AN INTRODUCTION TO THE WORLDLINE TECHNIQUE FOR QUANTUM FIELD THEORY CALCULATIONS* **

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(Received September 27, 1996)

These two lectures give a pedagogical introduction to the "string-inspired" worldline technique for perturbative calculations in quantum field theory. This includes an overview over the present range of its applications. Several examples are calculated in detail, up to the three-loop level. The emphasis is on photon scattering in quantum electrodynamics.

PACS numbers: 11.15. Kc, 12.20. Ds

Introduction

Many one-loop amplitudes in relativistic quantum field theory admit representations in terms of path integrals over the space of closed loops [1–7]. Those path integrals are closely related to the Feynman path integrals used in nonrelativistic quantum mechanics, though they have perhaps not reached quite the same practical importance. Still there exists a considerable amount of literature on applications of particle path integrals to quantum field theory (see *e.g.* [8–14]), to which I cannot possibly do justice here.

The subject of these lectures is a rather specific and novel way of evaluating this type of path integral. This method may be called "string–inspired" because it is analogous to calculations in string perturbation theory, and its development was triggered by efforts [15, 16] to use the peculiar organization of string amplitudes to improve on the efficiency of calculations in ordinary quantum field theory. However this is to some extent accidental, and the practical application of this technique requires no knowledge of

* Presented at the XXXVI Cracow School of Theoretical Physics, Zakopane, Poland, June 1–11, 1996.
** Supported by Deutsche Forschungsgemeinschaft.

string theory. The lectures are meant as an introduction for practicioners in quantum field theory. More effort will therefore be spent on explaining the calculation rules of the worldline formalism, than on rigorous derivations.

The first lecture is devoted to the theoretical side. I will first sketch the history of the subject, and then give an overview over the range of one-loop amplitudes in field theory which are presently accessible to this method. I concentrate then on the explanation of an even more recent multiloop generalization of this formalism [38–44].

The second lecture contains four detailed examples of calculations performed with this formalism, mainly taken from quantum electrodynamics:

1. One-loop QED vacuum polarization.
2. One-loop QED photon splitting.
3. The two-loop (scalar and spinor) QED β-functions.
4. The three-loop scalar master integral.

1. First lecture: Theory

1.1. The Bern–Kosower formalism

It has been realized not too long ago that string theory may be used as a guiding principle for finding new and perhaps more efficient parameter integral representations for amplitudes in ordinary quantum field theory. The relevance of string theory for this purpose resides in the well-known fact that, in a sense which can be made precise, string theory contains ordinary field theory in the limit where the tension of the string becomes infinite. In this limit, all massive modes of the string get suppressed, and one remains with the massless modes. Those can be identified with ordinary massless particles such as gauge bosons, gravitons, or massless spin-$\frac{1}{2}$ fermions.

To use this fact for actual calculations of amplitudes may appear an enormous detour. It is worthwhile nevertheless due to the different organization of string amplitudes, and of the different methods used to calculate them. For details on string perturbation theory, the reader may consult [45]; let us mention here just the following qualitative points which will be important later on:

1. Scattering amplitudes in string theory are usually calculated in first, not second quantization.
2. String theory calculations involve a much smaller number of different "Feynman diagrams".

The second point is illustrated in Fig. 1, depicting a two-point "Feynman diagram" for the closed string. In the infinite string tension limit this Riemann surface is squeezed to a graph, though not to a single one — two Feynman diagrams of different topologies emerge. The first actual calculation done along these lines was performed by Metsayev and Tseytlin [46] in 1988, who managed to extract the correct one-loop β-function coefficient for pure Yang–Mills theory from the partition function of an open string propagating in a Yang–Mills background. This yielded also some explanation for a fact which had been noted long before [47], namely that this β-function, when calculated in dimensional regularization, vanishes for $D = 26$, the critical dimension of the open string.

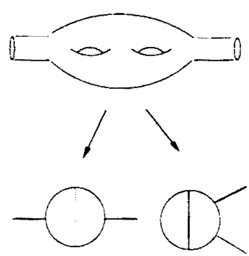

Fig. 1. Infinite string tension limit of a string diagram

A systematic investigation of the infinite string tension limit was undertaken in the following years by Bern and Kosower [15]. This was done again with a view on application to QCD, however now to the calculation of complete on-shell scattering amplitudes. Again the idea was to calculate, say, gauge boson scattering amplitudes in an appropriate (super) string model containing SU(n) gauge theory, up to the point where one has obtained a parameter integral representation for the amplitude considered. At this stage one performs the infinite string tension limit, which should eliminite all contributions due to propagating massive modes, and lead to a parameter integral representation for the corresponding field theory amplitude.

For the calculation of the string scattering amplitude, one may use the Polyakov path integral. In the simplest case, the closed bosonic string propagating in flat spacetime, this is of the form

$$\Big\langle V_1 \cdots V_N \Big\rangle \sim \sum_{\text{top}} \int \mathcal{D}h \int \mathcal{D}x(\sigma,\tau) V_1 \cdots V_N \, \mathrm{e}^{-S[x,h]} \,. \tag{1}$$

Here the $\int \mathcal{D}x(\sigma,\tau)$ is over the space of all embeddings of the string worldsheet with a fixed topology into spacetime. The $\int \mathcal{D}h$ is over the space of all worldsheet metrics, and the sum over topologies $\sum_{\rm top}$ corresponds to the loop expansion in field theory (Fig. 2). $S[x,h]$ is a free gaussian action,

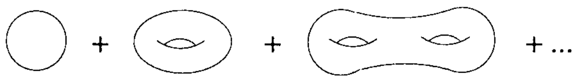

Fig. 2. The loop expansion in string perturbation theory

$$S[x,h] = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \sqrt{h} h^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu \ . \tag{2}$$

Here $\frac{1}{2\pi\alpha'}$ is the string tension, and the vertex operators $V_1,\ldots,V_N$ represent the scattering string states. In the case of the open string, which is the more relevant one for our purpose, the worldsheet has a boundary, and the vertex operators are inserted on this boundary. For instance, at the one-loop level the worldsheet is just an annulus, and a vertex operator may be integrated along either one of the two boundary components.

The vertex operators most relevant for us are of the form

$$V_S = \int d\tau \, \mathrm{e}^{ik\cdot x(\tau)} , \tag{3}$$

$$V_A = \int d\tau \, T^a \varepsilon_\mu \dot{x}^\mu \mathrm{e}^{ik\cdot x(\tau)}. \tag{4}$$

They represent a scalar and a gauge boson with definite momentum and polarization. T^a is a generator of the gauge group in some representation. The integration variable τ parametrizes the boundary in question. As the action is gaussian, $\int \mathcal{D}x$ can be performed by Wick contractions,

$$\Bigl\langle x^\mu(\tau_1)x^\nu(\tau_2)\Bigr\rangle = G(\tau_1,\tau_2)\eta^{\mu\nu} , \tag{5}$$

where G denotes the Green's function for the Laplacian on the annulus, restricted to its boundary.

Performance of the one-loop path integral then leads to the following *Bern–Kosower Master Formula* for the (single-trace partial contribution) to the one-loop N-gluon scattering amplitude in QCD:

$$A_{N;1} \sim (\alpha')^{\frac{N}{2}-2} Z \int_0^\infty \frac{dT}{T} [4\pi T]^{-\frac{D}{2}} e^{-m^2 T} \prod_{i=1}^{N-1} d\tau_i \, \theta(\tau_i - \tau_{i+1})$$
$$\times \exp\Bigg[\sum_{1\le i<j\le N} \alpha' G_{ij} p_i p_j + \sqrt{\alpha'} \dot{G}_{ij} (p_i \varepsilon_j - p_j \varepsilon_i) - \ddot{G}_{ij} \varepsilon_i \varepsilon_j \Bigg] \Big|_{\text{multi-linear}} \,. \tag{6}$$

Here it is understood that only the terms linear in all the polarization vectors $\varepsilon_1, \ldots, \varepsilon_N$ have to be kept. Dots generally denote a derivative acting on the first variable, $\dot{G}(\tau_1, \tau_2) = \frac{\partial}{\partial \tau_1} G(\tau_1, \tau_2)$, and we abbreviate $G_{ij} := G(\tau_i, \tau_j)$ etc. D is the spacetime dimension, and Z denotes the partition function of the free string. The T-integral is a remnant of the path integral over the space of all metrics.

The main point to be noted is that this formula serves as a unifying generating functional for all one-loop N-point gluon amplitudes — something for which no known analogue exists in standard field theory.

The analysis of the infinite string tension limit, for which I refer the reader to [15, 16], proceeds differently for the cases of a spin-0, spin-$\frac{1}{2}$, or spin-1 particle circulating in the loop. The result of this analysis can be summarized in the "Bern–Kosower Rules", which allow one to construct the final integral representations without referring to string theory any more. The relation of those parameter integrals to the ones originating from the corresponding Feynman diagram calculations has been clarified in [17]. Concerning the advantage of the Bern–Kosower rules over the Feynman rules, we mention the following points:

1. Superior organization of gauge invariance.
2. Absence of loop momentum, which reduces the number of kinematic invariants from the beginning, and allows for a particularly efficient use of the spinor helicity method.
3. The method combines nicely with spacetime supersymmetry.

