

UNIFICATION BEYOND GUTs:
GAUGE-YUKAWA UNIFICATION*

J. KUBO

College of Liberal Arts, Kanazawa University,
920-11 Kanazawa, Japan

M. MONDRAGON

Instituto de Física, UNAM
Apdo. Postal 20-364, México 01000 D.F. México

AND

G. ZOUPANOS

Physics Department, Nat. Technical University,
157 80 Zografou, Athens, Greece*(Received November 18, 1996)*

Gauge-Yukawa Unification (GYU) is a renormalization group invariant functional relation among gauge and Yukawa couplings which holds beyond the unification point in Grand Unified Theories (GUTs). We present here various models where GYU is obtained by requiring the principles of finiteness and reduction of couplings. We examine the consequences of these requirements for the low energy parameters, especially for the top quark mass. The predictions are such that they clearly distinguish already GYU from ordinary GUTs. It is expected that it will be possible to discriminate among the various GYUs when more accurate measurements of the top quark mass are available.

PACS numbers: 12.60. -i, 12.60. Jv, 11.10. Hi

* Based on lectures given by G. Zoupanos in the XXXIV and XXXVI Cracow Schools of Theoretical Physics, Zakopane, Poland 1993, 1996 and in the 2nd Bruno Pontecorvo School on Elementary Particle Physics, Capri 1996. Partially supported by the E.C. projects CHRX-CT93-0319, and ERBFMRXCT960090, the Greek project PENED/1170, and the Conacyt project IN110296.

1. Introduction

The standard model (SM) is very accurate in describing the elementary particles and their interactions, but it has a large number of free parameters whose values are determined only experimentally.

To reduce the number of free parameters of a theory, and thus render it more predictive, one is usually led to introduce a symmetry. Grand Unified Theories (GUTs) are very good examples of such a procedure [1–3]. For instance, in the case of minimal $SU(5)$ it was possible to reduce the gauge couplings by one and give a prediction for one of them. GUTs can also relate the Yukawa couplings among themselves, again $SU(5)$ provided an example of this by predicting the ratio M_τ/M_b [4] in SM. Unfortunately, requiring more gauge symmetry does not seem to help, since additional complications are introduced due to new degrees of freedom, in the ways and channels of breaking the symmetry, etc.

A natural extension of the GUT idea is to find a way to relate the gauge and Yukawa sectors of a theory, that is to achieve Gauge-Yukawa Unification (GYU). A symmetry which naturally relates the two sectors is supersymmetry, in particular $N = 2$ supersymmetry. It turns out, however, that $N = 2$ supersymmetric theories have serious phenomenological problems due to light mirror fermions. Also in superstring theories and in composite models there exist relations among the gauge and Yukawa couplings, but both kind of theories have phenomenological problems.

There have been other attempts to relate the gauge and Yukawa sectors. One was proposed by Decker, Pestieau, and Veltman [6]. By requiring the absence of quadratic divergences in the SM, they found a relationship between the squared masses appearing in the Yukawa and in the gauge sectors of the theory. A very similar relation is obtained by applying naively in the SM the general formula derived from demanding spontaneous supersymmetry breaking via F-terms [7]. In both cases a prediction for the top quark was possible only when it was permitted experimentally to neglect the M_H as compared to $M_{W,Z}$ with the result $M_t = 69$ GeV. Otherwise there is only a quadratic relation among M_t and M_H .

A well known relation among gauge and Yukawa couplings is the Pendleton-Ross (P-R) *infrared* fixed point [8]. The P-R proposal, involving the Yukawa coupling of the top quark g_t and the strong gauge coupling α_3 , was that the ratio α_t/α_3 , where $\alpha_t = g_t^2/4\pi$, has an infrared fixed point. This assumption predicted $M_t \sim 100$ GeV. In addition, it has been shown [9] that the P-R conjecture is not justified at two-loops, since then the ratio α_t/α_3 diverges in the infrared.

Another interesting conjecture, made by Hill [10], is that α_t itself develops a quasi-infrared fixed point, leading to the prediction $M_t \sim 280$ GeV.

The P-R and Hill conjectures have been done in the framework on the SM. The same conjectures within the minimal supersymmetric SM (MSSM) lead to the following relations:

$$M_t \simeq 140 \text{ GeV} \sin \beta \quad (\text{P-R}), \quad (1)$$

$$M_t \simeq 200 \text{ GeV} \sin \beta \quad (\text{Hill}), \quad (2)$$

where $\tan \beta = v_u/v_d$ is the ratio of the two VEV of the Higgs fields of the MSSM. We should stress that in this case there is no prediction for M_t , given that $\sin \beta$ is not fixed from other considerations.

In a series of papers [11–14, 63] we have proposed another way to relate the gauge and Yukawa sectors of a theory. It is based on the fact that within the framework of a renormalizable field theory, one can find renormalization group invariant (RGI) relations among parameters that can improve the calculability and the predictive power of a theory. We have considered models in which the GYU is achieved using the principles of reduction of couplings [17–21] and finiteness [11, 22–27, 33–36, 61]. These principles, which are formulated in perturbation theory, are not explicit symmetry principles, although they might imply symmetries. The former principle is based on the existence of RGI relations among couplings, which preserve perturbative renormalizability. Similarly, the latter one is based on the fact that it is possible to find RGI relations among couplings that keep finiteness in perturbation theory, even to all orders. Applying these principles one can relate the gauge and Yukawa couplings without introducing necessarily a symmetry, nevertheless improving the predictive power of a model.

It is worth noting that the above principles have been applied in supersymmetric GUTs for reasons that will be transparent in the following sections. We should also stress that our conjecture for GYU is by no means in conflict with the interesting proposals mentioned before (see also Ref. [60]), but it rather uses all of them, hopefully in a more successful perspective. For instance, the use of susy GUTs comprises the demand of the cancellation of quadratic divergences in the SM. Similarly, the very interesting conjectures about the infrared fixed points are generalized in our proposal, since searching for RGI relations among various couplings corresponds to searching for *fixed points* of the coupled differential equations obeyed by the various couplings of a theory.

2. Unification of couplings by the RGI method

Let us next briefly outline the idea of reduction of couplings. Any RGI relation among couplings (which does not depend on the renormalization scale μ explicitly) can be expressed, in the implicit form $\Phi(g_1, \dots, g_A) =$

const., which has to satisfy the partial differential equation (PDE)

$$\mu \frac{d\Phi}{d\mu} = \vec{\nabla} \cdot \vec{\beta} = \sum_{a=1}^A \beta_a \frac{\partial \Phi}{\partial g_a} = 0, \quad (3)$$

where β_a is the β -function of g_a . This PDE is equivalent to a set of ordinary differential equations, the so-called reduction equations (REs) [18],

$$\beta_g \frac{dg_a}{dg} = \beta_a, \quad a = 1, \dots, A, \quad (4)$$

where g and β_g are the primary coupling and its β -function, and the counting on a does not include g . Since maximally $(A - 1)$ independent RGI “constraints” in the A -dimensional space of couplings can be imposed by the Φ_a ’s, one could in principle express all the couplings in terms of a single coupling g . The strongest requirement is to demand power series solutions to the REs,

$$g_a = \sum_{n=0} \rho_a^{(n)} g^{2n+1}, \quad (5)$$

which formally preserve perturbative renormalizability. Remarkably, the uniqueness of such power series solutions can be decided already at the one-loop level [18]. To illustrate this, let us assume that the β -functions have the form

$$\begin{aligned} \beta_a &= \frac{1}{16\pi^2} \left[\sum_{b,c,d \neq g} \beta_a^{(1)bcd} g_b g_c g_d + \sum_{b \neq g} \beta_a^{(1)b} g_b g^2 \right] + \dots, \\ \beta_g &= \frac{1}{16\pi^2} \beta_g^{(1)} g^3 + \dots, \end{aligned} \quad (6)$$

where \dots stands for higher order terms, and $\beta_a^{(1)bcd}$ ’s are symmetric in b, c, d . We then assume that the $\rho_a^{(n)}$ ’s with $n \leq r$ have been uniquely determined. To obtain $\rho_a^{(r+1)}$ ’s, we insert the power series (5) into the REs (4) and collect terms of $O(g^{2r+3})$ and find

$$\sum_{d \neq g} M(r)_a^d \rho_d^{(r+1)} = \text{lower order quantities},$$

where the r.h.s. is known by assumption, and

$$M(r)_a^d = 3 \sum_{b,c \neq g} \beta_a^{(1)bcd} \rho_b^{(1)} \rho_c^{(1)} + \beta_a^{(1)d} - (2r+1) \beta_g^{(1)} \delta_a^d, \quad (7)$$

$$0 = \sum_{b,c,d \neq g} \beta_a^{(1)bcd} \rho_b^{(1)} \rho_c^{(1)} \rho_d^{(1)} + \sum_{d \neq g} \beta_a^{(1)d} \rho_d^{(1)} - \beta_g^{(1)} \rho_a^{(1)}. \quad (8)$$

Therefore, the $\rho_a^{(n)}$'s for all $n > 1$ for a given set of $\rho_a^{(1)}$'s can be uniquely determined if $\det M(n)_a^d \neq 0$ for all $n \geq 0$.

As it will be clear later by examining specific examples, the various couplings in supersymmetric theories have easily the same asymptotic behaviour. Therefore searching for a power series solution of the form (5) to the REs (4) is justified. This is not the case in non-supersymmetric theories.

The possibility of coupling unification described in this section is without any doubt attractive because the "completely reduced" theory contains only one independent coupling, but it can be unrealistic. Therefore, one often would like to impose fewer RGI constraints, and this is the idea of partial reduction [19].

3. Partial reduction in $N = 1$ supersymmetric gauge theories

Let us consider a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with gauge coupling constant g . The superpotential of the theory is given by

$$W = \frac{1}{2} m_{ij} \phi_i \phi_j + \frac{1}{6} C_{ijk} \phi_i \phi_j \phi_k, \quad (9)$$

where m_{ij} and C_{ijk} are gauge invariant tensors and the matter field ϕ_i transforms according to the irreducible representation R_i of the gauge group G . The renormalization constants associated with the superpotential (9), assuming that supersymmetry is preserved, are

$$\phi_i^0 = (Z_i^j)^{(1/2)} \phi_j, \quad (10)$$

$$m_{ij}^0 = Z_{ij}^{i'j'} m_{i'j'}, \quad (11)$$

$$C_{ijk}^0 = Z_{ijk}^{i'j'k'} C_{i'j'k'}. \quad (12)$$

The $N = 1$ non-renormalization theorem [32] ensures that there are no mass and cubic-interaction-term infinities and therefore

$$\begin{aligned} Z_{ijk}^{i'j'k'} Z_{i'j'}^{1/2 i''} Z_{j'}^{1/2 j''} Z_{k'}^{1/2 k''} &= \delta_{(i}^{i''} \delta_{j}^{j''} \delta_{k)}^{k''}, \\ Z_{ij}^{i'j'} Z_{i'j'}^{1/2 i''} Z_{j'}^{1/2 j''} &= \delta_{(i}^{i''} \delta_{j)}^{j''}. \end{aligned} \quad (13)$$

As a result the only surviving possible infinities are the wave-function renormalization constants Z_i^j , i.e., one infinity for each field. The one-loop β -function of the gauge coupling g is given by [22]

$$\beta_g^{(1)} = \frac{dg}{dt} = \frac{g^3}{16\pi^2} \left[\sum_i l(R_i) - 3C_2(G) \right], \quad (14)$$

where $l(R_i)$ is the Dynkin index of R_i and $C_2(G)$ is the quadratic Casimir of the adjoint representation of the gauge group G . The β -functions of C_{ijk} , by virtue of the non-renormalization theorem, are related to the anomalous dimension matrix γ_{ij} of the matter fields ϕ_i as:

$$\beta_{ijk} = \frac{dC_{ijk}}{dt} = C_{ijl} \gamma_k^l + C_{ikl} \gamma_j^l + C_{jkl} \gamma_i^l. \quad (15)$$

At one-loop level γ_{ij} is [22]

$$\gamma_{ij}^{(1)} = \frac{1}{32\pi^2} \left[C^{ikl} C_{jkl} - 2g^2 C_2(R_i) \delta_{ij} \right], \quad (16)$$

where $C_2(R_i)$ is the quadratic Casimir of the representation R_i , and $C^{ijk} = C_{ijk}^*$. Since dimensional coupling parameters such as masses and couplings of cubic scalar field terms do not influence the asymptotic properties of a theory on which we are interested here, it is sufficient to take into account only the dimensionless supersymmetric couplings such as g and C_{ijk} . So we neglect the existence of dimensional parameters, and assume furthermore that C_{ijk} are real so that C_{ijk}^2 always are positive numbers. For our purposes, it is convenient to work with the square of the couplings and to arrange C_{ijk} in such a way that they are covered by a single index i ($i = 1, \dots, n$):

$$\alpha = \frac{|g|^2}{4\pi}, \quad \alpha_i = \frac{|g_i|^2}{4\pi}. \quad (17)$$

The evolution equations of α 's in perturbation theory then take the form

$$\begin{aligned} \frac{d\alpha}{dt} &= \beta = -\beta^{(1)} \alpha^2 + \dots, \\ \frac{d\alpha_i}{dt} &= \beta_i = -\beta_i^{(1)} \alpha_i \alpha + \sum_{j,k} \beta_{i,jk}^{(1)} \alpha_j \alpha_k + \dots, \end{aligned} \quad (18)$$

where \dots denotes the contributions from higher orders, and $\beta_{i,jk}^{(1)} = \beta_{i,kj}^{(1)}$.

