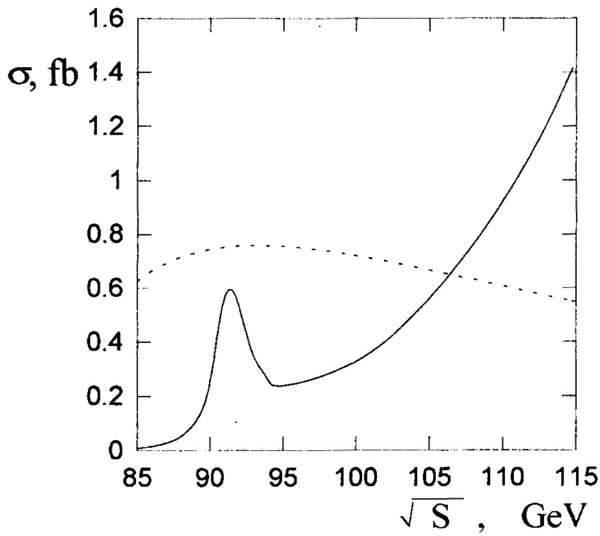

Fig. 3. Standard (solid line) and non–standard (dashed line) contributions to the cross sections (fb) of the reaction $\mu^-\mu^+ \to e\bar{\nu}_e W^+$ in dependence on the total cms energy. The standard contribution is evaluated with the cut $-q^2 > m^2$.

In what follows, we will show that the non–standard piece of the cross section $d\sigma_{\text{non-stand}}$ (see Eq. (20)) is *completely* determined by the sequence of processes (40), (41) for real muonic neutrinos. Such interpretation provides a simple and transparent derivation of the main results of the previous sections, gives a clear physical interpretation, and leads to a clearer understanding of the region of applicability of the result.

In view of the realistic situation at future muon colliders, we assume that the longitudinal l and the transverse a sizes of the colliding beams satisfy two inequalities:

$$l \ll \gamma c \tau, \qquad \gamma = \frac{E}{mc^2}, \tag{42}$$

$$a \ll c \tau, \tag{43}$$

where τ is the muon life time in its rest frame, $c\tau = 660$ m. Because of the condition (42), the number of muons can be assumed to be a constant during the time which is required for the colliding beams to cross each other. Note, that conditions (42)–(43) are perfectly fulfilled for the projects of muon colliders which are discussed in the literature [1]. Using the beam parameters from Ref. [1] and the total energy $2E = 100$ GeV one gets:

$$\frac{a}{c\tau} \sim \frac{l}{\gamma c\tau} \sim 10^{-8}\,. \tag{44}$$

To get a qualitative understanding of the phenomena, it is convenient to consider a collision of a single muon μ^- with the beam of muons of the positive charge μ^+ in the rest frame of the μ^-. Being unstable, this muon is surrounded by a "cloud" of ν_μ's which appear in the μ^- decay (certainly, there are also $\bar{\nu}_e$ and e "clouds", but they are not interesting for us at the moment). The density of neutrinos in this cloud decreases as $1/r^2$ with the growth of the distance r from the μ^-, its time dependence corresponds to the exponential decay law, and the angular distribution is isotropic. Therefore, one finds

$$n_\nu(\boldsymbol{r}, t) = \frac{\theta(t - r/c)}{4\pi c\tau r^2} \exp\left(-\frac{t - r/c}{\tau}\right). \tag{45}$$

This density is normalized by the condition

$$\int_0^\infty c\, n_\nu(\boldsymbol{r}, t)\, 4\pi r^2\, dt = 1$$

which means that the total number of ν_μ's which cross a sphere of radius r is equal to unity.

We shall see below that the main contribution to the $\sigma_{\text{non-stand}}$ comes from the distance $r \sim a$. The typical time of collision is therefore $\Delta t \sim a/c$. If the collision occurs at the time t which satisfies the inequality $\Delta t \ll t \ll \tau$, the neutrino density can be taken to be time independent in the collision region (these assumptions are justified by the obtained result):

$$n_\nu(\boldsymbol{r}, t) = \frac{1}{4\pi c\tau r^2}. \tag{46}$$

The distribution of the neutrinos over impact parameters ϱ of the $\mu^+\mu^-$ collisions[2] is given by

$$dN_\nu = \frac{d^2\varrho}{4\pi c\tau} \int_{-\infty}^{+\infty} \frac{dz}{z^2 + \varrho^2} = \frac{1}{4c\tau}\frac{d^2\varrho}{\varrho}, \qquad \boldsymbol{r} = (\boldsymbol{\varrho}, z)\,. \tag{47}$$

[2] The z-axis is antiparallel to the momentum $\boldsymbol{p}_2$ of the μ^+ .

We note that the main contribution to this integral is given by the longitudinal distances of the order of

$$|z| \lesssim \varrho .$$

It follows from the above equation that the scale for the distribution over impact parameters is set by the quantity $c\tau$ which is *macroscopically* large: $c\tau = 660$ m. In contrast, the incident μ^+ beam has much more moderate sizes: for the projects of the muon collider the transverse size of the beam is of the order of $a \sim 10^{-3}$ cm. Hence, it is clear that a larger part of the neutrinos is outside the range of the incident beam and can not collide with μ^+'s.