This has been demonstrated in both gluon [18] and graviton [19, 20] scattering calculations.

However, we are interested here mainly in a different property of this formalism. To demonstrate this property, it suffices to consider the simplest possible case of a Bern–Kosower type formula, the one-loop N-point amplitude for ϕ^3-theory. For this case, which was already considered in [5], one has (up to a coupling constant factor)

$$\Gamma^{(1)}(p_1,\dots,p_N)=\int_0^\infty\frac{dT}{T}[4\pi T]^{-\frac{D}{2}}\prod_{i=1}^{N}\int_0^T d\tau_i\exp\Big[\sum_{1\le i<j\le N}G_B(\tau_i,\tau_j)p_ip_j\Big], \tag{7}$$

where

$$G_B(\tau_i,\tau_j)=\mid\tau_i-\tau_j\mid-\frac{(\tau_i-\tau_j)^2}{T}. \tag{8}$$

This parameter integral can, for any given fixed ordering of the integration variables $\tau_{i_1}>\tau_{i_2}>\cdots>\tau_{i_N}$, easily be identified with the corresponding Feynman parameter integral [15, 27] (the usual α_i-parameters just correspond to the differences $\tau_{i_k}-\tau_{i_{k+1}}$). The complete integral therefore does not represent any particular Feynman diagram, with a fixed ordering of the external legs, but *the sum of them* (Fig. 3):

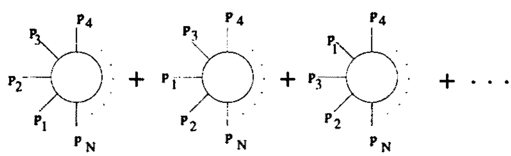

Fig. 3. Sum of one-loop diagrams in ϕ^3 – theory.

This may not seem particularly relevant at the one-loop level. However Eq.(7) holds off-shell, so that one can sew together, say, legs number 1 and N, and obtain the following sum of two-loop $(N-2)$-point diagrams (Fig. 4)

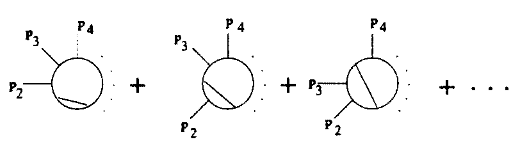

Fig. 4. Sum of two-loop diagrams with different topologies

Now this is obviously interesting, as we have at hand a single integral formula for a sum containing diagrams of different topologies. We may think of this as a remnant of the "less fragmented" nature of string perturbation theory mentioned before (Fig. 1).

Moreover, it calls certain well-known cancellations to mind which happen in gauge theory due to the fact that the Feynman diagram calculation splits a gauge invariant amplitude into gauge non-invariant pieces. For instance, to obtain the 3-loop β-function coefficient for quenched QED, one needs to calculate the sum of diagrams shown in Fig. 5.

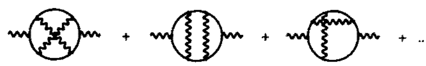

Fig. 5. Sum of diagrams contributing to the 3-loop QED β-function.

Performing this calculation in, say, dimensional regularization, one finds that

(i) All poles higher than $1/\varepsilon$ cancel.

(ii) Individual diagrams give contributions to the β-function proportional to $\zeta(3)$ which cancel in the sum.

The first property is known to hold to all orders of perturbation theory [48], and according to recent knot-theoretical arguments [49] which link both properties, the same should then also be true for the second one.

It seems therefore very natural to apply the Bern–Kosower formalism to this type of calculation. However, in its original version the Bern–Kosower formalism was confined to tree-level and one-loop amplitudes. A multiloop generalization along the original lines has not yet been constructed, though partial results exist [21, 22]. In the absence of better ideas one could, of course, always start at the one-loop level, and pursue the explicit sewing procedure indicated above. This turns out to be unnecessarily clumsy, though; there is a more efficient way of inserting propagators, which will be explained in the third part of this lecture. It is based on a purely field-theoretical approach to the Bern–Kosower formalism, which was, at the one-loop level, proposed by Strassler [23]. This approach uses a representation of one-loop amplitudes in terms of first-quantized worldline path integrals, of a well-known type. I will first demonstrate this method for the example of photon scattering in scalar QED.

The one-loop effective action for a Maxwell background induced by a scalar loop may be written in terms of the following first-quantized particle path integral (see again [5]):

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int \mathcal{D}x \exp\left[-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2 + ieA_\mu\dot{x}^\mu\right)\right]. \tag{9}$$

In this formula, T is the usual Schwinger proper–time parameter. At fixed proper-time T, we have a path integral over the space of periodic functions $x^\mu(\tau)$ with period T, describing all possible embeddings of the circle with circumference T into spacetime. We will generally use dimensional regularization, and therefore work in D dimensions from the beginning. The spacetime metric is taken Euclidean.

The essence of the method is simple: We will evaluate this path integral in a one-dimensional perturbation theory. If we expand the "interaction exponential",

$$\exp\Big[-\int_0^T d\tau\, ieA_\mu\dot{x}^\mu\Big] = \sum_{N=0}^{\infty}\frac{(-ie)^N}{N!}\prod_{i=0}^{N}\int_0^T d\tau_i\Big[\dot{x}^\mu(\tau_i)A_\mu(x(\tau_i))\Big], \tag{10}$$

the individual terms correspond to Feynman diagrams describing a fixed number of interactions of the scalar loop with the external field (Fig. 6):

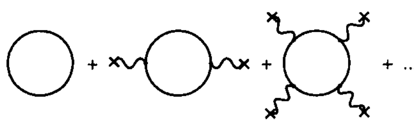

Fig. 6. Expanding the path integral in powers of the background field

The corresponding N-photon scattering amplitude is then obtained by specializing to a background consisting of a sum of plane waves with definite polarizations,

$$A_\mu(x) = \sum_{i=1}^{N}\varepsilon_\mu^{(i)}\mathrm{e}^{ik_i\cdot x}, \tag{11}$$

and picking out the term containing every $\varepsilon^{(i)}$ once (this also removes the $\frac{1}{N!}$ in Eq.(10)). We find thus exactly the same photon vertex operator used in string perturbation theory, Eqs. (3),(4), inserted on a circle instead on the boundary of the annulus. But in the infinite string tension limit the annulus gets squeezed into a circle; we may therefore think of the path integral Eq. (9) as the infinite string tension limit of the corresponding Polyakov path integral described above. Again the evaluation of the path integral at fixed values of the parameters reduces to Wick-contracting

$$\Big\langle \dot{x}_1^{\mu_1}\mathrm{e}^{ik_1\cdot x_1}\cdots\dot{x}_N^{\mu_N}\mathrm{e}^{ik_N\cdot x_N}\Big\rangle\,. \tag{12}$$

The Green's function to be used is now simply the one for the second-derivative operator, acting on periodic functions. To derive this Green's function, first observe that $\int\mathcal{D}x(\tau)$ contains the constant functions, which we must get rid of to obtain a well-defined inverse. Let us therefore restrict our integral over the space of all loops by fixing the average position x_0^μ of the loop:

$$\int \mathcal{D}x = \int dx_0 \int \mathcal{D}y, \tag{13}$$

$$x^\mu(\tau) = x_0^\mu + y^\mu(\tau), \tag{14}$$

$$\int_0^T d\tau\, y^\mu(\tau) = 0\,. \tag{15}$$

For effective action calculations this reduces the effective action to the effective Lagrangian,

$$\Gamma = \int d^D x_0\, \mathcal{L}(x_0). \tag{16}$$

In scattering amplitude calculations, the integral over x_0 just gives momentum conservation,

$$\int d^D x_0\, \mathrm{e}^{ix_0 \cdot \sum_{i=1}^N k_i} = (2\pi)^D \delta\Bigl(\sum_{i=1}^N k_i\Bigr)\,. \tag{17}$$

The reduced path integral $\int \mathcal{D}y(\tau)$ has an invertible kinetic operator. This inverse is easily constructed using the eigenfunctions of the derivative operator on the circle with circumference T, $\{\mathrm{e}^{2\pi i n \frac{\tau}{T}}, n \in \mathbf{Z}\}$, and leads to

$$2\langle \tau_1 \mid \Bigl(\frac{d}{d\tau}\Bigr)^{-2} \mid \tau_2 \rangle = 2T \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{(2\pi i n)^2} \mathrm{e}^{2\pi i n \frac{\tau_1 - \tau_2}{T}} = G_B(\tau_1, \tau_2) - \frac{T}{6}. \tag{18}$$

Here G_B is the function which was introduced already in Eq. (8). The constant $-T/6$ drops out of all calculations once momentum conservation is applied, and is therefore usually deleted at the beginning.

Working out Eq. (12) by Wick-contractions using the function G_B leads to a parameter integral representation for the one-loop N-photon scattering amplitude identical with the one encoded in the Bern–Kosower master formula for this special case. We will look at this parameter integral in more detail later on.

Alternatively, one can also apply this method for calculating the higher derivative expansion of the effective action $\Gamma(A)$ [24–26]. This can be done in a manifestly gauge invariant way using Fock–Schwinger gauge centered at x_0.