Given the set of the evolution equations (18), we investigate the asymptotic properties, as follows. First we define [17, 18]

$$\tilde{\alpha}_i \equiv \frac{\alpha_i}{\alpha}, \quad i = 1, \dots, n, \quad (19)$$

and derive from Eq. (18)

$$\begin{aligned} \alpha \frac{d\tilde{\alpha}_i}{d\alpha} &= -\tilde{\alpha}_i + \frac{\beta_i}{\beta} = \left(-1 + \frac{\beta_i^{(1)}}{\beta^{(1)}} \right) \tilde{\alpha}_i \\ &\quad - \sum_{j,k} \frac{\beta_{i,jk}^{(1)}}{\beta^{(1)}} \tilde{\alpha}_j \tilde{\alpha}_k + \sum_{r=2} \left(\frac{\alpha}{\pi} \right)^{r-1} \tilde{\beta}_i^{(r)}(\tilde{\alpha}), \end{aligned} \quad (20)$$

where $\tilde{\beta}_i^{(r)}(\tilde{\alpha})$ ($r = 2, \dots$) are power series of $\tilde{\alpha}$'s and can be computed from the r -th loop β -functions. Next we search for fixed points ρ_i of Eq. (19) at $\alpha = 0$. To this end, we have to solve

$$\left(-1 + \frac{\beta_i^{(1)}}{\beta^{(1)}}\right) \rho_i - \sum_{j,k} \frac{\beta_{i,jk}^{(1)}}{\beta^{(1)}} \rho_j \rho_k = 0, \quad (21)$$

and assume that the fixed points have the form

$$\rho_i = 0 \text{ for } i = 1, \dots, n'; \quad \rho_i > 0 \text{ for } i = n' + 1, \dots, n. \quad (22)$$

We then regard $\tilde{\alpha}_i$ with $i \leq n'$ as small perturbations to the undisturbed system which is defined by setting $\tilde{\alpha}_i$ with $i \leq n'$ equal to zero. As we have seen, it is possible to verify at the one-loop level [18] the existence of the unique power series solution

$$\tilde{\alpha}_i = \rho_i + \sum_{r=2} \rho_i^{(r)} \alpha^{r-1}, \quad i = n' + 1, \dots, n \quad (23)$$

of the reduction equations (20) to all orders in the undisturbed system. These are RGI relations among couplings and keep formally perturbative renormalizability of the undisturbed system. So in the undisturbed system there is only *one independent* coupling, the primary coupling α .

The small perturbations caused by nonvanishing $\tilde{\alpha}_i$ with $i \leq n'$ enter in such a way that the reduced couplings, *i.e.*, $\tilde{\alpha}_i$ with $i > n'$, become functions not only of α but also of $\tilde{\alpha}_i$ with $i \leq n'$. It turned out that, to investigate such partially reduced systems, it is most convenient to work with the partial differential equations

$$\left\{ \tilde{\beta} \frac{\partial}{\partial \alpha} + \sum_{a=1}^{n'} \tilde{\beta}_a \frac{\partial}{\partial \tilde{\alpha}_a} \right\} \tilde{\alpha}_i(\alpha, \tilde{\alpha}) = \tilde{\beta}_i(\alpha, \tilde{\alpha}),$$

$$\tilde{\beta}_{i(a)} = \frac{\beta_{i(a)}}{\alpha^2} - \frac{\beta}{\alpha^2} \tilde{\alpha}_{i(a)}, \quad \tilde{\beta} \equiv \frac{\beta}{\alpha}, \quad (24)$$

which are equivalent to the reduction equations (20), where we let a, b run from 1 to n' and i, j from $n' + 1$ to n in order to avoid confusion. We then look for solutions of the form

$$\tilde{\alpha}_i = \rho_i + \sum_{r=2} \left(\frac{\alpha}{\pi}\right)^{r-1} f_i^{(r)}(\tilde{\alpha}_a), \quad i = n' + 1, \dots, n, \quad (25)$$

where $f_i^{(r)}(\tilde{\alpha}_a)$ are supposed to be power series of $\tilde{\alpha}_a$. This particular type of solution can be motivated by requiring that in the limit of vanishing perturbations we obtain the undisturbed solutions (23) [21, 28]. Again it is possible to obtain the sufficient conditions for the uniqueness of $f_i^{(r)}$ in terms of the lowest order coefficients.

4. The minimal asymptotically free SU(5) model

The minimal N=1 supersymmetric SU(5) model [29] is particularly interesting, being the the simplest GUT supported by the LEP data [5]. Here we will consider it as an attractive example of a partially reduced model. Its particle content is well defined and has the following transformation properties under SU(5): three ($\bar{\mathbf{5}} + \mathbf{10}$)-supermultiplets which accommodate three fermion families, one ($\mathbf{5} + \bar{\mathbf{5}}$) to describe the two Higgs supermultiplets appropriate for electroweak symmetry breaking and a $\mathbf{24}$ -supermultiplet required to provide the spontaneous symmetry breaking of SU(5) down to SU(3)×SU(2)×U(1).

Since we are neglecting the dimensional parameters and the Yukawa couplings of the first two generations, the superpotential of the model is exactly given by

$$W = \frac{1}{2} g_t \mathbf{10}_3 \mathbf{10}_3 H + g_b \bar{\mathbf{5}}_3 \mathbf{10}_3 \bar{H} + g_\lambda (\mathbf{24})^3 + g_f \bar{H} \mathbf{24} H, \quad (26)$$

where H, \bar{H} are the $\mathbf{5}, \bar{\mathbf{5}}$ -Higgs supermultiplets and we have suppressed the SU(5) indices. According to the notation introduced in Eq. (19), Eqs. (20) become

$$\begin{aligned} \alpha \frac{d\tilde{\alpha}_t}{d\alpha} &= \frac{27}{5} \tilde{\alpha}_t - 3 \tilde{\alpha}_t^2 - \frac{4}{3} \tilde{\alpha}_t \tilde{\alpha}_b - \frac{8}{5} \tilde{\alpha}_t \tilde{\alpha}_f, \\ \alpha \frac{d\tilde{\alpha}_b}{d\alpha} &= \frac{23}{5} \tilde{\alpha}_b - \frac{10}{3} \tilde{\alpha}_b^2 - \tilde{\alpha}_b \tilde{\alpha}_t - \frac{8}{5} \tilde{\alpha}_b \tilde{\alpha}_f, \\ \alpha \frac{d\tilde{\alpha}_\lambda}{d\alpha} &= 9 \tilde{\alpha}_\lambda - \frac{21}{5} \tilde{\alpha}_\lambda^2 - \tilde{\alpha}_\lambda \tilde{\alpha}_f, \\ \alpha \frac{d\tilde{\alpha}_f}{d\alpha} &= \frac{83}{15} \tilde{\alpha}_f - \frac{53}{15} \tilde{\alpha}_f^2 - \tilde{\alpha}_f \tilde{\alpha}_t - \frac{4}{3} \tilde{\alpha}_f \tilde{\alpha}_b - \frac{7}{5} \tilde{\alpha}_f \tilde{\alpha}_\lambda \end{aligned} \quad (27)$$

in the one-loop approximation. Given the above equations describing the evolution of the four independent couplings (α_i , $i = t, b, \lambda, f$), there exist $2^4 = 16$ non-degenerate solutions corresponding to vanishing ρ 's as well as non-vanishing ones given by Eq. (25). The possibility to predict the top quark mass depends on a nontrivial interplay between the vacuum expectation value of the two SU(2) Higgs doublets involved in the model and the known masses of the third generation (m_b , m_τ). It is clear that only the solutions of the form

$$\rho_t, \rho_b \neq 0 \quad (28)$$

can predict the top and bottom quark masses.

There exist exactly four such solutions. The first solution is ruled out since it is inconsistent with Eq. (17), and the second one is ruled out since it does not satisfy the criteria to be asymptotically free. We are left with

two asymptotically free solutions, which we label 3 and 4 (or AFUT3 and AFUT4, for asymptotically free unified theory). According to the criteria of section 3, these two solutions give the possibility to obtain partial reductions. To achieve this, we look for solutions [12] of the form Eq. (23) to both 3 and 4.

We present now the computation of some lower order terms within the one-loop approximation for the solutions. For solution 3:

$$\tilde{\alpha}_i = \eta_i + f_i^{(r_\lambda=1)} \tilde{\alpha}_\lambda + f_i^{(r_\lambda=2)} \tilde{\alpha}_\lambda^2 + \dots \quad \text{for } i = t, b, f, \quad (29)$$

where

$$\begin{aligned} \eta_{t,b,f} &= \frac{2533}{2605}, \frac{1491}{2605}, \frac{560}{521}, \\ f_{t,b,f}^{(r_\lambda=1)} &\simeq 0.018, 0.012, -0.131, \\ f_{t,b,f}^{(r_\lambda=2)} &\simeq 0.005, 0.004, -0.021. \end{aligned} \quad (30)$$

For the solution 4,

$$\begin{aligned} \tilde{\alpha}_i &= \eta_i + f_i^{(r_f=1)} \tilde{\alpha}_f + f_i^{(r_\lambda=1)} \tilde{\alpha}_\lambda + f_i^{(r_f=1, r_\lambda=1)} \tilde{\alpha}_f \tilde{\alpha}_\lambda \\ &+ f_i^{(r_f=2)} \tilde{\alpha}_f^2 + f_i^{(r_\lambda=2)} \tilde{\alpha}_\lambda^2 \dots \quad \text{for } i = t, b, \end{aligned} \quad (31)$$

where

$$\begin{aligned} \eta_{t,b} &= \frac{89}{65}, \frac{63}{65}, \quad f_i^{(r_\lambda=1)} = f_i^{(r_\lambda=2)} = 0, \\ f_{t,b}^{(r_f=1)} &\simeq -0.258, -0.213, \quad f_{t,b}^{(r_f=1)} \simeq -0.258, -0.213, \\ f_{t,b}^{(r_f=2)} &\simeq -0.055, -0.050, \quad f_{t,b}^{(r_f=1, r_\lambda=1)} \simeq -0.021, -0.018. \end{aligned} \quad (32)$$

In the solutions (29) and (31) we have suppressed the contributions from the Yukawa couplings of the first two generations because they are negligibly small.

Presumably, both solutions are related; a numerical analysis on the solutions [12] suggests that the solution 3 is a “boundary” of 4. If it is really so, then there is only one unique reduction solution in the minimal supersymmetric GUT that provides us with the possibility of predicting α_t . Note furthermore that not only α_t but also α_b is predicted in this reduction solution.

Just below the unification scale we would like to obtain the MSSM $SU(3) \times SU(2) \times U(1)$ and one pair of Higgs doublets, and assume that all the superpartners are degenerate at the supersymmetry breaking scale, where

the MSSM will be broken to the normal SM. Then the standard model should be spontaneously broken down to $SU(3) \times U(1)_{em}$ due to VEV of the two Higgs $SU(2)$ -doublets contained in the $\mathbf{5}, \bar{\mathbf{5}}$ -super-multiplets.

One way to obtain the correct low energy theory is to add to the Lagrangian soft supersymmetry breaking terms and to arrange the mass parameters in the superpotential along with the soft breaking terms so that the desired symmetry breaking pattern of the original $SU(5)$ is really the preferred one, all the superpartners are unobservable at present energies, there is no contradiction with proton decay, and so forth. Then we study the evolution of the couplings at two loops respecting all the boundary conditions at M_{GUT} .