The number of muonic neutrinos which collide with the μ^+ beam of the radius a is therefore

$$N_\nu \sim \int\limits_0^a \frac{d^2\varrho}{4c\tau\varrho} = \frac{\pi}{2}\frac{a}{c\tau}. \tag{48}$$

The non–standard piece of the cross section of the process (1) is proportional to the number of neutrinos (48) and can be estimated as

$$\sigma_{\text{non-stand}} \sim \frac{a}{c\tau} \left\langle \sigma_{\nu\mu\to W} \right\rangle , \tag{49}$$

where $\langle \sigma_{\nu\mu\to W} \rangle$ corresponds to the cross section of the process (41) averaged over the ν_μ energy spectrum (see Eq. (18)). The estimate (49) corresponds to the $\sigma_{\text{non-stand}}$ obtained in (20).

Having performed the estimate, we can proceed further with the exact calculation. We base the calculation on the same idea as the estimate presented in the previous section. Namely, let us consider a collision of the muonic neutrino with the incident μ^+ beam. The source of these neutrinos is the decaying μ^-. Therefore we assume that the distribution of these neutrinos can be determined by analyzing the exponential decay law of the muon. After the distribution is known, we apply the standard equation for the number of events: cross section multiplied by luminosity. In our case, however, the number of produced $W's$ is defined by the cross section of the reaction $\nu_\mu\mu^+ \to W^+$ and the luminosity of the $\nu_\mu\mu^+$ collision. It is this luminosity which is sensitive to the details of the neutrino distribution in space and time.

It turns out [4] that the results derived in this way are completely equivalent to the beam–size dependent part of the cross section (20) presented in the previous part of this paper. From this comparison one concludes that the qualitative picture presented above is *exact* and the discussed effect corresponds to the scattering of the beam of positively charged muons on the halo of real neutrinos which are produced in the course of μ^- decay.

6. Conclusions

In this paper we have developed a theory of the processes with the t–channel singularity in the physical region and discussed a physical meaning of the linear beam size effect at muon colliders. Our analysis clearly shows that this singularity is fictitious and corresponds to the appearance of a real particle in an intermediate state. When standard formulas of the scattering theory are used for the calculation, one finds infinite cross sections in these cases due to the fact that both the "life time" of such intermediate state is infinite and the flux of the incident particles (determined by the plane waves) is infinite too.

Though the first of these "infinities" is a physical one, the second actually is not. The reason is that in the realistic situation which takes place at high energy colliders the colliding particles are confined to beams of macroscopical but *finite* sizes.

To describe this situation we have considered the collision of two wave packets which correspond to the incident muon beams and obtained the finite cross section proportional to the transverse size of the beam. Based on this exact consideration, a simple physical picture of the effect has been developed.

Let us describe this picture in the center of mass frame of the colliding beams. In this reference frame the decay products of the muon escape into a small angle $\vartheta \sim 1/\gamma$ around muon direction of motion. Due to this property, we can take into account only those decays of the muon which occur during the flight through the last straight part of the muon's trajectory before it reaches the interaction point. Let $L_s \geq 1$ m be the length of this part. Around each muon there appears a disk of the neutrinos. The radius of this disk is $\sim L_s\vartheta \sim L_s\gamma^{-1}$. The number of neutrinos in the disk is $\sim L_s/(c\gamma\tau) \ll 1$. The disks from individual muons form a "cloud" of neutrinos which follow the muon beam. An opposite beam, having a transverse radius a, cuts a cylinder of the same radius in the neutrino cloud. Only neutrinos inside this cylinder participate in the collision. The cylinder of the radius a is filled with the neutrinos which are produced at the distance

$$l_f \sim \frac{a}{\vartheta} \sim \gamma\, a, \tag{50}$$

which one can consider as a formation length of neutrino cloud.

From this one obtains the number of neutrinos ν_μ which participate in the collision. It can be estimated as

$$N_\mu\, \frac{l_f}{c\gamma\tau} \sim N_\mu\, \frac{a}{c\tau} \sim 10^4 \tag{51}$$

for $N_\mu \sim 10^{12}$. Therefore, at the formation length the number of muons does not change. Hence, the neutrino density inside muon beam is almost a constant during the collision time. Equation (51) also shows the level of statistical fluctuations which one expects for the $\nu_\mu \mu^+$ collisions.

It is instructive to compare a contribution of real and virtual ν_μ's to the total cross section. First, we remind the reader, that because of the fixing of the region of the final phase space, the cross section $\sigma_{\rm stand}$ (see Eq. (8) and the discussion after it) is completely determined by virtual ν_μ's with $q^2 < -m^2c^2$. In contrast, the contribution of real ν_μ's in the intermediate state absolutely dominates in $\sigma_{\rm non-stand}$ — the relative contribution of virtual neutrinos to this piece of the cross section can be estimated as $\sim \hbar/(a\sqrt{t_0}) \sim \hbar/(amc) \sim 10^{-10}$. Let us stress, however, that the approach described in the first part of this paper allows us to calculate the contribution of virtual neutrinos to $\sigma_{\rm non-stand}$ as well as the contribution which appears due to the interference of real and virtual neutrinos.

In summary, comparing the results of the calculation of the number of events in $\nu_\mu \mu^+$ collisions with the exact calculation based on the notion of the colliding wave packets, we conclude that the above qualitative picture is precise. Therefore, *the linear beam-size effect corresponds to the scattering of the* μ^+ *beam on the "cloud" of real neutrinos* ν_μ *produced in the* $\mu^- \to e\bar{\nu}_e\nu_\mu$ *decay.*

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