1.2. One-loop wordline path integrals and correlators

The representation of amplitudes in quantum field theory in terms of first-quantized particle path integrals is an old and well-studied subject, and I cannot possibly survey the literature here. Let me just mention a few contributions which were particularly important for the development of the subject:

1. R.P. Feynman 1950 [1] (path integral representation of the scalar propagator).
2. E.S. Fradkin 1967 [2] (path integral representation of the electron propagator).
3. Berezin and Marinov 1977 [50], L. Brink, P. Di Vecchia and P. Howe 1977 [51] (superparticle Lagrangians).
4. L. Alvarez-Gaumé 1983 [10] (application of superparticle path integrals to anomaly calculations).

The following overview is restricted to those cases which are presently relevant for the "string-inspired" technique.

The free Gaussian path integral

With our conventions, the free coordinate path integral is

$$\int \mathcal{D}y \exp\left[-\int_0^T \left(\frac{\dot{y}^2}{4}\right)\right] = \left(4\pi T\right)^{-D/2} . \tag{19}$$

All other free path integrals are normalized to unity.

Scalar field theory with a self-interaction potential $V(\phi)$

For a self-interacting (real) scalar field, the path integral representation of the one-loop effective action analogous to Eq.(9) reads

$$\Gamma[\phi] = \frac{1}{2}\int_0^\infty \frac{dT}{T}\, \mathrm{e}^{-m^2T} \int \mathcal{D}x \exp\left[-\int_0^T \left(\frac{\dot{x}^2}{4} + V''\left(\phi(x)\right)\right)\right] . \tag{20}$$

For example, for $V(\phi) = \frac{\lambda}{3!}\phi^3$ one has an interaction exponential

$$\exp\left[-\lambda \int_0^T d\tau \phi(x(\tau))\right] ,$$

leading to a scalar vertex operator V_S as in Eq. (3). Wick-contraction of N such operators, using

$$\langle y^\mu(\tau_1)\, y^\nu(\tau_2)\rangle = -g^{\mu\nu} G_B(\tau_1,\tau_2) \tag{21}$$

then leads to Eq. (7).

Photon scattering in quantum electrodynamics

We have already discussed photon scattering in scalar QED. Eq. (9) generalizes to the Dirac fermion loop as follows (see *e.g.* [5]):

$$\begin{aligned}\Gamma[A] = \; & -2\int_0^\infty \frac{dT}{T} e^{-m^2 T} \int \mathcal{D}x\, \mathcal{D}\psi \\ & \times \exp\left[-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi\dot{\psi} ieA_\mu \dot{x}^\mu ie\psi^\mu F_{\mu\nu}\psi^\nu\right)\right] .\end{aligned} \tag{22}$$

The "bosonic" coordinate path integral $\mathcal{D}y$ is identical with the scalar QED case. The additional "fermionic" $\mathcal{D}\psi$ – integration is over anticommuting Grassmann-valued functions, obeying antiperiodic boundary conditions, $\psi^\mu(T) = -\psi^\mu(0)$. We may think of this double path integral as breaking up a Dirac spinor into a "convective part" and a "spin part". The global factor of (-2) compared to the complex scalar loop case is due to the different statistics and degrees of freedom. The photon vertex operator acquires an additional Grassmann piece,

$$V_A = \int_0^T d\tau \left(\varepsilon_\mu \dot{x}^\mu + 2i\varepsilon_\mu \psi^\mu k_\nu \psi^\nu\right) e^{ik\cdot x(\tau)} . \tag{23}$$

The Grassmann path integral is performed using the Wick contraction rule

$$\begin{aligned}\langle \psi^\mu(\tau_1)\, \psi^\nu(\tau_2)\rangle &= \tfrac{1}{2} g^{\mu\nu} G_F(\tau_1,\tau_2), && (24)\\ G_F(\tau_1,\tau_2) &= \mathrm{sign}(\tau_1,\tau_2). && (25)\end{aligned}$$

Before Wick-contracting two Grassmann fields, they must be made adjacent using the anticommutativity. The worldline action has a "worldline supersymmetry", namely

$$\delta_\eta x^\mu = -2\eta\psi^\mu, \quad \delta_\eta \psi^\mu = \eta\dot{x}^\mu, \tag{26}$$

with some Grassmann constant η. It makes therefore sense to combine y and ψ into a superfield. Defining

$$X^\mu = x^\mu + \sqrt{2}\theta\psi^\mu = x_0^\mu + Y^\mu, \quad D = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial\tau}, \quad \int d\theta\, \theta = 1 , \tag{27}$$

Eq. (22) can be rewritten as a super path integral,

$$\Gamma[A] = -2\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}\int \mathcal{D}X \times\exp\left\{-\int_0^T d\tau\int d\theta\,\left[-\frac{1}{4}X\,D^3X - ieDX^\mu A_\mu(X)\right]\right\}. \quad (28)$$

The two Wick contraction rules are then combined into a single rule for the field Y,

$$\langle Y^\mu(\tau_1,\theta_1)\,Y^\nu(\tau_2,\theta_2)\rangle = -g^{\mu\nu}\,\hat{G}(\tau_1,\theta_1;\tau_2,\theta_2), \quad (29)$$

$$\hat{G}(\tau_1,\theta_1;\tau_2,\theta_2) = G_B(\tau_1,\tau_2) + \theta_1\theta_2 G_F(\tau_1,\tau_2). \quad (30)$$

The photon vertex operator Eq. (23) for the spinor loop is rewritten as

$$V_A = -\int_0^T d\tau d\theta\varepsilon_\mu DX^\mu\exp[ikX]. \quad (31)$$

This superfield reformulation makes no difference for the final integral representations obtained, but is often useful for keeping intermediate expressions compact.

Spinor QED in a constant external field $\bar{F}$

An additional background field $\bar{A}^\mu(x)$, with constant field strength tensor $\bar{F}_{\mu\nu}$, can be completely taken into account by changing the worldline Green's functions, and the path integral determinant [34, 35, 36]. The Green's functions become [36]

$$\begin{aligned}
\mathcal{G}_B(\tau_1,\tau_2) &= \frac{1}{2(e\bar{F})^2}\left(\frac{e\bar{F}}{\sin(e\bar{F}T)}\mathrm{e}^{-ie\bar{F}T\dot{G}_{B12}} + ie\bar{F}\dot{G}_{B12} - \frac{1}{T}\right),\\
\dot{\mathcal{G}}_B(\tau_1,\tau_2) &= \frac{i}{e\bar{F}}\left(\frac{e\bar{F}}{\sin(e\bar{F}T)}\mathrm{e}^{-ie\bar{F}T\dot{G}_{B12}} - \frac{1}{T}\right),\\
\ddot{\mathcal{G}}_B(\tau_1,\tau_2) &= 2\delta_{12} - 2\frac{e\bar{F}}{\sin(e\bar{F}T)}\mathrm{e}^{-ie\bar{F}T\dot{G}_{B12}},\\
\mathcal{G}_F(\tau_1,\tau_2) &= G_{F12}\frac{\mathrm{e}^{-ie\bar{F}T\dot{G}_{B12}}}{\cos(e\bar{F}T)}.
\end{aligned} \quad (32)$$

These expressions should be understood as power series in the field strength matrix $\bar{F}$ (note that Eqs. (32) do *not* assume invertibility of $\bar{F}$). The background field breaks the Lorentz invariance, so that the generalized Green's functions are nontrivial Lorentz matrices in general. Accordingly, the Wick contraction rules Eqs. (21), (24) have to be rewritten as

$$\begin{aligned}\langle y^\mu(\tau_1)y^\nu(\tau_2)\rangle &= -\mathcal{G}_B^{\mu\nu}(\tau_1,\tau_2),\\ \langle \psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle &= \tfrac{1}{2}\mathcal{G}_F^{\mu\nu}(\tau_1,\tau_2).\end{aligned} \tag{33}$$

Again momentum conservation leads to the freedom to subtract from $\mathcal{G}_B$ its constant coincidence limit,

$$\mathcal{G}_B(\tau,\tau) = \frac{1}{2(e\bar{F})^2}\left(e\bar{F}\cot(e\bar{F}T) - \frac{1}{T}\right). \tag{34}$$

To correctly obtain this and other coincidence limits, one has to set

$$\mathrm{sign}(\tau,\tau) = 0,$$

and adopt the general rule that coincidence limits must always be taken *after* derivatives. It follows that, for instance,

$$\dot{G}_B(\tau,\tau) = 0, \quad \dot{G}_B^2(\tau,\tau) = 1. \tag{35}$$

The change of the path integral determinant Eq. (19) induced by the external field is

$$(4\pi T)^{-D/2} \to (4\pi T)^{-D/2}\mathrm{det}^{-1/2}\left[\frac{\sin(e\bar{F}T)}{e\bar{F}T}\right] \qquad \text{(Scalar QED)}, \tag{36}$$

$$(4\pi T)^{-D/2} \to (4\pi T)^{-D/2}\mathrm{det}^{-1/2}\left[\frac{\tan(e\bar{F}T)}{e\bar{F}T}\right] \qquad \text{(Spinor QED)}. \tag{37}$$

Scalar or spinor loop contribution to gluon scattering

The path integrals for the external gluon case differ from Eqs. (9),(22) only by path-ordering of the exponentials, and the addition of a global colour trace. For a given N-gluon amplitude, this trace factors out as $\mathrm{tr}(T^{a_1}\cdots T^{a_N})$, where the T^{a_i} are the colour matrices carried by the gluon vertex operators, Eq. (4). The path-ordering leads to ordered τ-integrals $\int\prod_{i=1}^{N-1} d\tau_i\theta(\tau_i-\tau_{i+1})$ such as in Eq. (6) (translation invariance in τ is used for setting $\tau_N = 0$).