5. Finiteness in $N = 1$ SUSY gauge theories

According to the discussion in Chapter 3, the non-renormalization theorem ensures there are no extra mass and cubic-interaction-term renormalizations, implying that the β -functions of C_{ijk} can be expressed as linear combinations of the anomalous dimensions γ_{ij} of ϕ^i . Therefore, all the one-loop β -functions of the theory vanish if $\beta_g^{(1)}$ and $\gamma_{ij}^{(1)}$, given in Eqs. (14) and (16) respectively, vanish, *i.e.*

$$\sum_i \ell(R_i) = 3C_2(G), \quad (33)$$

$$C^{ikl}C_{jkl} = 2\delta_j^i g^2 C_2(R_i). \quad (34)$$

A very interesting result is that the conditions (33,34) are necessary and sufficient for finiteness at the two-loop level [22].

In case supersymmetry is broken by soft terms, one-loop finiteness of the soft sector imposes further constraints on it [24]. In addition, the same set of conditions that are sufficient for one-loop finiteness of the soft breaking terms render the soft sector of the theory two-loop finite [25].

The one- and two-loop finiteness conditions (33), (34) restrict considerably the possible choices of the irreps. R_i for a given group G as well as the Yukawa couplings in the superpotential (9). Note in particular that the finiteness conditions cannot be applied to the supersymmetric standard model (SSM), since the presence of a $U(1)$ gauge group is incompatible with the condition (33), due to $C_2[U(1)] = 0$. This naturally leads to the expectation that finiteness should be attained at the grand unified level only, the SSM being just the corresponding, low-energy, effective theory.

Another important consequence of one- and two-loop finiteness is that supersymmetry (most probably) can only be broken by soft breaking terms.

Indeed, due to the unacceptability of gauge singlets, F-type spontaneous symmetry breaking [30] terms are incompatible with finiteness, as well as D-type [31] spontaneous breaking which requires the existence of a U(1) gauge group.

A natural question to ask is what happens at higher loop orders. The answer is contained in a theorem [33] which states the necessary and sufficient conditions to achieve finiteness at all orders. Before we discuss the theorem let us make some introductory remarks. The finiteness conditions impose relations between gauge and Yukawa couplings. To require such relations which render the couplings mutually dependent at a given renormalization point is trivial. What is not trivial is to guarantee that relations leading to a reduction of the couplings hold at any renormalization point. As we have seen, the necessary, but also sufficient, condition for this to happen is to require that such relations are solutions to the REs

$$\beta_g \frac{d\lambda_{ijk}}{dg} = \beta_{ijk} \quad (35)$$

and hold at all orders. As we have seen, remarkably the existence of all-order solutions to (35) can be decided at the one-loop level.

Let us now turn to the all-order finiteness theorem [33], which states when a $N = 1$ supersymmetric gauge theory can become finite to all orders in the sense of vanishing β -functions, that is of physical scale invariance. It is based on (a) the structure of the supercurrent in $N = 1$ SYM [39, 40, 41], and on (b) the non-renormalization properties of $N = 1$ chiral anomalies [33, 34]. Details on the proof can be found in Refs. [33] and further discussion in Refs. [34–36]. Here, following mostly Ref. [36] we present a comprehensible sketch of the proof.

Consider a $N = 1$ supersymmetric gauge theory, with simple Lie group G . The content of this theory is given at the classical level by the matter supermultiplets S_i , which contain a scalar field ϕ_i and a Weyl spinor $\psi_{i\alpha}$, and the gauge fields V_a , which contain a gauge vector field A_μ^a and a gaugino Weyl spinor λ_α^a .

Let us first recall certain facts about the theory:

(1) A massless $N = 1$ supersymmetric theory is invariant under a U(1) chiral transformation R under which the various fields transform as follows

$$A'_\mu = A_\mu, \quad \lambda'_\alpha = \exp(-i\theta)\lambda_\alpha, \quad \phi' = \exp(-i\frac{2}{3}\theta)\phi, \quad \psi'_\alpha = \exp(-i\frac{1}{3}\theta)\psi_\alpha, \quad \dots \quad (36)$$

The corresponding axial Noether current $J_R^\mu(x)$,

$$J_R^\mu(x) = \bar{\lambda}\gamma^\mu\gamma^5\lambda + \dots, \quad (37)$$

is conserved classically, while in the quantum case is violated by the axial anomaly

$$\partial J_R^\mu = r(\epsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} + \dots). \quad (38)$$

From its known topological origin in ordinary gauge theories [37], one would expect the axial vector current J_R^μ to satisfy the Adler–Bardeen theorem [38] and receive corrections only at the one-loop level. Indeed it has been shown that the same non-renormalization theorem holds also in supersymmetric theories [34]. Therefore

$$r = \hbar\beta_g^{(1)}. \quad (39)$$

(2) The massless theory we consider is scale invariant at the classical level and, in general, there is a scale anomaly due to radiative corrections. The scale anomaly appears in the trace of the energy momentum tensor $T_{\mu\nu}$, which is traceless classically. It has the form

$$T_\mu^\mu = \beta_g F^{\mu\nu} F_{\mu\nu} + \dots \quad (40)$$

(3) Massless, $N = 1$ supersymmetric gauge theories are classically invariant under the supersymmetric extension of the conformal group – the superconformal group. Examining the superconformal algebra, it can be seen that the subset of superconformal transformations consisting of translations, supersymmetry transformations, and axial R transformations is closed under supersymmetry, *i.e.* these transformations form a representation of supersymmetry. It follows that the conserved currents corresponding to these transformations make up a supermultiplet represented by an axial vector superfield called supercurrent [39] J ,

$$J \equiv \{J_R'^\mu, Q_\alpha^\mu, T_\nu^\mu, \dots\}, \quad (41)$$

where $J_R'^\mu$ is the current associated to R invariance, Q_α^μ is the one associated to supersymmetry invariance, and T_ν^μ the one associated to translational invariance (energy-momentum tensor).

The anomalies of the R current $J_R'^\mu$, the trace anomalies of the supersymmetry current, and the energy-momentum tensor, form also a second supermultiplet, called the supertrace anomaly

$$\begin{aligned} S &= \{\text{Re } S, \text{Im } S, S_\alpha\} \\ &= \{T_\mu^\mu, \partial_\mu J_R'^\mu, \sigma_{\alpha\beta}^\mu \bar{Q}_\mu^{\dot{\beta}} + \dots\}, \end{aligned} \quad (42)$$

where T_μ^μ in Eq. (40) and

$$\partial_\mu J_R'^\mu = \beta_g \epsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} + \dots \quad (43)$$

$$\sigma_{\alpha\beta}^\mu \bar{Q}_\mu^{\dot{\beta}} = \beta_g \lambda^\beta \sigma_{\alpha\beta}^{\mu\nu} F_{\mu\nu} + \dots \quad (44)$$

(4) It is very important to note that the Noether current defined in (37) is not the same as the current associated to R invariance that appears in the supermultiplet J in (41), but they coincide in the tree approximation. So starting from a unique classical Noether current $J_{R(\text{class})}^\mu$, the Noether current J_R^μ is defined as the quantum extension of $J_{R(\text{class})}^\mu$ which allows for the validity of the non-renormalization theorem. On the other hand $J_R'^\mu$, is defined to belong to the supercurrent J , together with the energy-momentum tensor. The two requirements cannot be fulfilled by a single current operator at the same time.

Although the Noether current J_R^μ which obeys (38) and the current $J_R'^\mu$ belonging to the supercurrent multiplet J are not the same, there is a relation [33] between quantities associated with them

$$r = \beta_g(1 + x_g) + \beta_{ijk}x^{ijk} - \gamma_A r^A, \quad (45)$$

where r was given in Eq. (39). The r^A are the non-renormalized coefficients of the anomalies of the Noether currents associated to the chiral invariances of the superpotential, and — like r — are strictly one-loop quantities. The γ_A 's are linear combinations of the anomalous dimensions of the matter fields, and x_g , and x^{ijk} are radiative correction quantities. The structure of equality (45) is independent of the renormalization scheme.

One-loop finiteness, *i.e.* vanishing of the β -functions at one-loop, implies that the Yukawa couplings λ_{ijk} must be functions of the gauge coupling g . To find a similar condition to all orders it is necessary and sufficient for the Yukawa couplings to be a formal power series in g , which is solution of the REs (35).

We can now state the theorem for all-order vanishing β -functions.

Theorem:

Consider an $N = 1$ supersymmetric Yang-Mills theory, with simple gauge group. If the following conditions are satisfied

1. There is no gauge anomaly.
2. The gauge β -function vanishes at one-loop

$$\beta_g^{(1)} = 0 = \sum_i l(R_i) - 3C_2(G). \quad (46)$$

3. There exist solutions of the form

$$\lambda_{ijk} = \rho_{ijk}g, \quad \rho_{ijk} \in \mathbb{C} \quad (47)$$

to the conditions of vanishing one-loop matter fields anomalous dimensions

$$\gamma_j^{i(1)} = 0 = \frac{1}{32\pi^2} [C^{ikl} C_{jkl} - 2g^2 C_2(R_i) \delta_{ij}]. \quad (48)$$

4. These solutions are isolated and non-degenerate when considered as solutions of vanishing one-loop Yukawa β -functions:

$$\beta_{ijk} = 0. \quad (49)$$

Then, each of the solutions (47) can be uniquely extended to a formal power series in g , and the associated super Yang-Mills models depend on the single coupling constant g with a β function which vanishes at all-orders.

It is important to note a few things: The requirement of isolated and non-degenerate solutions guarantees the existence of a formal power series solution to the reduction equations. The vanishing of the gauge β -function at one-loop, $\beta_g^{(1)}$, is equivalent to the vanishing of the R current anomaly (38). The vanishing of the anomalous dimensions at one-loop implies the vanishing of the Yukawa couplings β -functions at that order. It also implies the vanishing of the chiral anomaly coefficients r^A . This last property is a necessary condition for having β functions vanishing at all orders.

Proof:

Insert β_{ijk} as given by the REs into the relationship (45) between the axial anomalies coefficients and the β -functions. Since these chiral anomalies vanish, we get for β_g an homogeneous equation of the form

$$0 = \beta_g(1 + O(\hbar)). \quad (50)$$

The solution of this equation in the sense of a formal power series in \hbar is $\beta_g = 0$, order by order. Therefore, due to the REs (35), $\beta_{ijk} = 0$ too.

Thus we see that finiteness and reduction of couplings are intimately related.

6. Finite SU(5) model

As a realistic example of the concepts presented in the previous section we consider a Finite Unified Model Based on SU(5). From the classification of theories with vanishing one-loop β function for the gauge coupling [23], one can see that using SU(5) as gauge group there exist only two candidate models which can accommodate three fermion generations. These models contain the chiral supermultiplets **5**, $\bar{\mathbf{5}}$, **10**, $\bar{\mathbf{5}}$, **24** with the multiplicities

(6, 9, 4, 1, 0) and (4, 7, 3, 0, 1), respectively. Only the second one contains a **24**-plet which can be used for spontaneous symmetry breaking (SSB) of $SU(5)$ down to $SU(3) \times SU(2) \times U(1)$. (For the first model one has to incorporate another way, such as the Wilson flux breaking to achieve the desired SSB of $SU(5)$ [11]). Therefore, we would like to concentrate only on the second model.

To simplify the situation, we neglect the intergenerational mixing among the lepton and quark supermultiplets and consider the following $SU(5)$ invariant cubic superpotential for the (second) model:

$$W = \sum_{i=1}^3 \sum_{\alpha=1}^4 \left[\frac{1}{2} g_{i\alpha}^u \mathbf{10}_i \mathbf{10}_i H_\alpha + g_{i\alpha}^d \mathbf{10}_i \bar{\mathbf{5}}_i \bar{H}_\alpha \right] + \sum_{\alpha=1}^4 g_\alpha^f H_\alpha \mathbf{24} \bar{H}_\alpha + \frac{g^\lambda}{3} (\mathbf{24})^3, \text{ with } g_{i\alpha}^{u,d} = 0 \text{ for } i \neq \alpha, \quad (51)$$

where the $\mathbf{10}_i$'s and $\bar{\mathbf{5}}_i$'s are the usual three generations, and the four ($\mathbf{5} + \bar{\mathbf{5}}$) Higgses are denoted by H_α , \bar{H}_α . The superpotential is not the most general one, but by virtue of the non-renormalization theorem, this does not contradict the philosophy of the coupling unification by the reduction method (a RG invariant fine tuning is a solution of the reduction equation). In the case at hand, however, one can find a discrete symmetry that can be imposed on the most general cubic superpotential to arrive at the non-intergenerational mixing [11]. This is given in Table I.