In the fermion loop case, the worldline Lagrangian Eq. (22) now contains a term $\psi^\mu[A_\mu, A_\nu]\psi^\nu$ which, in the component formalism, forces one to introduce an additional two-gluon vertex operator besides Eq. (23) [23]. This is not necessary in the superfield formalism, where the super vertex operator Eq. (31) remains sufficient. Here the only change is that a suitable supersymmetric generalization of the above θ-functions is needed. This is

$$\theta(\hat{\tau}_{ij}) = \theta(\tau_i - \tau_j) + \theta_i\theta_j\delta(\tau_i - \tau_j), \tag{38}$$

where $\hat{\tau}_{ij} \equiv \tau_i - \tau_j + \theta_i\theta_j$. The nonabelian commutator terms above are then generated by the δ-function terms [52].

Gluon loop contribution to gluon scattering

The treatment of the gluon loop case in this formalism is considerably more involved than the scalar and spinor loop cases. This is due to the fact that the consistent coupling of a spin-1 path integral to a spin-1 background requires the introduction of auxiliary degrees of freedom, whose contributions have to be removed later on. I will just write down the appropriate path integral, and refer the reader to Ref. [23, 36] both for its derivation and evaluation. It reads

$$\Gamma[A] = -\frac{1}{2}\lim_{C\to\infty}\int_0^\infty \frac{dT}{T}\exp\Bigl[-CT\Bigl(\frac{D}{2}-1\Bigr)\Bigr]\int_P \mathcal{D}x^\mu\,\frac{1}{2}\Bigl(\int_A - \int_P\Bigr)\mathcal{D}\psi^\mu\mathcal{D}\bar{\psi}^\mu$$
$$\times \mathrm{tr}\,\mathcal{P}\exp\Bigl\{-\int_0^T d\tau\Bigl[\frac{1}{4}\dot{x}^2 + igA_\mu\dot{x}^\mu - \bar{\psi}^\mu\Bigl[(\partial_\tau - C)\delta_{\mu\nu} - 2igF_{\mu\nu}\Bigr]\psi^\nu\Bigr]\Bigr\}. \tag{39}$$

The Grassmann path integral now appears both with antiperiodic ("A") and periodic ("P") boundary conditions. $\mathcal{P}$ denotes the path-ordering operator. This path integral actually describes a whole multiplet of p – forms, $p = 1,\ldots,D$ circulating in the loop; the role of the limit $C \to \infty$ is to suppress all contributions from $p \geq 2$, and the contributions from the zero form cancel out in the combination $\int_A - \int_P$.

Pseudoscalar Yukawa coupling

We consider now a pseudoscalar ϕ interacting with a Dirac fermion ψ via the $\bar{\psi}\phi\gamma_5\psi$-vertex. In contrast to the gauge path integral, the following path integral representation for the one-loop effective action induced for the pseudoscalar field by a spinor loop was found only recently [28]:

$$\Gamma(\phi) = -2\int_0^\infty \frac{dT}{T}\int \mathcal{D}x\,\mathcal{D}\psi\,\mathcal{D}\psi_5\,\mathrm{e}^{-S_{\mathrm{Yps}}[x,\psi,\psi_5]},$$

$$S_{\rm Yps}[x,\psi,\psi_5] = \int_0^T d\tau \left\{ \frac{\dot{x}^2}{4} + \frac{1}{2}\psi\dot{\psi} + \frac{1}{2}\psi_5\dot{\psi}_5 + m^2 + \lambda^2\phi^2(x) - 2i\lambda\psi_5\psi\cdot\partial\phi(x) \right\} . \tag{40}$$

Besides y and ψ there is now a second fermionic field ψ_5. It is a Lorentz scalar, and has the same correlator as ψ:

$$\langle \psi_5(\tau_1)\,\psi_5(\tau_2)\rangle = \tfrac{1}{2}\,G_F(\tau_1,\tau_2)\,. \tag{41}$$

Scalar Yukawa coupling

The case of the scalar-fermion coupling $\bar{\psi}\phi\psi$ differs from the pseudoscalar case only by the following change of the action,

$$S_{Yps} \to S_{Ys} = S_{Yps} + 2\lambda m\phi(x)\,. \tag{42}$$

For the combination of both couplings see [28], for the nonabelian case [29, 30].

To conclude this section, let us mention that the representation of axial vector couplings on the worldline has been studied in [31–33]. A treatment of external fermions along the present lines is still lacking.

1.3. Multiloop generalization

If one wishes to generalize Strassler's approach beyond one loop, obviously one needs to know how the one-loop Green's function G_B, defined on the circle, generalizes to higher order graphs. As a first step in the construction of such "multiloop worldline Green's functions" [38], we will ask the following simple question: How does the Green's function $G_B(\tau_1,\tau_2)$ between two fixed points τ_1,τ_2 on the circle change, if we insert, between two other points τ_a and τ_b, a (scalar) propagator of fixed proper-time length $\bar{T}$ (Fig. 7).

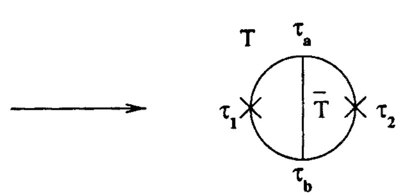

Fig. 7. Change of the one-loop Green's function by a propagator insertion

To answer this question, let us start with the worldline-path integral representation for the one-loop two-point – amplitude in ϕ^3 – theory, and sew together the two external legs. The result is, of course, the vacuum path integral with a propagator insertion

$$\Gamma^{(2)}_{\text{vac}} = \int_0^\infty \frac{dT}{T} \mathrm{e}^{-m^2 T} \int \mathcal{D}x \int_0^T d\tau_a \int_0^T d\tau_b \, \langle \phi(x(\tau_a))\phi(x(\tau_b))\rangle \exp\Bigl[-\int_0^T d\tau \frac{\dot{x}^2}{4}\Bigr] . \tag{43}$$

Here $\langle \phi(x(\tau_a))\phi(x(\tau_b))\rangle$ is the x-space scalar propagator in D dimensions, which, if we specialize to the massless case for a moment, would read

$$\langle \phi(x(\tau_a))\phi(x(\tau_b))\rangle = \frac{\Gamma(D/2-1)}{4\pi^{D/2}\Bigl[(x_a - x_b)^2\Bigr]^{(D/2-1)}} . \tag{44}$$

Clearly this form of the propagator is not suitable for calculations in our auxiliary one-dimensional field theory. We will therefore again make use of the Schwinger proper-time representation,

$$\langle \phi(x(\tau_a))\phi(x(\tau_b))\rangle = \int_0^\infty d\bar{T} e^{-\bar{m}^2 \bar{T}} (4\pi\bar{T})^{-D/2} \exp\Biggl[-\frac{(x(\tau_a) - x(\tau_b))^2}{4\bar{T}}\Biggr] , \tag{45}$$

where the mass was also reinstated. We have then the following path integral representation for the two-loop vacuum amplitude:

$$\begin{aligned} \Gamma^{(2)}_{\text{vac}} &= \int_0^\infty \frac{dT}{T} \mathrm{e}^{-m^2 T} \int_0^\infty d\bar{T} (4\pi\bar{T})^{-D/2} e^{-\bar{m}^2 \bar{T}} \int_0^T d\tau_a \int_0^T d\tau_b \\ &\quad \times \int \mathcal{D}x \, \exp\Biggl[-\int_0^T d\tau \frac{\dot{x}^2}{4} + \frac{(x(\tau_a) - x(\tau_b))^2}{4\bar{T}}\Biggr] . \end{aligned} \tag{46}$$

The propagator insertion has, for fixed parameters $\bar{T}, \tau_a, \tau_b$, just produced an additional contribution to the original free worldline action. Moreover, this term is quadratic in x, so we may hope to absorb it into the free worldline Green's function. For this purpose, it is useful to introduce an integral operator B_{ab} with integral kernel

$$B_{ab}(\tau_1, \tau_2) = \Bigl[\delta(\tau_1 - \tau_a) - \delta(\tau_1 - \tau_b)\Bigr]\Bigl[\delta(\tau_a - \tau_2) - \delta(\tau_b - \tau_2)\Bigr] \tag{47}$$

(B_{ab} acts trivially on Lorentz indices). We may then rewrite

$$(x(\tau_a) - x(\tau_b))^2 = \int_0^T d\tau_1 \int_0^T d\tau_2\, x(\tau_1) B_{ab}(\tau_1,\tau_2) x(\tau_2)\,. \tag{48}$$

Obviously, the presence of the additional term corresponds to changing the defining equation for G_B, Eq. (18), to

$$G_B^{(1)}(\tau_1,\tau_2) = 2\langle \tau_1 \mid \Bigl(\frac{d^2}{d\tau^2} - \frac{B_{ab}}{T}\Bigr)^{-1} \mid \tau_2 \rangle. \tag{49}$$

After eliminating the zero-mode as before, this modified equation can be solved simply by summing a geometric series:

$$\begin{aligned}\Bigl(\frac{d^2}{d\tau^2} - \frac{B_{ab}}{T}\Bigr)^{-1} &= \Bigl(\frac{d}{d\tau}\Bigr)^{-2} + \Bigl(\frac{d}{d\tau}\Bigr)^{-2}\frac{B_{ab}}{T}\Bigl(\frac{d}{d\tau}\Bigr)^{-2} \\ &\quad + \Bigl(\frac{d}{d\tau}\Bigr)^{-2}\frac{B_{ab}}{T}\Bigl(\frac{d}{d\tau}\Bigr)^{-2}\frac{B_{ab}}{T}\Bigl(\frac{d}{d\tau}\Bigr)^{-2} + \cdots,\end{aligned} \tag{50}$$

leading to [38]

$$\begin{aligned}&G_B^{(1)}(\tau_1,\tau_2) = \\ &G_B(\tau_1,\tau_2) + \frac{1}{2}\frac{[G_B(\tau_1,\tau_a) - G_B(\tau_1,\tau_b)][G_B(\tau_a,\tau_2) - G_B(\tau_b,\tau_2)]}{T + G_B(\tau_a,\tau_b)}\,.\end{aligned} \tag{51}$$

The worldline Green's function between points τ_1 and τ_2 is thus simply the one-loop Green's function plus one additional piece, which takes the effect of the insertion into account. Observe that this piece can still be written in terms of the various one-loop Green's functions G_{Bij}. However it is not a function of $\tau_1 - \tau_2$ any more, nor is its coincidence limit a constant.

Knowledge of this Green's function is not quite enough for performing two-loop calculations. We also need to know how the path integral determinant is changed by the propagator insertion. Using the log det=tr log formula, this can again be calculated easily, and yields

$$\begin{aligned}\frac{\int \mathcal{D}y \exp\Bigl[-\int_0^T d\tau \frac{\dot{y}^2}{4} - \frac{(y(\tau_a)-y(\tau_b))^2}{4T}\Bigr]}{\int \mathcal{D}y \exp\Bigl[-\int_0^T d\tau \frac{\dot{y}^2}{4}\Bigr]} &= \frac{\mathrm{Det}'\bigl(\frac{d^2}{d\tau^2} - B_{ab}\bigr)^{-D/2}}{\mathrm{Det}'\bigl(\frac{d^2}{d\tau^2}\bigr)^{-D/2}} \\ &= \Bigl(1 + \frac{G_{Bab}}{T}\Bigr)^{-D/2}.\end{aligned} \tag{52}$$

As usual, the prime denotes the omission of the zero mode from a determinant.

To summarize, the insertion of a scalar propagator into a scalar loop can, for fixed values of the proper-time parameters, be completely taken into account by changing the path integral normalization, and replacing G_B by $G_B^{(1)}$. The vertex operators remain unchanged.

In the case of a photon insertion, instead of Eq. (44) one has, in Feynman gauge,

$$-\frac{e^2}{2}\frac{\Gamma\left(\frac{D}{2}-1\right)}{4\pi^{D/2}}\frac{\dot{x}(\tau_a)\cdot\dot{x}(\tau_b)}{\left([x(\tau_a)-x(\tau_b)]^2\right)^{D/2-1}}\,. \tag{53}$$

The denominator of the propagator is treated as above, *i.e.*exponentiated and absorbed into the worldline Green's function, while the numerator $\dot{x}_a\cdot\dot{x}_b$ remains, and participates in the Wick contractions.

The whole procedure extends without difficulty to the case of m propagator insertions. The determinant factor Eq. (52) generalizes to

$$\left(\det A^{(m)}\right)^{D/2}, \tag{54}$$

and the two-loop Green's function to

$$G_B^{(m)}(\tau_1,\tau_2)=G_B(\tau_1,\tau_2) \\ +\frac{1}{2}\sum_{k,l=1}^{m}[G_B(\tau_1,\tau_{a_k})-G_B(\tau_1,\tau_{b_k})]A_{kl}^{(m)}[G_B(\tau_{a_l},\tau_2)-G_B(\tau_{b_l},\tau_2)]\,. \tag{55}$$

Here $A^{(m)}$ is the symmetric $m\times m$ – matrix defined by

$$\begin{aligned} A^{(m)} &= \left[\bar{T}-\frac{C}{2}\right]^{-1}, \\ \bar{T}_{kl} &= \bar{T}_k\delta_{kl}\,, \\ C_{kl} &= G_B(\tau_{a_k},\tau_{a_l})-G_B(\tau_{a_k},\tau_{b_l})-G_B(\tau_{b_k},\tau_{a_l})+G_B(\tau_{b_k},\tau_{b_l})\,, \end{aligned} \tag{56}$$

and $\bar{T}_1,\dots\bar{T}_m$ denote the proper-time lengths of the inserted propagators.

Note that our formulas give the Green's function only between points on the loop. This suffices, for instance, to calculate single spinor-loop contributions to photon scattering in QED, *i.e.* sums of diagrams as shown in

Fig. 5, or to the analogous (pseudo-) scalar scattering diagrams in models with Yukawa couplings (the generalization to the case of several fermion loops interconnected by photon propagators requires no new concepts [44]).

In ϕ^3-theory or Yang–Mills theory there are, of course, also contributions to the amplitude with external legs on the propagator insertions. The extension of formula (55) involving points located on the insertion was given in [38] for the two–loop case, and obtained for the general case by Roland and Sato [42]. This knowledge then is sufficient to write down worldline representations for all ϕ^3 graphs which have the topology of a loop with insertions. According to graph theory, for the first few orders of perturbation theory such a loop, or "Hamiltonian circuit", can always be found — all trivalent graphs with less than 34 vertices do have this property. ϕ^4 graphs are reduced to ϕ^3 graphs in the usual way by introducing an auxiliary field (which can be again represented on the worldline [44]). A multiloop generalization for Yang–Mills theory still waits to be constructed (some preliminary steps have been taken in [36]).

Roland and Sato also provided a link back to string theory by analyzing the infinite string tension limit of the Green's function $G_B^{RS(m)}$ of the corresponding Riemann surface, and identifying $G_B^{(m)}$ with the leading order term of $G_B^{RS(m)}$ in the $\frac{1}{\alpha'}$ - expansion:

$$G_B^{RS(m)}(z_1,z_2) \quad \overset{\alpha'\to 0}{\longrightarrow} \quad \frac{1}{\alpha'}G_B^{(m)}(\tau_1,\tau_2) + \text{finite}\,. \tag{57}$$

2. Second lecture: Examples

2.1. One-loop QED vacuum polarization

For a warm-up, let us recalculate the one-loop vacuum polarization tensors in scalar and spinor quantum electrodynamics.

A) Scalar QED

According to Eqs. (9–11), this amplitude can be written as

$$\begin{aligned} \Gamma^{\mu\nu}_{\rm scal}[k_1,k_2] \;=\; & (-ie)^2\int_0^\infty \frac{dT}{T}e^{-m^2T}\int \mathcal{D}x\int_0^T d\tau_1\int_0^T d\tau_2 \\ & \times \dot x^\mu(\tau_1)\mathrm{e}^{ik_1\cdot x(\tau_1)}\dot x^\nu(\tau_2)\mathrm{e}^{ik_2\cdot x(\tau_2)}\mathrm{e}^{-\int_0^T d\tau\frac{1}{4}\dot x^2}\,. \end{aligned} \tag{58}$$

Separating off the zero mode according to Eqs. (13), (17), one obtains

$$\Gamma^{\mu\nu}_{\rm scal}[k_1,k_2] = -(2\pi)^D\delta(k_1+k_2)e^2\int_0^\infty\frac{dT}{T}e^{-m^2T}\int_0^T d\tau_1\int_0^T d\tau_2 \times\int\mathcal{D}y\,\dot y^\mu(\tau_1)\mathrm{e}^{ik_1\cdot y(\tau_1)}\dot y^\nu(\tau_2)\mathrm{e}^{ik_2\cdot y(\tau_2)}\mathrm{e}^{-\int_0^T d\tau\frac{1}{4}\dot y^2}\,. \tag{59}$$

We use the energy-momentum conservation factor $(2\pi)^D\delta(k_1+k_2)$ for setting $k_1=-k_2=k$, and then omit it. We now need to Wick-contract the two photon vertex operators, starting from the basic rule

$$\langle y^\mu(\tau_1)y^\nu(\tau_2)\rangle = -g^{\mu\nu}G_B(\tau_1,\tau_2)\,. \tag{60}$$

The rules for Wick-contracting expressions involving both elementary fields and exponentials are the following:

(i) Contract fields with each other as usual, and fields with exponentials according to

$$\Bigl\langle y^\mu(\tau_1)\mathrm{e}^{ik\cdot y(\tau_2)}\Bigr\rangle = i\langle y^\mu(\tau_1)y^\nu(\tau_2)\rangle k_\nu\mathrm{e}^{ik\cdot y(\tau_2)} \tag{61}$$

(the field disappears, the exponential stays in the game).