TABLE I

The charges of the $Z_7 \times Z_3$ symmetry

	$\mathbf{10}_1$	$\mathbf{10}_2$	$\mathbf{10}_3$	$\bar{\mathbf{5}}_1$	$\bar{\mathbf{5}}_2$	$\bar{\mathbf{5}}_3$	H_1	H_2	H_3	H_4
Z_7	1	2	4	4	1	2	5	3	6	0
Z_3	1	2	0	0	0	0	1	2	0	0

Given the superpotential W , we can compute the β functions of the model. We denote the gauge coupling by g (with the vanishing one-loop β function), and our normalization of the β functions is as usual, *i.e.*,

$$dg_i/d\ln\mu = \beta_i^{(1)}/16\pi^2 + O(g^5),$$

where μ is the renormalization scale. We find:

$$\begin{aligned}
\beta_g^{(1)} &= 0, \\
\beta_{i\alpha}^u{}^{(1)} &= \frac{1}{16\pi^2} \left[-\frac{96}{5} g^2 + 6 \sum_{\beta=1}^4 (g_{i\beta}^u)^2 + 3 \sum_{j=1}^3 (g_{j\alpha}^u)^2 + \frac{24}{5} (g_\alpha^f)^2 \right. \\
&\quad \left. + 4 \sum_{\beta=1}^4 (g_{i\beta}^d)^2 \right] g_{i\alpha}^u, \\
\beta_{i\alpha}^d{}^{(1)} &= \frac{1}{16\pi^2} \left[-\frac{84}{5} g^2 + 3 \sum_{\beta=1}^4 (g_{i\beta}^u)^2 + \frac{24}{5} (g_\alpha^f)^2 + 4 \sum_{j=1}^3 (g_{j\alpha}^d)^2 \right. \\
&\quad \left. + 6 \sum_{\beta=1}^4 (g_{i\beta}^d)^2 \right] g_{i\alpha}^d, \\
\beta^\lambda{}^{(1)} &= \frac{1}{16\pi^2} \left[-30 g^2 + \frac{63}{5} (g^\lambda)^2 + 3 \sum_{\alpha=1}^4 (g_\alpha^f)^2 \right] g^\lambda, \\
\beta_\alpha^f{}^{(1)} &= \frac{1}{16\pi^2} \left[-\frac{98}{5} g^2 + 3 \sum_{i=1}^3 (g_{i\alpha}^u)^2 + 4 \sum_{i=1}^3 (g_{i\alpha}^d)^2 + \frac{48}{5} (g_\alpha^f)^2 \right. \\
&\quad \left. + \sum_{\beta=1}^4 (g_\beta^f)^2 + \frac{21}{5} (g^\lambda)^2 \right] g_\alpha^f.
\end{aligned} \tag{52}$$

We then regard the gauge coupling g as the primary coupling and solve the reduction equations (4) with the power series ansatz. One finds that the power series,

$$\begin{aligned}
(g_{ii}^u)^2 &= \frac{8}{5} g^2 + \dots, \quad (g_{ii}^d)^2 = \frac{6}{5} g^2 + \dots, \quad (g^\lambda)^2 = \frac{15}{7} g^2 + \dots, \\
(g_4^f)^2 &= g^2, \quad (g_\alpha^f)^2 = 0 + \dots \quad (\alpha = 1, 2, 3),
\end{aligned} \tag{53}$$

exists uniquely, where \dots indicates higher order terms and all the other couplings have to vanish. As we have done in the previous section, we can easily verify that the higher order terms can be uniquely computed.

Consequently, all the one-loop β functions of the theory vanish. Moreover, all the one-loop anomalous dimensions for the chiral supermultiplets,

$$\begin{aligned}
\gamma_{10i}^{(1)} &= \frac{1}{16\pi^2} \left[-\frac{36}{5} g^2 + 3 \sum_{\beta=1}^4 (g_{i\beta}^u)^2 + 2 \sum_{\beta=1}^4 (g_{i\beta}^d)^2 \right], \\
\gamma_{\bar{5}i}^{(1)} &= \frac{1}{16\pi^2} \left[-\frac{24}{5} g^2 + 4 \sum_{\beta=1}^4 (g_{i\beta}^d)^2 \right],
\end{aligned}$$

$$\begin{aligned}
 \gamma_{H_\alpha}^{(1)} &= \frac{1}{16\pi^2} \left[-24g^2 + 3 \sum_{i=1}^3 (g_{i\alpha}^u)^2 + \frac{24}{5} (g_\alpha^f)^2 \right], \\
 \gamma_{\bar{H}_\alpha}^{(1)} &= \frac{1}{16\pi^2} \left[-24g^2 + 4 \sum_{i=1}^3 (g_{i\alpha}^d)^2 + \frac{24}{5} (g_\alpha^f)^2 \right], \\
 \gamma_{24}^{(1)} &= \frac{1}{16\pi^2} \left[-\frac{10}{5} g^2 + \sum_{\alpha=1}^4 (g_\alpha^f)^2 + \frac{21}{5} (g^\lambda)^2 \right]
 \end{aligned} \tag{54}$$

also vanish in the reduced system. As it has already been mentioned before, these conditions are necessary and sufficient for finiteness at the two-loop level [22].

In most of the previous studies of the present model [26, 27], however, the complete reduction of the Yukawa couplings, which is necessary for all-order-finiteness, was ignored. They have used the freedom offered by the degeneracy in the one- and two-loop approximations in order to make specific ansätze that could lead to phenomenologically acceptable predictions. In the above model, we found a diagonal solution for the Yukawa couplings, with each family coupled to a different Higgs. However, we may use the fact that mass terms do not influence the RG functions in a certain class of renormalization schemes, and introduce appropriate mass terms that permit us to perform a rotation in the Higgs sector such that only one pair of Higgs doublets, coupled to the third family, remains light and acquires a non-vanishing VEV [27]. Note that the effective coupling of the Higgs doublets to the first family after the rotation is very small avoiding in this way a potential problem with the proton lifetime [42]. Thus, effectively, we have at low energies the Minimal Supersymmetric Standard Model (MSSM) with only one pair of Higgs doublets satisfying the boundary conditions at M_{GUT}

$$g_t^2 = \frac{8}{5}g^2 + O(g^4), \quad g_b^2 = g_\tau^2 = \frac{6}{5}g^2 + O(g^4), \tag{55}$$

where g_i ($i = t, b, \tau$) are the top, bottom and tau Yukawa couplings of the MSSM, and the other Yukawa couplings should be regarded as free.

Adding soft breaking terms (which are supposed not to influence the β functions beyond M_{GUT}), we can obtain supersymmetry breaking. The conditions on the soft breaking terms to preserve one-loop finiteness have been given already some time ago [24]. Recently, the same problem in two-loop orders has been addressed [25]. It is an open problem whether there exists a suitable set of conditions on the soft terms for all-loop finiteness.

7. Predictions of low energy parameters

In this section we will refine the predictions of the AFUT and FUT models, taking into account certain corrections and we will compare them with the experimental data.

As mentioned before, at low energies we want the MSSM, with one pair of Higgs doublets, and we will assume that at the supersymmetry breaking scale all the superpartners are degenerate.

Since the gauge symmetry is spontaneously broken below M_{GUT} , the finiteness conditions in the case of the FUT model do not restrict the renormalization property at low energies, and all it remains is a boundary condition on the gauge and Yukawa couplings at M_{GUT} , *i.e.*, Eq. (53). Clearly the same holds also in the AFUT models. So we examine the evolution of these couplings according to their renormalization group equations at two-loops with the corresponding boundary conditions at M_{GUT} .

Below M_{GUT} their evolution is assumed to be governed by the MSSM. We further assume a unique threshold M_{SUSY} for all superpartners of the MSSM so that below M_{SUSY} the SM is the correct effective theory. We recall that $\tan\beta$ is usually determined in the Higgs sector, which however strongly depends on the supersymmetry breaking terms. Here we avoid this by using the tau mass M_τ as input, which means that we partly fix the Higgs sector indirectly. That is, assuming that

$$M_Z \ll M_t \ll M_{\text{SUSY}} , \quad (56)$$

we require the matching condition at M_{SUSY} [43],

$$\begin{aligned} \alpha_t^{\text{SM}} &= \alpha_t \sin^2 \beta , \quad \alpha_b^{\text{SM}} = \alpha_b \cos^2 \beta , \quad \alpha_\tau^{\text{SM}} = \alpha_\tau \cos^2 \beta , \\ \alpha_\lambda &= \frac{1}{4} \left(\frac{3}{5} \alpha_1 + \alpha_2 \right) \cos^2 2\beta \end{aligned} \quad (57)$$

to be satisfied, where α_i^{SM} ($i = t, b, \tau$) are the SM Yukawa couplings and α_λ is the Higgs coupling. The MSSM threshold corrections to this matching condition [44, 45] will be discussed later. This is our definition of $\tan\beta$, and Eq. (57) fixes $\tan\beta$, because with a given set of the input parameters [46],

$$M_\tau = 1.777 \text{ GeV} , \quad M_Z = 91.188 \text{ GeV} , \quad (58)$$

with [47]

$$\begin{aligned} \alpha_{\text{EM}}^{-1}(M_Z) &= 127.9 + \frac{8}{9\pi} \log \frac{M_t}{M_Z} , \\ \sin^2 \theta_W(M_Z) &= 0.2319 - 3.03 \times 10^{-5} T - 8.4 \times 10^{-8} T^2 , \\ T &= M_t/[\text{GeV}] - 165 , \end{aligned} \quad (59)$$

the matching condition (57) and the GYU boundary condition at M_{GUT} can be satisfied only for a specific value of $\tan \beta$. Here M_τ, M_t, M_Z are pole masses, and the couplings are defined in the $\overline{\text{MS}}$ scheme with six flavors. The translation from a Yukawa coupling into the corresponding mass follows according to

$$m_i = \frac{1}{\sqrt{2}} g_i(\mu) v(\mu), \quad i = t, b, \tau \quad \text{with} \quad v(M_Z) = 246.22 \text{ GeV}, \quad (60)$$

where $m_i(\mu)$'s are the running masses satisfying the respective evolution equation of two-loop order. The pole masses can be calculated from the running ones of course. For the top mass, we use [43, 44]

$$M_t = m_t(M_t) \left[1 + \frac{4}{3} \frac{\alpha_3(M_t)}{\pi} + 10.95 \left(\frac{\alpha_3(M_t)}{\pi} \right)^2 + k_t \frac{\alpha_t(M_t)}{\pi} \right], \quad (61)$$

where $k_t \simeq -0.3$ for the range of parameters we are concerned with in this paper [44]. Note that both sides of Eq. (61) contain M_t so that M_t is defined only implicitly. Therefore, its determination requires an iteration method. As for the tau and bottom masses, we assume that $m_\tau(\mu)$ and $m_b(\mu)$ for $\mu \leq M_Z$ satisfy the evolution equation governed by the $\text{SU}(3)_C \times \text{U}(1)_{\text{EM}}$ theory with five flavors and use

$$M_b = m_b(M_b) \left[1 + \frac{4}{3} \frac{\alpha_{3(5f)}(M_b)}{\pi} + 12.4 \left(\frac{\alpha_{3(5f)}(M_b)}{\pi} \right)^2 \right],$$

$$M_\tau = m_\tau(M_\tau) \left[1 + \frac{\alpha_{\text{EM}(5f)}(M_\tau)}{\pi} \right], \quad (62)$$

where the experimental value of $m_b(M_b)$ is $(4.1 - 4.5) \text{ GeV}$ [46]. The couplings with five flavors entered in Eq. (30) $\alpha_{3(5f)}$ and $\alpha_{\text{EM}(5f)}$ are related to α_3 and α_{EM} by

$$\alpha_{3(5f)}^{-1}(M_Z) = \alpha_3^{-1}(M_Z) - \frac{1}{3\pi} \ln \frac{M_t}{M_Z},$$

$$\alpha_{\text{EM}(5f)}^{-1}(M_Z) = \alpha_{\text{EM}}^{-1}(M_Z) - \frac{8}{9\pi} \ln \frac{M_t}{M_Z}. \quad (63)$$

Using the input values given in eqs. (58) and (60), we find

$$m_\tau(M_\tau) = 1.771 \text{ GeV}, \quad m_\tau(M_Z) = 1.746 \text{ GeV},$$

$$\alpha_{\text{EM}(5f)}^{-1}(M_\tau) = 133.7, \quad (64)$$

and from Eq. (60) we obtain

$$\alpha_\tau^{\text{SM}}(M_Z) = \frac{g_\tau^2}{4\pi} = 8.005 \times 10^{-6}, \quad (65)$$

which we use as an input parameter instead of M_τ .