(ii) Once all elementary fields have disappeared, contraction of the remaining exponentials yields a universal factor

$$\Bigl\langle \mathrm{e}^{ik_1\cdot y_1}\cdots\mathrm{e}^{ik_N\cdot y_N}\Bigr\rangle = \exp\Bigl[-\frac{1}{2}\sum_{i,j=1}^N k_\mu\langle y^\mu(\tau_1)y^\nu(\tau_2)\rangle k_\nu\Bigr]\,. \tag{62}$$

For the case at hand this produces two terms,

$$\Bigl\langle \dot y^\mu(\tau_1)\mathrm{e}^{ik\cdot y(\tau_1)}\dot y^\nu(\tau_2)\mathrm{e}^{-ik\cdot y(\tau_2)}\Bigr\rangle = \Bigl\{g^{\mu\nu}\ddot G_{B12}-k^\mu k^\nu\dot G^2_{B12}\Bigr\}\mathrm{e}^{-k^2G_{B12}}\,. \tag{63}$$

Now one could just write out G_B and its derivatives,

$$\dot G_B(\tau_1,\tau_2) = \mathrm{sign}(\tau_1-\tau_2)-2\frac{(\tau_1-\tau_2)}{T}\,, \tag{64}$$

$$\ddot G_B(\tau_1,\tau_2) = 2\delta(\tau_1-\tau_2)-\frac{2}{T}\,, \tag{65}$$

and calculate the parameter integrals. It is useful, though, to first perform a partial integration in the variable τ_1 or τ_2. The integrand then turns into

$$\left\{g^{\mu\nu}k^2 - k^\mu k^\nu\right\}\dot{G}^2_{B12}\mathrm{e}^{-k^2 G_{B12}} . \tag{66}$$

Note that this makes the gauge invariance of the vacuum polarization manifest. We rescale to the unit circle, $\tau_i = Tu_i, i = 1,2$, and use translation invariance in τ to fix the zero to be at the location of the second vertex operator. We have then

$$G_B(\tau_1, 0) = Tu_1(1-u_1), \dot{G}_B(\tau_1,\tau_2) = 1 - 2u_1. \tag{67}$$

Taking the free determinant factor Eq. (19) into account, and performing the global proper-time integration, one finds

$$\begin{aligned}
\Omega^{\mu\nu}_{\mathrm{scal}}[k] &= e^2\left[g^{\mu\nu}k^2 - k^\mu k^\nu\right]\int_0^\infty \frac{dT}{T}e^{-m^2T}(4\pi T)^{-D/2}T^2 \\
&\quad\times\int_0^1 du(1-2u)^2\mathrm{e}^{-Tu(1-u)k^2} \\
&= \frac{e^2}{(4\pi)^{D/2}}\left[g^{\mu\nu}k^2 - k^\mu k^\nu\right]\Gamma\left(2-\tfrac{D}{2}\right) \\
&\quad\times\int_0^1 du(1-2u)^2\left[m^2 + u(1-u)k^2\right]^{D/2-2} .
\end{aligned} \tag{68}$$

It is easy to verify that this agrees with the result reached by calculating the sum of the corresponding two field theory diagrams in dimensional regularization.

B) Spinor QED

For the fermion loop, the path integral for the two-photon amplitude becomes, in the component formalism,

$$\begin{aligned}
\Gamma^{\mu\nu}_{\mathrm{spin}}[k_1,k_2] &= -2(-ie)^2\int_0^\infty\frac{dT}{T}e^{-m^2T}\int\mathcal{D}x\int\mathcal{D}\psi\int_0^T d\tau_1\int_0^T d\tau_2 \\
&\quad\left(\dot{x}_1^\mu + 2i\psi_1^\mu\psi_1\cdot k_1\right)\left(\dot{x}_2^\nu + 2i\psi_2^\nu\psi_2\cdot k_2\right)\mathrm{e}^{-\int_0^T d\tau\frac{1}{4}\dot{x}^2}
\end{aligned}$$

(69)

The calculation of $\mathcal{D}x$ is identical with the scalar QED calculation. Only the calculation of $\mathcal{D}\psi$ is new, and amounts to a single Wick contraction,

$$(2i)^2\Big\langle \psi_1^\mu\psi_1\cdot k_1\psi_2^\nu\psi_2\cdot k_2\Big\rangle = -G_{F12}^2\Big[g^{\mu\nu}k^2-k^\mu k^\nu\Big] = -\Big[g^{\mu\nu}k^2-k^\mu k^\nu\Big]\,. \tag{70}$$

Adding this to the bosonic result shows that, up to the global normalization, the parameter integral for the spinor loop is obtained from the one for the scalar loop simply by substituting, in Eq. (66),

$$\dot{G}_{B12}^2 \to \dot{G}_{B12}^2 - G_{F12}^2 = -\frac{4}{T}G_{B12}\,. \tag{71}$$

The complete change thus amounts to supplying Eq. (68) with a global factor of -2, and replacing $(1-2u)^2$ by $-4u(1-u)$. This leads to

$$\begin{aligned}\Omega^{\mu\nu}_{\text{spin}}[k] &= 8\frac{e^2}{(4\pi)^{D/2}}\Big[g^{\mu\nu}k^2 - k^\mu k^\nu\Big]\Gamma(2-\tfrac{D}{2})\\ &\quad\times\int_0^1 du u(1-u)\Big[m^2+u(1-u)k^2\Big]^{D/2-2}\,,\end{aligned} \tag{72}$$

again in agreement with the result of the field theory calculation.

The qualitative features of this calculation generalize to the N – photon amplitude. It is obvious that the Wick contraction of N photon vertex operators will lead to an integrand of the form

$$P_N(\dot{G}_{Bij},\ddot{G}_{Bij},G_{Fij})\exp\Big[\sum_{k<l}G_{Bkl}k_k\cdot k_l\Big] \tag{73}$$

with some polynomial P_N. What is remarkable is that the substitution above can be promoted to the following general substitution rule. As Bern and Kosower have shown, it is always possible to remove all second derivatives $\ddot{G}_{Bij}$ appearing in the integrand by a suitable chain of partial integrations. Once this has been done, the integrand for the spinor loop case can be obtained from the one for the scalar loop by simultaneously replacing every closed cycle of $\dot{G}_B$'s appearing, say $\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1}$, by its "supersymmetrization",

$$\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1} \to \dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1} - G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1} \tag{74}$$

(up to the global factor of -2). Note that an expression is considered a cycle already if it can be put into cycle form using the antisymmetry of $\dot{G}_B$ (*e.g.* $\dot{G}_{Bab}\dot{G}_{Bab} = -\dot{G}_{Bab}\dot{G}_{Bba}$). This rule may be understood as a remnant of worldsheet supersymmetry [16].

The close connection between the scalar and fermion loop calculations may appear surprising, as it appears to have no analogue in standard field theory. It arises because the treatment of fermions in the worldline formalism (and in string theory) does not correspond to the usual first order formalism, but to a second order formalism for fermions. The Feynman rules for spinor QED in the second order formalism (see [53] and references therein) are, up to statistics and degrees of freedom, the ones for scalar QED with the addition of a third vertex (Fig. 8) *.

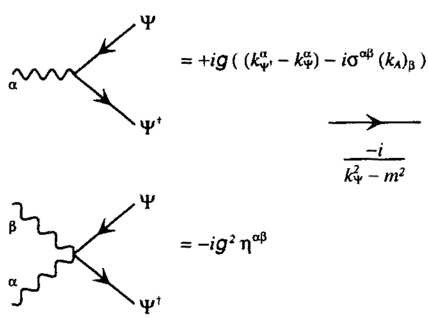

Fig. 8. The second order Feynman rules for fermion QED. They correspond to the ones for scalar QED with a third vertex added. This vertex involves $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and corresponds to the $\psi^\mu F_{\mu\nu}\psi^\nu$ – term in the worldline Lagrangian. For the details and for the nonabelian case see [53].

2.2. Photon splitting in QED

Next let us consider an application involving constant external fields, namely the one-loop three photon amplitude in a constant magnetic field. This amplitude gives rise to photon splitting, a process of potential astrophysical interest, and was first calculated exactly by Adler [54] in 1971.

We will again do both the scalar and the spinor QED cases. This amplitude is finite, so that we can work in $D = 4$. First we have to specialize our formulas for the general constant background field to the pure magnetic

* I thank A. Morgan for providing this figure.

field case. We choose the B-field along the z-axis, and introduce matrices $\boldsymbol{I}_{03}$ and $\boldsymbol{I}_{12}$ projecting on the t, z- and x, y-planes, so that

$$F \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \boldsymbol{I}_{03} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \boldsymbol{I}_{12} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{75}$$

We may then rewrite the determinant factors Eqs. (37) as

$$\det{}^{-1/2}\left[\frac{\sin(eFT)}{eFT}\right] = \frac{(eBT)}{\sinh(eBT)}, \tag{76}$$

$$\det{}^{-1/2}\left[\frac{\tan(eFT)}{eFT}\right] = \frac{(eBT)}{\tanh(eBT)}. \tag{77}$$

The Green's functions Eq. (32) specialize to

$$\begin{aligned}
\bar{\mathcal{G}}_B(\tau_1,\tau_2) &= G_{B12}\boldsymbol{I}_{03} - \frac{T}{2}\frac{\left[\cosh(z\dot{G}_{B12}) - \cosh(z)\right]}{z\sinh(z)}\boldsymbol{I}_{12} \\
&\quad + \frac{T}{2z}\left(\frac{\sinh(z\dot{G}_{B12})}{\sinh(z)} - \dot{G}_{B12}\right)i\hat{\boldsymbol{F}}, \\
\dot{\mathcal{G}}_B(\tau_1,\tau_2) &= \dot{G}_{B12}\boldsymbol{I}_{03} + \frac{\sinh(z\dot{G}_{B12})}{\sinh(z)}\boldsymbol{I}_{12} - \left(\frac{\cosh(z\dot{G}_{B12})}{\sinh(z)} - \frac{1}{z}\right)i\hat{\boldsymbol{F}}, \\
\ddot{\mathcal{G}}_B(\tau_1,\tau_2) &= \ddot{G}_{B12}\boldsymbol{I}_{03} + 2\left(\delta_{12} - \frac{z\cosh(z\dot{G}_{B12})}{T\sinh(z)}\right)\boldsymbol{I}_{12} + 2\frac{z\sinh(z\dot{G}_{B12})}{T\sinh(z)}i\hat{\boldsymbol{F}}, \\
\mathcal{G}_F(\tau_1,\tau_2) &= G_{F12}\boldsymbol{I}_{03} + G_{F12}\frac{\cosh(z\dot{G}_{B12})}{\cosh(z)}\boldsymbol{I}_{12} - G_{F12}\frac{\sinh(z\dot{G}_{B12})}{\cosh(z)}i\hat{\boldsymbol{F}}.
\end{aligned} \tag{78}$$

We have now introduced $z = eBT$, and $\hat{\boldsymbol{F}} = \frac{F}{B}$. In writing $\mathcal{G}_B$ we have already subtracted its coincidence limit, which is indicated by the "bar". The coincidence limits for $\dot{\mathcal{G}}_B$ and $\mathcal{G}_F$ are also needed,

$$\dot{\mathcal{G}}_B(\tau,\tau) = -\Bigl(\coth(z) - \frac{1}{z}\Bigr) i\hat{F}\,,$$
$$\mathcal{G}_F(\tau,\tau) = -\tanh(z) i\hat{F}\,. \tag{79}$$

To obtain the photon splitting amplitude, we will use these correlators for the Wick contraction of three vertex operators V_0 and $V_{1,2}$, representing the incoming and the two outgoing photons.

A) Scalar QED

The calculation is greatly simplified by the peculiar kinematics of this process. Energy–momentum conservation $k_0+k_1+k_2=0$ forces collinearity of all three four–momenta, so that, writing $-k_0 \equiv k \equiv \omega n$,

$$k_1 = \frac{\omega_1}{\omega}k, k_2 = \frac{\omega_2}{\omega}k;\; k^2 = k_1^2 = k_2^2 = k\cdot k_1 = k\cdot k_2 = k_1\cdot k_2 = 0. \tag{80}$$

Moreover, Adler [54] was able to show on general grounds that there is only one non-vanishing polarization case. This is the case where the magnetic vector $\hat{\mathbf{k}} \times \hat{\varepsilon_0}$ of the incoming photon is parallel to the plane containing the external field and the direction of propagation $\hat{\mathbf{k}}$, and those of the outgoing ones are both perpendicular to this plane. An appropriate choice of $\varepsilon_{0,1,2}$ leads to the further vanishing relations

$$\varepsilon_{1,2}\cdot\varepsilon_0 = \varepsilon_{1,2}\cdot k = \varepsilon_{1,2}\cdot F = 0\,. \tag{81}$$

This leaves us with the following small number of nonvanishing Wick contractions:

$$\langle V_0V_1V_2\rangle = -i\prod_{i=0}^{2}\int_0^T d\tau_i \exp\Bigl[\frac{1}{2}\sum_{i,j=0}^{2}\bar{\omega}_i\bar{\omega}_j n\bar{\mathcal{G}}_{Bij}n\Bigr]\,\varepsilon_1\ddot{\mathcal{G}}_{B12}\varepsilon_2\sum_{i=0}^{2}\bar{\omega}_i\varepsilon_0\dot{\mathcal{G}}_{B0i}n\,. \tag{82}$$

For compact notation we have defined $\bar{\omega}_0=\omega, \bar{\omega}_{1,2}=-\omega_{1,2}$.

Performing the Lorentz contractions, and taking the determinant factor Eq. (76) into account, one obtains the following simple parameter integral for this amplitude:

$$C_{\rm scal}\,[\omega,\omega_1,\omega_2,B] = \frac{m^8}{8\omega\omega_1\omega_2}\int_0^\infty dT\,T\frac{{\rm e}^{-m^2T}}{z^2{\rm sinh}^2(z)}\int_0^T d\tau_1\,d\tau_2\,\ddot{G}_{B12}$$
$$\times\Bigl[\sum_{i=0}^2 \bar\omega_i\cosh(z\dot{G}_{B0i})\Bigr]\exp\Bigl\{-\frac{1}{2}\sum_{i,j=0}^2\bar\omega_i\bar\omega_j\Bigl[G_{Bij}+\frac{T}{2z}\frac{\cosh(z\dot{G}_{Bij})}{\sinh(z)}\Bigr]\Bigr\}. \tag{83}$$

Translation invariance in τ has been used to set the position τ_0 of the incoming photon equal to T. For better comparison with the literature, we have normalized the amplitude as in [54].

B) Spinor QED

For the fermion loop case, we will now make use of the superfield formalism. As in the case without a background field, $\mathcal{G}_B$ and $\mathcal{G}_F$ can be combined into a super propagator,

$$\hat{\mathcal{G}}(\tau_1,\theta_1;\tau_2,\theta_2)\equiv\mathcal{G}_{B12}+\theta_1\theta_2\mathcal{G}_{F12}\,. \tag{84}$$

This allows us to write the result of the Wick contraction for the spinor loop in complete analogy to the scalar loop result Eq. (82),

$$\begin{aligned}\langle V_0V_1V_2\rangle &= -i\prod_{i=0}^2\int_0^T d\tau_i\int d\theta_i\,\exp\Bigl[\frac{1}{2}\sum_{i,j=0}^2\bar\omega_i\bar\omega_j n\hat{\bar{\mathcal{G}}}_{ij}n\Bigr]\\ &\quad\times\varepsilon_1D_1D_2\hat{\mathcal{G}}_{12}\varepsilon_2\sum_{i=0}^2\bar\omega_i\varepsilon_0D_0\hat{\mathcal{G}}_{0i}n\,.\end{aligned} \tag{85}$$

Performing the θ – integrations and Lorentz contractions, we obtain [37]:

$$C_{\rm spin}[\omega,\omega_1,\omega_2,B] = \frac{m^8}{4\omega\omega_1\omega_2}\int_0^\infty dT\,T\frac{{\rm e}^{-m^2T}}{z^2{\rm sinh}(z)}$$
$$\times\int_0^T d\tau_1\,d\tau_2\,\exp\Bigl\{-\frac{1}{2}\sum_{i,j=0}^2\bar\omega_i\bar\omega_j\Bigl[G_{Bij}+\frac{T}{2z}\frac{\cosh(z\dot{G}_{Bij})}{\sinh(z)}\Bigr]\Bigr\}$$
$$\times\Bigl\{\Bigl[-\cosh(z)\ddot{G}_{B12}+\omega_1\omega_2\Bigl(\cosh(z)-\cosh(z\dot{G}_{B12})\Bigr)\Bigr]$$

$$\times\left[\frac{\omega}{\sinh(z)\cosh(z)}-\omega_1\frac{\cosh(z\dot{G}_{B01})}{\sinh(z)}-\omega_2\frac{\cosh(z\dot{G}_{B02})}{\sinh(z)}\right]$$
$$+\frac{\omega\omega_1\omega_2 G_{F12}}{\cosh(z)}\left[\sinh(z\dot{G}_{B01})\Big(\cosh(z)-\cosh(z\dot{G}_{B02})\Big)-(1\leftrightarrow 2)\right]\Bigg\}. \tag{86}$$

Numerical analysis of this three-parameter integral has shown it to be in agreement with other known integral representations of this amplitude [37]. However, the method of calculation improves in several respects over standard field theory methods. In particular, it allowed us to make use of the vanishing relations for the kinematic invariants essentially on line one of the calculation to reduce the number of terms.

2.3. The two-loop QED β-functions

We proceed to the two-loop level, and to a recalculation of the two-loop β-function coefficients for scalar and spinor QED. In field theory, the fermion QED calculation requires consideration of the diagrams shown in Fig. 9, while for scalar QED there are some more diagrams involving the seagull vertex. Of the possible counterdiagrams contributions those from electron wave function and vertex renormalization cancel on account of the QED Ward identity, however mass renormalization must be taken into account. The worldline formalism applies only to the calculation of the bare regularized amplitude; the renormalization has to be performed in field theory.