The matching condition (57) suffers from the threshold corrections coming from the MSSM superpartners:

$$\alpha_i^{\text{SM}} \rightarrow \alpha_i^{\text{SM}} (1 + \Delta_i^{\text{SUSY}}), \quad i = 1, 2, \dots, \tau. \quad (66)$$

It was shown that these threshold effects to the gauge couplings can be effectively parametrized by just one energy scale [48]. Accordingly, we can identify our M_{SUSY} with that defined in Ref. [48]. This ensures that there are no further one-loop threshold corrections to $\alpha_3(M_Z)$ when we calculate it as a function of $\alpha_{\text{EM}}(M_Z)$ and $\sin^2 \theta_W(M_Z)$.

The same scale M_{SUSY} does not describe threshold corrections to the Yukawa couplings, and they could cause large corrections to the fermion mass prediction [44, 45]¹. For m_b , for instance, the correction can be as large as 50% for very large values of $\tan \beta$, especially in models with radiative gauge symmetry breaking and with supersymmetry softly broken by the universal breaking terms. As we will see, the SU(5)-FUT and AFUT models predict (with these corrections suppressed) values for the bottom quark mass that are rather close to the experimentally allowed region so that there is room only for small corrections. Consequently, if we want to break SU(2)×U(1) gauge symmetry radiatively, the models favor non-universal soft breaking terms [49].

To get an idea about the magnitude of the correction, we consider the case that all the superpartners have the same mass $M_{\text{SUSY}} = 500$ GeV with $M_{\text{SUSY}} \gg \mu_H$ and $\tan \beta \geq 50$. Using Δ 's given in Ref. [45], we find that the MSSM correction to the M_t prediction is ~ -1 % for this case. Comparing with the results of [45, 50], this may appear to be underestimated for other cases. Note, however, that there is a nontrivial interplay among the corrections between the M_t and M_b predictions for a given GYU boundary condition at M_{GUT} and the fixed pole tau mass, which has not been taken into account in refs. [45, 50]. In the following discussion, therefore, we regard the MSSM threshold correction to the M_t prediction as unknown and denote it by

$$\delta^{\text{MSSM}} M_t. \quad (67)$$

In the case of the AFUT models, the non-observation of proton decay favours a solution close to AFUT3.

In Table II we present the predictions for M_t for various M_{SUSY} , in the case of the FUT model.

¹ It is possible to compute the MSSM correction to M_t directly, *i.e.*, without constructing an effective theory below M_{SUSY} . In this approach, too, large corrections have been reported [50]. In the present paper, evidently, we are following the effective theory approach as *e.g.* Refs. [44, 45].

TABLE II

The predictions for different M_{SUSY} for FUT

M_{SUSY} [GeV]	$\alpha_3(M_Z)$	$\tan \beta$	M_{GUT} [GeV]	$m_b(M_b)$ [GeV]	M_t [GeV]
300	0.123	54.2	2.08×10^{16}	4.54	183.5
500	0.122	54.3	1.77×10^{16}	4.54	184.0
10^3	0.120	54.4	1.42×10^{16}	4.54	184.4

As we can see from the table, only negative MSSM corrections of at most $\sim 10\%$ to $m_b(M_b)$ are allowed ($m_b^{\text{exp}}(M_b) = (4.1 - 4.5) \text{ GeV}$), implying that FUT favors non-universal soft symmetry breaking terms as announced. The predicted M_t values are well below the infrared value [51], for instance 194 GeV for $M_{\text{SUSY}} = 500 \text{ GeV}$, so that the M_t prediction must be sensitive against the change of the boundary condition.

We recall that if one includes the threshold effects of superheavy particles [52], the GUT scale M_{GUT} at which α_1 and α_2 are supposed to meet is related to the mass of the superheavy $\text{SU}(3)_C$ -triplet Higgs supermultiplets contained in H_α and \bar{H}_α . These effects have therefore influence on the GYU boundary conditions.

TABLE III

The predictions for the AFUT model

m_{SUSY} [GeV]	$\alpha_3(M_Z)$	$\tan \beta$	M_{GUT} [GeV]	m_b [GeV]	m_t [GeV]
300	0.120	47.7	1.8×10^{16}	5.4	179.7
500	0.118	47.7	1.39×10^{16}	5.3	178.9

In Table III we present the predictions for the AFUT viable model (AFUT3). For these model the corrections mentioned above have been calculated [16] and are of the order of $\leq 2\%$. The threshold effects of the superheavy particles were estimated to be of the same order as in the gauge sector, which leads to an uncertainty of $\sim \pm 0.4 \text{ GeV}$ in M_t . The structure of the threshold effects in FUT is involved, but they are not arbitrary and probably determinable to a certain extent, because the mixing of the superheavy Higgses is strongly dictated by the fermion mass matrix of the MSSM. To bring these threshold effects under control is challenging. Here we assume that the magnitude of these effects is $\sim \pm 4 \text{ GeV}$ in M_t , which is estimated by comparing the minimal GYU model based on $\text{SU}(5)$ [16].

Thus, for the FUT model the prediction for M_t [16] will be

$$M_t = (183 + \delta^{\text{MSSM}} M_t \pm 5) \text{ GeV} , \quad (68)$$

where the finite corrections coming from the conversion from the dimensional reduction scheme to the ordinary $\overline{\text{MS}}$ in the gauge sector [62] are included, and those in the Yukawa sector are included as an uncertainty of $\sim \pm 1$ GeV. The MSSM threshold correction is denoted $\delta^{\text{MSSM}} M_t$ which has been discussed in the previous section.

In the case of the AFUT model the prediction is [16]

$$M_t = (181 + \delta^{\text{MSSM}} M_t \pm 3) \text{ GeV} . \quad (69)$$

Comparing the M_t prediction above with the most recent experimental values [53],

$$\begin{aligned} M_{top} &= 176.8 \pm 4.4_{\text{stat}} \pm 4.8_{\text{syst}} \text{ GeV} & \text{CDF} \\ M_{top} &= 169.0 \pm 8.0_{\text{stat}} \pm 8.0_{\text{syst}} \text{ GeV} & \text{D0} \end{aligned} \quad (70)$$

we see it is consistent with the experimental data.

It is interesting to note that the consistency of the finiteness hypothesis is closely related to the fine structure of supersymmetry breaking and also to the Higgs sector, because these superpartner corrections to m_b can be kept small for appropriate supersymmetric spectrum characterized by very heavy squarks and/or small μ_H describing the mixing of the two Higgs doublets in the superpotential ².

The predictions for M_t versus M_{SUSY} for the two sets of boundary conditions given above (AFUT3 and AFUT4) together with the corresponding predictions of the FUT model, are given in Figure 1. In a recent study [16], we have considered the proton decay constraint [55] to further reduce the parameter space of the model. It has been found that the model consistent with the non-observation of the proton decay should be very close to AFUT3, implying a better possibility to discriminate between the FUT and AFUT models, as one can see from Figure 1.

² The solution with small μ_H is favored by the experimental data and cosmological constraints [49]. The sign of this correction is determined by the relative sign of μ_H and the gluino mass parameter and is correlated with the chargino exchange contribution to the $b \rightarrow s \gamma$ decay [44]. The later has the same sign as the Standard Model and the charged Higgs contributions when the supersymmetric corrections to m_b are negative.

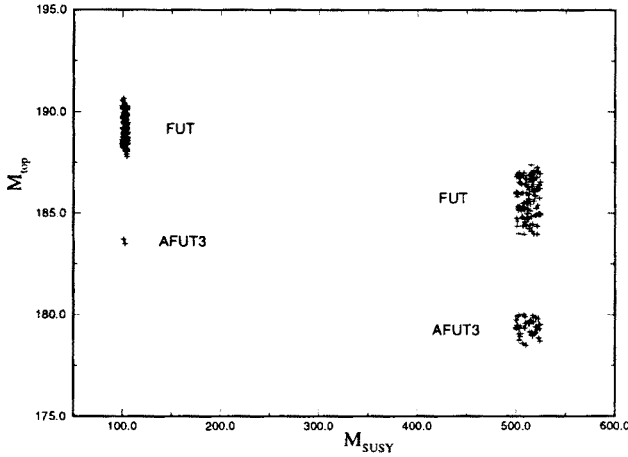


Fig. 1. M_t predictions of SU(5) FUT and AFUT3 models, for given M_{SUSY} around 100 and 500 GeV. For the FUT model $\tilde{\alpha}_t = 1.6$, $\tilde{\alpha}_b = 1.2$, and for AFUT3 $\tilde{\alpha}_t = 0.97$, $\tilde{\alpha}_b = 0.57$.

8. Asymptotically non-free supersymmetric Pati-Salam model

We present now a model where the reduction of couplings is applied, but that does not have a single gauge group, but a product of simple groups. In order for the RGI method for the gauge coupling unification to work, the gauge couplings should have the same asymptotic behavior. Note that this common behavior is absent in the standard model with three families. A way to achieve a common asymptotic behavior of all the different gauge couplings is to embed $SU(3)_C \times SU(2)_L \times U(1)_Y$ to some non-abelian gauge group, as it was done in the previous sections. However, in this case still a major rôle in the GYU is due to the group theoretical aspects of the covering GUT. Here we would like to examine the power of RGI method by considering theories without covering GUTs. We found [13] that the minimal phenomenologically viable model is based on the gauge group of Pati and Salam [1]– $\mathcal{G}_{\text{PS}} \equiv SU(4) \times SU(2)_R \times SU(2)_L$. We recall that $N = 1$ supersymmetric models based on this gauge group have been studied with renewed interest because they could in principle be derived from superstring [56].

In our supersymmetric, Gauge-Yukawa unified model based on \mathcal{G}_{PS} [13], three generations of quarks and leptons are accommodated by six chiral supermultiplets, three in $(4, 2, 1)$ and three $(\bar{4}, 1, 2)$, which we denote by

$\Psi^{(I)\mu i_R}$ and $\bar{\Psi}^{(I)i_L}$. (I runs over the three generations, and $\mu, \nu (= 1, 2, 3, 4)$ are the $SU(4)$ indices while $i_R, i_L (= 1, 2)$ stand for the $SU(2)_{L,R}$ indices.) The Higgs supermultiplets in $(\mathbf{4}, \mathbf{2}, \mathbf{1})$, $(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1})$ and $(\mathbf{15}, \mathbf{1}, \mathbf{1})$ are denoted by $H^{\mu i_R}$, $\bar{H}_{\mu i_R}$ and Σ_{ν}^{μ} , respectively. They are responsible for the spontaneous symmetry breaking (SSB) of $SU(4) \times SU(2)_R$ down to $SU(3)_C \times U(1)_Y$. The SSB of $U(1)_Y \times SU(2)_L$ is then achieved by the nonzero VEV of $h_{i_R i_L}$ which is in $(\mathbf{1}, \mathbf{2}, \mathbf{2})$. In addition to these Higgs supermultiplets, we introduce $G_{\nu i_R i_L}^{\mu} (\mathbf{15}, \mathbf{2}, \mathbf{2})$, $\phi (\mathbf{1}, \mathbf{1}, \mathbf{1})$ and $\Sigma'_{\nu}{}^{\mu} (\mathbf{15}, \mathbf{1}, \mathbf{1})$. The $G_{\nu i_R i_L}^{\mu}$ is introduced to realize the $SU(4) \times SU(2)_R \times SU(2)_L$ version of the Georgi-Jarlskog type ansatz [57] for the mass matrix of leptons and quarks while ϕ is supposed to mix with the right-handed neutrino supermultiplets at a high energy scale. With these things in mind, we write down the superpotential of the model W , which is the sum of the following superpotentials:

$$\begin{aligned}
 W_Y &= \sum_{I,J=1}^3 g_{IJ} \bar{\Psi}_{\mu}^{(I)i_R} \Psi^{(J)\mu i_L} h_{i_R i_L}, \\
 W_{GJ} &= g_{GJ} \bar{\Psi}_{\mu}^{(2)i_R} G_{\nu i_R j_L}^{\mu} \Psi^{(2)\nu j_L}, \\
 W_{NM} &= \sum_{I=1,2,3} g_{I\phi} \epsilon_{i_R j_R} \bar{\Psi}_{\mu}^{(I)i_R} H^{\mu j_R} \phi, \\
 W_{SB} &= g_H \bar{H}_{\mu i_R} \Sigma_{\nu}^{\mu} H^{\nu i_R} + \frac{g_{\Sigma}}{3} \text{Tr} [\Sigma^3] + \frac{g_{\Sigma'}}{2} \text{Tr} [(\Sigma')^2 \Sigma], \\
 W_{TDS} &= \frac{g_G}{2} \epsilon^{i_R j_R} \epsilon^{i_L j_L} \text{Tr} [G_{i_R i_L} \Sigma G_{j_R j_L}], \\
 W_M &= m_h h^2 + m_G G^2 + m_{\phi} \phi^2 + m_H \bar{H} H + m_{\Sigma} \Sigma^2 + m_{\Sigma'} (\Sigma')^2.
 \end{aligned} \tag{71}$$

Although W has the parity, $\phi \rightarrow -\phi$ and $\Sigma' \rightarrow -\Sigma'$, it is not the most general potential, but, as we already mentioned, this does not contradict the philosophy of the coupling unification by the RGI method.

We denote the gauge couplings of $SU(4) \times SU(2)_R \times SU(2)_L$ by α_4 , α_{2R} and α_{2L} , respectively. The gauge coupling for $U(1)_Y$, α_1 , normalized in the usual GUT inspired manner, is given by $1/\alpha_1 = 2/5\alpha_4 + 3/5\alpha_{2R}$. In principle, the primary coupling can be any one of the couplings. But it is more convenient to choose a gauge coupling as the primary one because the one-loop β functions for a gauge coupling depends only on its own gauge coupling. For the present model, we use α_{2L} as the primary one. Since the gauge sector for the one-loop β functions is closed, the solutions of the fixed point equations (21) are independent on the Yukawa and Higgs couplings. One easily obtains $\rho_4^{(1)} = 8/9$, $\rho_{2R}^{(1)} = 4/5$, so that the RGI relations (25)

at the one-loop level become

$$\tilde{\alpha}_4 = \frac{\alpha_4}{\alpha_{2L}} = \frac{8}{9}, \quad \tilde{\alpha}_1 = \frac{\alpha_1}{\alpha_{2L}} = \frac{5}{6}. \quad (72)$$

The solutions in the Yukawa-Higgs sector strongly depend on the result of the gauge sector. After slightly involved algebraic computations, one finds that most predictive solutions contain at least three vanishing $\rho_i^{(1)}$'s. Out of these solutions, there are two that exhibit the most predictive power and moreover they satisfy the neutrino mass relation $m_{\nu\tau} > m_{\nu\mu}, m_{\nu e}$.

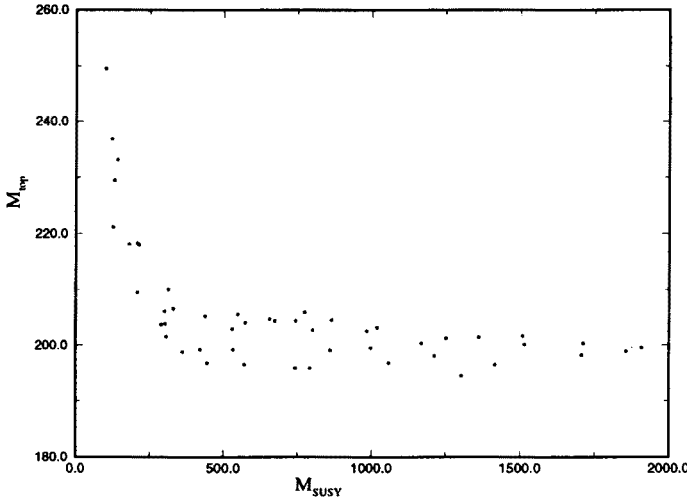


Fig. 2. The values for M_t predicted by the Pati-Salam model for different M_{SUSY} scales. Only the ones with M_{SUSY} beyond 400 GeV are realistic.

For the first solution we have $\rho_{1\phi}^{(1)} = \rho_{2\phi}^{(1)} = \rho_{\Sigma}^{(1)} = 0$, while for the second solution, $\rho_{1\phi}^{(1)} = \rho_{2\phi}^{(1)} = \rho_G^{(1)} = 0$, and one finds that for the cases above the power series solutions (25) take the form

$$\begin{aligned} \tilde{\alpha}_{GJ} &\simeq \begin{cases} 1.67 - 0.05\tilde{\alpha}_{1\phi} + 0.004\tilde{\alpha}_{2\phi} - 0.90\tilde{\alpha}_{\Sigma} + \dots \\ 2.20 - 0.08\tilde{\alpha}_{2\phi} - 0.05\tilde{\alpha}_G + \dots \end{cases}, \\ \tilde{\alpha}_{33} &\simeq \begin{cases} 3.33 + 0.05\tilde{\alpha}_{1\phi} + 0.21\tilde{\alpha}_{2\phi} - 0.02\tilde{\alpha}_{\Sigma} + \dots \\ 3.40 + 0.05\tilde{\alpha}_{1\phi} - 1.63\tilde{\alpha}_{2\phi} - 0.001\tilde{\alpha}_G + \dots \end{cases}, \\ \tilde{\alpha}_{3\phi} &\simeq \begin{cases} 1.43 - 0.58\tilde{\alpha}_{1\phi} - 1.43\tilde{\alpha}_{2\phi} - 0.03\tilde{\alpha}_{\Sigma} + \dots \\ 0.88 - 0.48\tilde{\alpha}_{1\phi} + 8.83\tilde{\alpha}_{2\phi} + 0.01\tilde{\alpha}_G + \dots \end{cases}, \end{aligned}$$

$$\begin{aligned}
\tilde{\alpha}_H &\simeq \begin{cases} 1.08 - 0.03\tilde{\alpha}_{1\phi} + 0.10\tilde{\alpha}_{2\phi} - 0.07\tilde{\alpha}_\Sigma + \cdots \\ 2.51 - 0.04\tilde{\alpha}_{1\phi} - 1.68\tilde{\alpha}_{2\phi} - 0.12\tilde{\alpha}_G + \cdots \end{cases}, \\
\tilde{\alpha}_\Sigma &\simeq \begin{cases} - - - \\ 0.40 + 0.01\tilde{\alpha}_{1\phi} - 0.45\tilde{\alpha}_{2\phi} - 0.10\tilde{\alpha}_G + \cdots \end{cases}, \\
\tilde{\alpha}_{\Sigma'} &\simeq \begin{cases} 4.91 - 0.001\tilde{\alpha}_{1\phi} - 0.03\tilde{\alpha}_{2\phi} - 0.46\tilde{\alpha}_\Sigma + \cdots \\ 8.30 + 0.01\tilde{\alpha}_{1\phi} + 1.72\tilde{\alpha}_{2\phi} - 0.36\tilde{\alpha}_G + \cdots \end{cases}, \\
\tilde{\alpha}_G &\simeq \begin{cases} 5.59 + 0.02\tilde{\alpha}_{1\phi} - 0.04\tilde{\alpha}_{2\phi} - 1.33\tilde{\alpha}_\Sigma + \cdots \\ - - - \end{cases}. \quad (73)
\end{aligned}$$

We have assumed that the Yukawa couplings g_{IJ} except for g_{33} vanish. They can be included into RGI relations as small perturbations, but their numerical effects will be rather small.

The number N_H of the Higgses lighter than M_{SUSY} could vary from one to four while the number of those to be taken into account above M_{SUSY} is fixed at four. We have assumed here that $N_H = 1$. The dependence of the top mass on M_{SUSY} in this model is shown in Figure 2.

9. Asymptotically non-free $SO(10)$ model

We will show in this section a model based on $SO(10)$ in which also the reduction of couplings can be applied [14].

We denote the hermitean $SO(10)$ -gamma matrices by Γ_α , $\alpha = 1, \dots, 10$. The charge conjugation matrix C satisfies $C = C^{-1}$, $C^{-1} \Gamma_\alpha^T C = -\Gamma_\alpha$, and the Γ_{11} is defined as $\Gamma_{11} \equiv (-i)^5 \prod_{\alpha=1}^{10} \Gamma_\alpha$ with $(\Gamma_{11})^2 = 1$. The chiral projection operators are given by $\mathcal{P}_\pm = \frac{1}{2}(1 \pm \Gamma_{11})$.

In $SO(10)$ GUTs [3, 58], three generations of quarks and leptons are accommodated by three chiral supermultiplets in **16** which we denote by

$$\Psi^I(\mathbf{16}) \text{ with } \mathcal{P}_+ \Psi^I = \Psi^I, \quad (74)$$

where I runs over the three generations and the spinor index is suppressed. To break $SO(10)$ down to $SU(3)_C \times SU(2)_L \times U(1)_Y$, we use the following set of chiral superfields:

$$S_{\{\alpha\beta\}}(\mathbf{54}), A_{[\alpha\beta]}(\mathbf{45}), \phi(\mathbf{16}), \bar{\phi}(\overline{\mathbf{16}}). \quad (75)$$

The two $SU(2)_L$ doublets which are responsible for the spontaneous symmetry breaking (SSB) of $SU(2)_L \times U(1)_Y$ down to $U(1)_{\text{EM}}$ are contained in $H_\alpha(\mathbf{10})$. We further introduce a singlet φ which after the SSB of $SO(10)$ will mix with the right-handed neutrinos so that they will become superheavy.

The superpotential of the model is given by

$$W = W_Y + W_{SB} + W_{HS} + W_{NM} + W_M, \quad (76)$$

where

$$\begin{aligned}
 W_Y &= \frac{1}{2} \sum_{I,J=1}^3 g_{IJ} \Psi^I C \Gamma_\alpha \Psi^J H_\alpha , \\
 W_{SB} &= \frac{g_\phi}{2} \bar{\phi} \Gamma_{[\alpha\beta]} \phi A_{[\alpha\beta]} + \frac{g_S}{3!} \text{Tr} S^3 + \frac{g_A}{2} \text{Tr} A^2 S , \\
 W_{HS} &= \frac{g_{HS}}{2} H_\alpha S_{\{\alpha\beta\}} H_\beta , \quad W_{NM}^I = \sum_{I=1}^3 g_{INM} \Psi^I \bar{\phi} \varphi , \\
 W_M &= \frac{m_H}{2} H^2 + m_\varphi \varphi^2 + m_\phi \bar{\phi} \phi + \frac{m_S}{2} S^2 + \frac{m_A}{2} A^2 , \quad (77)
 \end{aligned}$$

and $\Gamma_{[\alpha\beta]} = i(\Gamma_\alpha \Gamma_\beta - \Gamma_\beta \Gamma_\alpha)/2$. As in the case of the SU(5) minimal model, the superpotential is not the most general one, but this does not contradict the philosophy of the coupling unification by the reduction method. W_{SB} is responsible for the SSB of SO(10) down to $SU(3)_C \times SU(2)_W \times U(1)_Y$, and this can be achieved without breaking supersymmetry, while W_{HS} is responsible for the triplet-doublet splitting of H . The right-handed neutrinos obtain a superheavy mass through W_{NM} after the SSB, and the Yukawa couplings for the leptons and quarks are contained in W_Y . We assume that there exists a choice of soft supersymmetry breaking terms so that all the vacuum expectation values necessary for the desired SSB corresponds to the minimum of the potential.

Given the supermultiplet content and the superpotential W , we can compute the β functions of the model. The gauge coupling of SO(10) is denoted by g , and our normalization of the β functions is as usual, *i.e.*, $dg_i/d\ln\mu = \beta_i^{(1)}/16\pi^2 + O(g^5)$, where μ is the renormalization scale. We find:

$$\begin{aligned}
 \beta_g^{(1)} &= 7g^3 , \\
 \beta_{g_T}^{(1)} &= g_T (14|g_T|^2 + \frac{27}{5}|g_{HS}|^2 + |g_{3NM}|^2 - \frac{63}{2}g^2) , \\
 \beta_{g_\phi}^{(1)} &= g_\phi (53|g_\phi|^2 + \frac{48}{5}|g_A|^2 + \frac{1}{2}|g_{1NM}|^2 + \frac{1}{2}|g_{2NM}|^2 + \frac{1}{2}|g_{3NM}|^2 - \frac{77}{2}g^2) , \\
 \beta_S^{(1)} &= g_S (\frac{84}{5}|g_S|^2 + 12|g_A|^2 + \frac{3}{2}|g_{HS}|^2 - 60g^2) , \\
 \beta_A^{(1)} &= g_A (16|g_\phi|^2 + \frac{28}{5}|g_S|^2 + \frac{116}{5}|g_A|^2 + \frac{1}{2}|g_{HS}|^2 - 52g^2) , \\
 \beta_{HS}^{(1)} &= g_{HS} (8|g_T|^2 + \frac{28}{5}|g_S|^2 + 4|g_A|^2 + \frac{113}{10}|g_{HS}|^2 - 38g^2) , \\
 \beta_{1NM}^{(1)} &= g_{1NM} (\frac{45}{2}|g_\phi|^2 + 9|g_{1NM}|^2 + \frac{17}{2}|g_{2NM}|^2 + \frac{17}{2}|g_{3NM}|^2 - \frac{45}{2}g^2) ,
 \end{aligned}$$

$$\begin{aligned}
\beta_{2NM}^{(1)} &= g_{2NM} \left(\frac{45}{2} |g_\phi|^2 + \frac{17}{2} |g_{1NM}|^2 + 9 |g_{2NM}|^2 + \frac{17}{2} |g_{3NM}|^2 - \frac{45}{2} g^2 \right), \\
\beta_{3NM}^{(1)} &= g_{3NM} \left(5 |g_T|^2 + \frac{45}{2} |g_\phi|^2 + \frac{17}{2} |g_{1NM}|^2 + \frac{17}{2} |g_{2NM}|^2 \right. \\
&\quad \left. + 9 |g_{3NM}|^2 - \frac{45}{2} g^2 \right). \tag{78}
\end{aligned}$$

We have assumed that the Yukawa couplings g_{IJ} except for $g_T \equiv g_{33}$ vanish. They can be included as small perturbations. Needless to say that the soft susy breaking terms do not alter the β functions above.

We find that there exist two independent solutions, A and B , that have the most predictive power, where we have chosen the $SO(10)$ gauge coupling as the primary coupling:

$$\begin{aligned}
\rho_T &= \begin{cases} 163/60 & \simeq 2.717 \\ 0 \end{cases}, \quad \rho_\phi = \begin{cases} 5351/9180 & \simeq 0.583 \\ 1589/2727 & \simeq 0.583 \end{cases}, \\
\rho_S &= \begin{cases} 152335/51408 & \simeq 2.963 \\ 850135/305424 & \simeq 2.783 \end{cases}, \quad \rho_A = \begin{cases} 31373/22032 & \simeq 1.424 \\ 186415/130896 & \simeq 1.424 \end{cases}, \\
\rho_{HS} &= \begin{cases} 7/81 & \simeq 0.086 \\ 170/81 & \simeq 2.099 \end{cases}, \quad \rho_{1NM} = \rho_{2NM} = \begin{cases} 191/204 & \simeq 0.936 \\ 191/303 & \simeq 0.630 \end{cases}, \\
\rho_{3NM} &= \begin{cases} 0 \\ 191/303 & \simeq 0.630 \end{cases} \quad \text{for } \begin{cases} A \\ B \end{cases}. \tag{79}
\end{aligned}$$

Clearly, the solution B has less predictive power because $\rho_T = 0$. So, we consider below only the solution A , in which the coupling α_{3NM} should be regarded as a small perturbation because $\rho_{3NM} = 0$.

Given this solution it is possible to show, as in the case of $SU(5)$, that the ρ 's can be uniquely computed in any finite order in perturbation theory.

The corrections to the reduced couplings coming from the small perturbations up to and including terms of $O(\tilde{\alpha}_{3NM}^2)$:

$$\begin{aligned}
\tilde{\alpha}_T &= (163/60 - 0.108 \cdots \tilde{\alpha}_{3NM} + 0.482 \cdots \tilde{\alpha}_{3NM}^2 + \cdots) + \cdots, \\
\tilde{\alpha}_\phi &= (5351/9180 + 0.316 \cdots \tilde{\alpha}_{3NM} + 0.857 \cdots \tilde{\alpha}_{3NM}^2 + \cdots) + \cdots, \\
\tilde{\alpha}_S &= (152335/51408 + 0.573 \cdots \tilde{\alpha}_{3NM} + 5.7504 \cdots \tilde{\alpha}_{3NM}^2 + \cdots) + \cdots, \\
\tilde{\alpha}_A &= (31373/22032 - 0.591 \cdots \tilde{\alpha}_{3NM} - 4.832 \cdots \tilde{\alpha}_{3NM}^2 + \cdots) + \cdots, \\
\tilde{\alpha}_{HS} &= (7/81 - 0.00017 \cdots \tilde{\alpha}_{3NM} + 0.056 \cdots \tilde{\alpha}_{3NM}^2 + \cdots) + \cdots, \\
\tilde{\alpha}_{1NM} &= \tilde{\alpha}_{2NM} = (191/204 - 4.473 \cdots \tilde{\alpha}_{3NM} + 2.831 \cdots \tilde{\alpha}_{3NM}^2 + \cdots) + \cdots, \tag{80}
\end{aligned}$$

where \cdots indicates higher order terms which can be uniquely computed. In the partially reduced theory defined above, we have two independent couplings, α and α_{3NM} (along with the Yukawa couplings α_{IJ} , $I, J \neq T$).

At the one-loop level, Eq. (80) defines a line parametrized by $\tilde{\alpha}_{3NM}$ in the 7 dimensional space of couplings. A numerical analysis shows that this line blows up in the direction of $\tilde{\alpha}_S$ at a finite value of $\tilde{\alpha}_{3NM}$ [14]. So if we require $\tilde{\alpha}_S$ to remain within the perturbative regime (*i.e.*, $g_S \leq 2$, which means $\tilde{\alpha}_S \leq 8$ because $\alpha_{GUT} \sim 0.04$), the $\tilde{\alpha}_{3NM}$ should be restricted to be below ~ 0.067 . As a consequence, the value of $\tilde{\alpha}_T$ is also bounded

$$2.714 \leq \tilde{\alpha}_T \leq 2.736 . \quad (81)$$

This defines GYU boundary conditions holding at the unification scale M_{GUT} in addition to the group theoretic one, $\alpha_T = \alpha_t = \alpha_b = \alpha_\tau$. The value of $\tilde{\alpha}_T$ is practically fixed so that we may assume that $\tilde{\alpha}_T = 163/60 \simeq 2.72$, which is the unperturbed value.

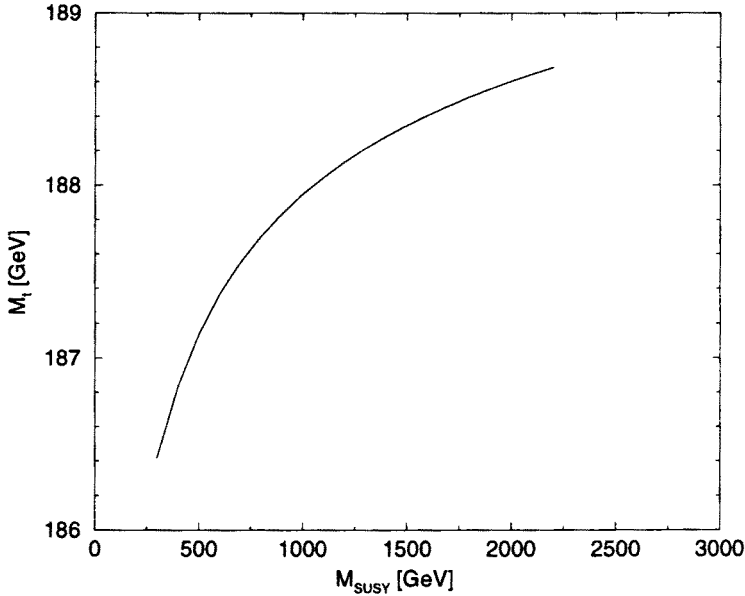


Fig. 3. M_t prediction versus M_{SUSY} for $\tilde{\alpha}_T = 2.717$.

Figure 3 shows the prediction for M_t in this model for different values of the supersymmetry breaking scale M_{SUSY} . It is worth noticing that the value for M_t predicted is below its infrared value ($M_{top-IR} \sim 189$ GeV) [14], but it is slightly above the recent experimental values (70).

10. Conclusions

As a natural extension of the unification of gauge couplings provided by all GUTs and the unification of Yukawa couplings, we have introduced the idea of Gauge–Yukawa Unification. GYU is a functional relationship among the gauge and Yukawa couplings provided by some principle. In our studies GYU has been achieved by applying the principles of reduction of couplings and finiteness. The consequence of GYU is that in the lowest order in perturbation theory the gauge and Yukawa couplings above M_{GUT} are related in the form

$$g_i = \kappa_i g_{\text{GUT}}, \quad i = 1, 2, 3, e, \dots, \tau, b, t, \quad (82)$$

where g_i ($i = 1, \dots, t$) stand for the gauge and Yukawa couplings, g_{GUT} is the unified coupling, and we have neglected the Cabibbo–Kobayashi–Maskawa mixing of the quarks. So, Eq. (82) exhibits a set of boundary conditions on the renormalization group evolution for the effective theory below M_{GUT} , which we have assumed to be the MSSM. We have shown [15, 16] that it is possible to construct some supersymmetric GUTs with GYU in the third generation that can predict the bottom and top quark masses in accordance with the recent experimental data [53]. This means that the top–bottom hierarchy could be explained in these models, in a similar way as the hierarchy of the gauge couplings of the SM can be explained if one assumes the existence of a unifying gauge symmetry at M_{GUT} .

It is clear that the GYU scenario is the most predictive scheme as far as the mass of the top quark is concerned. It may be worth recalling the predictions for M_t of ordinary GUTs, in particular of supersymmetric SU(5) and SO(10). The MSSM with SU(5) Yukawa boundary unification allows M_t to be anywhere in the interval between 100–200 GeV for varying $\tan \beta$, which is now a free parameter. Similarly, the MSSM with SO(10) Yukawa boundary conditions, *i.e.* $t - b - \tau$ Yukawa Unification gives M_t in the interval 160–200 GeV. We have analyzed [16] the infrared quasi-fixed-point behaviour of the M_t prediction in some detail. In particular we have seen that the *infrared value* for large $\tan \beta$ depends on $\tan \beta$ and its lowest value is ~ 188 GeV. Comparing this with the experimental value (70) we may conclude that the present data on M_t cannot be explained from the infrared quasi-fixed-point behaviour alone (see Figure 4).

Clearly, to exclude or verify different GYU models, the experimental as well as theoretical uncertainties have to be further reduced. One of the largest theoretical uncertainties in FUT results from the not-yet-calculated threshold effects of the superheavy particles. Since the structure of the superheavy particles is basically fixed, it will be possible to bring these threshold effects under control, which will reduce the uncertainty of the

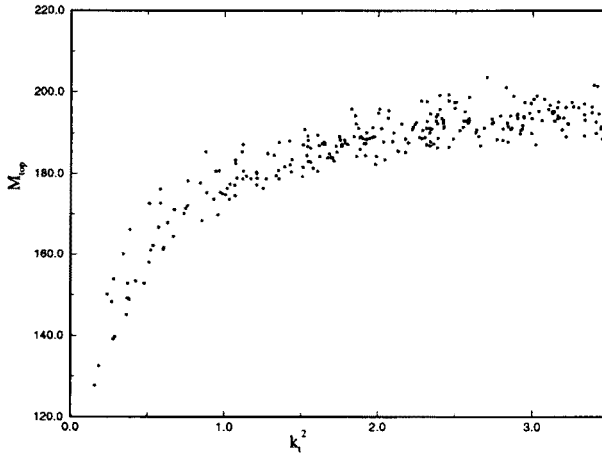


Fig. 4. The dependence of the top mass M_t with k_t^2 , at fixed $M_{\text{SUSY}} = 500$ GeV. As we can see, after $k_t^2 \sim 2.0$ the top mass goes to its infrared fixed point value.

M_t prediction. We have been regarding $\delta^{\text{MSSM}} M_t$ as unknown because we do not have sufficient information on the superpartner spectra. Recently, however, we have demonstrated [63] how to extend the principle of reduction of couplings in a way as to include the dimensionfull parameteres. As a result, it is in principle possible to predict the superpartner spectra as well as the rest of the massive parameters of a theory.

One of us (G.Z.) would like to thank the Organizing Committees of the Schools for their warm hospitality.

REFERENCES

- [1] J.C. Pati, A. Salam, *Phys. Rev. Lett.* **31**, 661 (1973).
- [2] H. Georgi, S.L. Glashow, *Phys. Rev. Lett.* **32**, 438 (1974); H. Georgi, H. Quinn, S. Weinberg, *Phys. Rev. Lett.* **33**, 451 (1974).
- [3] H. Fritzsch, P. Minkowski, *Ann. Phys.* **93**, 193 (1975); H. Georgi, in *Particles and Fields - 1974*, ed. C.E. Carlson, American Institute of Physics, New York.
- [4] A. Buras, J. Ellis, M.K. Gaillard, D. Nanopoulos, *Nucl. Phys.* **B135**, 66 (1978).

- [5] U. Amaldi, W. de Boer, H. Fürstenau, *Phys. Lett.* **B260**, 447 (1991).
- [6] R. Decker, J. Pestieau, *Lett. Nuovo Cim.* **29**, 560 (1980), M. Veltman, *Acta Phys. Pol.* **B12**, 437 (1981).
- [7] S. Ferrara, L. Girardello, F. Palumbo, *Phys. Rev.* **D20**, 403 (1979).
- [8] B. Pendleton, G.G. Ross, *Phys. Lett.* **B98**, 291 (1981).
- [9] W. Zimmermann, *Phys. Lett.* **B308**, 117 (1993).
- [10] C.T. Hill, *Phys. Rev.* **D24** (1981) 691; W.A. Bardeen, C.T. Hill, M. Lindner, *Phys. Rev.* **D411**, 1647 (1990).
- [11] D. Kapetanakis, M. Mondragón, G. Zoupanos, *Z. Phys.* **C60**, 181 (1993); M. Mondragón, G. Zoupanos, *Nucl. Phys. B* (Proc. Suppl.) **37C**, 98 (1995).
- [12] J. Kubo, M. Mondragón, G. Zoupanos, *Nucl. Phys.* **B424**, 291 (1994).
- [13] J. Kubo, M. Mondragón, N.D. Tracas and G. Zoupanos, *Phys. Lett.* **B342**, 155 (1995).
- [14] J. Kubo, M. Mondragón, S. Shoda, G. Zoupanos, *Nucl. Phys.* **B469**, 3 (1996).
- [15] J. Kubo, M. Mondragón, M. Olechowski, G. Zoupanos, Gauge-Yukawa Unification and the Top-Bottom Hierarchy, Proc. of the *Int. Europhysics Conf. on HEP*, Brussels 1995.
- [16] J. Kubo, M. Mondragón, M. Olechowski, G. Zoupanos, Testing Gauge-Yukawa-Unified Model by M_t , hep-ph/9512435, to be published in *Nucl. Phys. B*.
- [17] T.P. Cheng, E. Eichten, L.F. Li, *Phys. Rev.* **D9**, 2259 (1974); N.P. Chang, *Phys. Rev.* **D10**, 2706 (1974); E. Ma, *Phys. Rev.* **D17**, 623 (1978); *Phys. Rev.* **D31**, 1143 (1985).
- [18] W. Zimmermann, *Com. Math. Phys.* **97**, 211 (1985); R. Oehme, W. Zimmermann *Com. Math. Phys.* **97**, 569 (1985); R. Oehme, K. Sibold, W. Zimmermann, *Phys. Lett.* **B147**, 117 (1984); **B153**, 142 (1985); R. Oehme, *Prog. Theor. Phys. Suppl.* **86**, 215 (1986).
- [19] J. Kubo, K. Sibold, W. Zimmermann, *Nucl. Phys.* **B259**, 331 (1985); *Phys. Lett.* **B200**, 185 (1989).
- [20] J. Kubo, *Phys. Lett.* **B262**, 472 (1991).
- [21] J. Kubo, K. Sibold, W. Zimmermann, *Phys. Lett.* **B200**, 185 (1989).
- [22] A.J. Parkes, P.C. West, *Phys. Lett.* **B138** (1984) 99; *Nucl. Phys.* **B256**, 340 (1985); P. West, *Phys. Lett.* **B137**, 371 (1984); D.R.T. Jones, A.J. Parkes, *Phys. Lett.* **B160**, 267 (1985); D.R.T. Jones, L. Mezincescu, *Phys. Lett.* **B136**, 242 (1984); **B138**, 293 (1984); A.J. Parkes, *Phys. Lett.* **B156**, 73 (1985).
- [23] S. Hamidi, J. Patera, J.H. Schwarz, *Phys. Lett.* **B141**, 349 (1984); X.D. Jiang, X.J. Zhou, *Phys. Lett.* **B197**, 156 (1987); **B216**, 160 (1985).
- [24] D.R.T. Jones, L. Mezincescu, Y.-P. Yao, *Phys. Lett.* **B148**, 317 (1984).
- [25] I. Jack, D.R.T. Jones, *Phys. Lett.* **B333**, 372 (1994).
- [26] S. Hamidi, J.H. Schwarz, *Phys. Lett.* **B147**, 301 (1984); D.R.T. Jones, S. Raby, *Phys. Lett.* **B143**, 137 (1984); J.E. Bjorkman, D.R.T. Jones, S. Raby *Nucl. Phys.* **B259**, 503 (1985).
- [27] J. León et al, *Phys. Lett.* **B156**, 66 (1985).

- [28] W. Zimmermann, *Phys. Lett.* **B311**, 249 (1993)
- [29] S. Dimopoulos, H. Georgi, *Nucl. Phys.* **B193**, 150 (1981); N. Sakai, *Z. Phys.* **C11**, 153 (1981).
- [30] L. O’Raifeartaigh, *Nucl. Phys.* **B96**, 331 (1975).
- [31] P. Fayet, J. Iliopoulos, *Phys. Lett.* **B51**, 461 (1974).
- [32] J. Wess, B. Zumino, *Phys. Lett.* **B49**, 52 (1974); J. Iliopoulos, B. Zumino, *Nucl. Phys.* **B76**, 310 (1974); S. Ferrara, J. Iliopoulos, B. Zumino, *Nucl. Phys.* **B77**, 413 (1974); K. Fujikawa, W. Lang, *Nucl. Phys.* **B88**, 61 (1975).
- [33] C. Lucchesi, O. Piguet, K. Sibold, *Helv. Phys. Acta* **61**, 321 (1988); *Phys. Lett.* **B201**, 241 (1988).
- [34] O. Piguet, K. Sibold, *Int. Journ. Mod. Phys.* **A1**, 913 (1986); *Phys. Lett.* **B177**, 373 (1986); R. Ensign, K.T. Mahanthappa, *Phys. Rev.* **D36**, 3148 (1987).
- [35] C. Lucchesi, G. Zoupanos, All Order Finiteness in $N = 1$ SYM Theories: Criteria and Applications, hep-ph/9604216.
- [36] O. Piguet, Supersymmetry, Ultraviolet Finiteness and Grand Unification, hep-th/9606045.
- [37] L. Alvarez-Gaumé, P. Ginsparg, *Nucl. Phys.* **B243**, 449 (1984); W.A. Bardeen, B. Zumino, *Nucl. Phys.* **B243**, 421 (1984); B. Zumino, Y. Wu, A. Zee, *Nucl. Phys.* **B439**, 477 (1984).
- [38] S.L. Adler, W.A. Bardeen, *Phys. Rev.* **182**, 1517 (1969).
- [39] S. Ferrara, B. Zumino, *Nucl. Phys.* **B87**, 207 (1975).
- [40] O. Piguet, K. Sibold, *Nucl. Phys.* **B196**, 428 (1982); **B196**, 447 (1982).
- [41] O. Piguet, K. Sibold, *Renormalized Supersymmetry*, Birkhäuser, Boston 1986.
- [42] N. Deshpande, Xiao-Gang, He, E. Keith, *Phys. Lett.* **B332**, 88 (1994).
- [43] H. Arason *et al.*, *Phys. Rev.* **D46**, 3945 (1992); V. Barger, M.S. Berger, P. Ohmann, *Phys. Rev.* **D47**, 1093 (1993), and references therein.
- [44] L. Hall, R. Rattazzi, U. Sarid, *Phys. Rev.* **D50**, 7048 (1994); M. Carena, M. Olechowski, S. Pokorski, C.E.M. Wagner, *Nucl. Phys.* **B426**, 269 (1994).
- [45] B.D. Wright, Yukawa Coupling Thresholds: Application to the MSSM and the Minimal Supersymmetric SU(5) GUT, University of Wisconsin-Madison report, MAD/PH/812 (hep-ph/9404217).
- [46] Particle Data Group, L. Montanet *et al.*, *Phys. Rev.* **D50**, 1173 (1994).
- [47] P.H. Chankowski, Z. Plucienik, S. Pokorski, *Nucl. Phys.* **B439**, 23 (1995).
- [48] P. Langacker, N. Polonsky, *Phys. Rev.* **D47**, 4028 (1993).
- [49] F.M. Borzumati, M. Olechowski, S. Pokorski, *Phys. Lett.* **B349**, 311 (1995); H. Murayama, M. Olechowski, S. Pokorski, Viable $t - b - \tau$ Yukawa Unification in SUSY SO(10), MPI preprint No. MPI-PhT/95-100, hep-ph/9510327.
- [50] D. Pierce, in Proc. of SUSY 94, Ann Arbor, Michigan, 1994, eds. C. Kolda and J. Wells, p.418; A. Donini, One-loop Corrections to the Top, Stop and Gluino Masses in the MSSM, CERN report, CERN-TH-95-287 (hep-ph/9511289); J. Feng, N. Polonsky, S. Thomas, The Light Higgsino-Gaugino Window, University of Munich report, LMU-TPW-95-18.

- [51] W.A. Bardeen, M. Carena, S. Pokorski, C.E.M. Wagner, *Phys. Lett.* **B320**, 110 (1994); M. Carena, M. Olechowski, S. Pokorski, C.E.M. Wagner, *Nucl. Phys.* **B419**, 213 (1994).
- [52] J. Hisano, H. Murayama, T. Yanagida, *Phys. Rev. Lett.* **69**, 1992 (1993); J. Ellis, S. Kelley, D. V. Nanopoulos, *Nucl. Phys.* **B373**, 55 (1992); Y. Yamada, *Z. Phys.* **C60**, 83 (1993).
- [53] J. Lys, The CDF Collaboration, FERMILAB-CONF-96/409-E, Proceedings of the *28th International Conference on High Energy Physics (ICHEP'96)*; S. Protopopescu, D0 Collaboration, Proceedings of the 28th International Conference on High Energy Physics (ICHEP'96), Warsaw, Poland, July 25-31, 1996.
- [54] J. Kubo, M. Mondragón, G. Zoupanos, The Top-Bottom Hierarchy from Gauge-Yukawa Unification, to be published in the Proc. of the International Seminar '95, August 21-25, 1995 Kyoto.
- [55] J. Hisano, H. Murayama, T. Yanagida, *Nucl. Phys.* **B402**, 46 (1993).
- [56] I. Antoniadis, G.K. Leontaris, *Phys. Lett.* **B216**, 333 (1989); G. Leontaris, N. Tracas, *Z. Phys.* **C56**, 479 (1992); *Phys. Lett.* **B291**, 44 (1992).
- [57] H. Georgi, C. Jarlskog, *Phys. Lett.* **B86**, 297 (1979).
- [58] R. N. Mohapatra, Left-Right Symmetric Models of Weak Interactions in Proc. of a NATO ASI on Quarks, Leptons, and Beyond, September 5-16, 1983, Munich, eds. H. Fritsch *et al.* Plenum Press, New York 1983.
- [59] V. Barger *et al.*, *Phys. Lett.* **B313**, 351 (1993); M. Carena, S. Pokorski, C. Wagner, *Nucl. Phys.* **B406**, 59 (1993).
- [60] B. Schrempp, F. Schrempp, *Phys. Lett.* **B299**, 321 (1993); B. Schrempp, *Phys. Lett.* **B344**, 193 (1995); B. Schrempp, W. Wimmer, Top quark and Higgs boson masses: interplay between infrared and ultraviolet physics, DESY-960109 preprint, hep-ph/9606386.
- [61] A.V. Ermushev, D.I. Kazakov, O.V. Tarasov, *Nucl. Phys.* **B281**, 72 (1987); D.I. Kazakov, *Mod. Phys. Lett.* **A2**, 663 (1987); *Phys. Lett.* **B179**, 352 (1986); D.I. Kazakov, I.N. Kondrashuk, Low-energy predictions of Susy GUTs: minimal versus finite model, preprint E2-91-393.
- [62] I. Antoniadis, C. Kounnas, K. Tamvakis, *Phys. Lett.* **B119**, 377 (1982); A. Schüler, S. Sakakibatra, J.G. Körner, *Phys. Lett.* **B194**, 125 (1987).
- [63] J. Kubo, M. Mondragón, G. Zoupanos, Perturbative Unification of Soft Susy Breaking Terms, Preprint MPI-PhT/96-71, hep-ph/9609218; see also references therein. To be published in *Phys. Lett.* **B**