Fig. 9. Diagrams contributing to the two-loop vacuum polarization

In the following calculation, which is a variant of the one presented in [39], it will not be necessary to distinguish between individual graphs. We will again couple the two-loop path integral to some constant field background, and write down the effective action induced for the background field. The β-function coefficient will then be extracted from the divergence of the Maxwell term $\sim F^{\mu\nu}F_{\mu\nu}$, calculated in dimensional regularization. Of course we are free to choose any background we wish; we will therefore impose the condition $F^{\mu\nu}F_{\nu\lambda}\sim\delta^{\mu}_{\lambda}$, which leads to some simplifications.

A) Scalar QED

Applying the formalism developed in the first lecture to this case, we obtain, for the scalar QED case, immediately the following parameter integral for the two-loop effective Lagrangian:

$$\mathcal{L}^{(2)}_{\rm scal}[F] = (4\pi)^{-D}\Bigl(-\frac{e^2}{2}\Bigr)\int_0^\infty \frac{dT}{T}e^{-m^2T}T^{-D/2}\int_0^\infty d\bar{T}\int_0^T d\tau_a\int_0^T d\tau_b$$
$$\times\det{}^{-1/2}\biggl[\frac{\sin(eFT)}{eFT}\biggr]\det{}^{-1/2}\Bigl[\bar{T}+\tfrac{1}{2}\Bigl(\bar{\mathcal{G}}_{ab}+\bar{\mathcal{G}}_{ba}\Bigr)\Bigr]\langle\dot{y}_a\cdot\dot{y}_b\rangle\,. \tag{87}$$

Here T and $\bar{T}$ denote the electron and photon proper-times, and $\tau_{a,b}$ the endpoints of the photon insertion moving around the electron loop. The first determinant factor is the same as before, and represents the change of the free path integral determinant due to the external field; the second one represents its change due to the photon insertion, and generalizes Eq. (52) to the external field case. The two-loop Green's function Eq. (51) generalizes to [36]

$$\mathcal{G}_B^{(1)}(\tau_1,\tau_2)=\mathcal{G}_B(\tau_1,\tau_2)+\frac{1}{2}\frac{[\mathcal{G}_B(\tau_1,\tau_a)-\mathcal{G}_B(\tau_1,\tau_b)][\mathcal{G}_B(\tau_a,\tau_2)-\mathcal{G}_B(\tau_b,\tau_2)]}{\bar{T}+\frac{1}{2}\Bigl(\bar{\mathcal{G}}_{ab}+\bar{\mathcal{G}}_{ba}\Bigr)}\,. \tag{88}$$

We use this Green's function for Wick-contracting the "left over" numerator of the photon insertion, which gives

$$\langle\dot{y}_a\cdot\dot{y}_b\rangle = {\rm tr}\biggl[\ddot{\mathcal{G}}_{Bab}+\frac{1}{2}\frac{(\dot{\mathcal{G}}_{Baa}-\dot{\mathcal{G}}_{Bab})(\dot{\mathcal{G}}_{Bab}-\dot{\mathcal{G}}_{Bbb})}{\bar{T}+\frac{1}{2}\Bigl(\bar{\mathcal{G}}_{ab}+\bar{\mathcal{G}}_{ba}\Bigr)}\biggr]\,. \tag{89}$$

Care must be taken again with coincidence limits, as the derivatives should *not* act on the variables τ_a,τ_b explicitly appearing in the two-loop Green's function; again the correct rule in calculating $\langle\dot{y}_a\dot{y}_b\rangle$ is to first differentiate Eq. (88) with respect to τ_1,τ_2, and set $\tau_1=\tau_a,\tau_2=\tau_b$ afterwards.

Next note that with F chosen as above $\bar{\mathcal{G}}_{Bab}+\bar{\mathcal{G}}_{Bba}$ is a Lorentz scalar, so that

$$\det{}^{-1/2}\Bigl[\bar{T}+\tfrac{1}{2}\Bigl(\bar{\mathcal{G}}_{ab}+\bar{\mathcal{G}}_{ba}\Bigr)\Bigr] = \Bigl[\bar{T}+\tfrac{1}{2}\Bigl(\bar{\mathcal{G}}_{ab}+\bar{\mathcal{G}}_{ba}\Bigr)\Bigr]^{-D/2}, \tag{90}$$

and the $\bar{T}$-integration can be trivially performed. Note that the term containing δ_{ab} (see Eq. (32)) vanishes in dimensional regularization upon performance of the $\bar{T}$-integral (it corresponds to a tadpole insertion in field theory). After the usual rescaling $\tau_{a,b} = Tu_{a,b}$ one expands the integrand in a Taylor expansion in F, and picks out the coefficient of $\mathrm{tr}\,(F^2)$. This gives

$$\mathcal{L}^{(2)}_{\mathrm{scal}}[F] = \mathrm{tr}(F^2)\frac{e^2}{(4\pi)^D}\Big(-\frac{e^2}{2}\Big)\int\limits_0^\infty \frac{dT}{T}e^{-m^2T}T^{4-D}\int\limits_0^1 du_a\int\limits_0^1 du_b$$
$$\times\Big\{-\frac{1}{12}{G_{Bab}}^{-D/2}+\Big[\frac{1}{2}-\frac{D-4}{3(D-2)}-\frac{4}{3D}\Big]{G_{Bab}}^{1-D/2}+\Big[\frac{28}{3D}-\frac{8}{D-2}\Big]{G_{Bab}}^{2-D/2}\Big\}. \tag{91}$$

The scalar proper-time integral gives the usual global $\Gamma(4-D)$, and the remaining parameter integral produces Euler Beta-functions:

$$\int\limits_0^1 du_a\int\limits_0^1 du_b\,{G_{Bab}}^{n-D/2}=\int\limits_0^1 du_a[u_a(1-u_a)]^{n-D/2}=B\Big(n+1-\tfrac{D}{2},n+1-\tfrac{D}{2}\Big). \tag{92}$$

Expanding the result in $\varepsilon = D-4$, the $1/\varepsilon^2$-terms cancel, as expected. The $1/\varepsilon$-term becomes

$$\mathcal{L}^{(2)}_{\mathrm{scal}}[F] \sim \frac{1}{2\varepsilon}\frac{\alpha^2}{(4\pi)^2}F^{\mu\nu}F_{\mu\nu}\ . \tag{93}$$

So far this is a calculation of the bare regularized amplitude. It must still be supplemented with a contribution from one-loop mass renormalization, which involves only one-loop quantities (see [39]),

$$\Delta\mathcal{L}^{(2)}_{\mathrm{scal}}[F] = \delta {m^{(1)}}^2\frac{\partial}{\partial m^2}\mathcal{L}^{(1)}_{\mathrm{scal}}[F] \sim \frac{1}{2\varepsilon}\frac{\alpha^2}{(4\pi)^2}F_{\mu\nu}F^{\mu\nu}\ . \tag{94}$$

Adding up both contributions

$$\mathcal{L}^{(2)}_{\mathrm{scal}}[F] + \Delta\mathcal{L}^{(2)}_{\mathrm{scal}}[F] \sim \frac{1}{\varepsilon}\frac{\alpha^2}{(4\pi)^2}F_{\mu\nu}F^{\mu\nu} \tag{95}$$

and extracting the β-function coefficient in the usual way (see *e.g.* [55]), one finds the known result [56],

$$\beta^{(2)}_{\mathrm{scal}}(\alpha) = \frac{\alpha^3}{2\pi^2}\ . \tag{96}$$

B) Spinor QED

For the spinor loop in the superfield formalism one obtains again parameter integrals formally analogous to Eqs.(87),(89):

$$\begin{aligned}\mathcal{L}^{(2)}_{\rm spin}[F] &= (-2)(4\pi)^{-D}\Bigl(-\tfrac{e^2}{2}\Bigr)\int_0^\infty \frac{dT}{T^{1+D/2}}e^{-m^2T}\int_0^\infty d\bar{T}\int_0^T d\tau_a d\tau_b\int d\theta_a d\theta_b\\ &\times{\det}^{-1/2}\Bigl[\frac{\tan(eFT)}{eFT}\Bigr]{\det}^{-1/2}\Bigl[\bar{T}+\tfrac{1}{2}\Bigl(\bar{\hat{\mathcal{G}}}_{ab}+\bar{\hat{\mathcal{G}}}_{ba}\Bigr)\Bigr]\langle -D_aY_a\cdot D_bY_b\rangle\,,\end{aligned} \tag{97}$$

$$\langle -D_aY_a\cdot D_bY_b\rangle = {\rm tr}\Bigl[D_aD_b\hat{\mathcal{G}}_{Bab}+\frac{1}{2}\frac{D_a(\hat{\mathcal{G}}_{Baa}-\hat{\mathcal{G}}_{Bab})D_b(\hat{\mathcal{G}}_{Bab}-\hat{\mathcal{G}}_{Bbb})}{\bar{T}+\frac{1}{2}\Bigl(\bar{\hat{\mathcal{G}}}_{ab}+\bar{\hat{\mathcal{G}}}_{ba}\Bigr)}\Bigr]\,, \tag{98}$$

The further calculation also parallels the scalar case. The only point to be mentioned is that it is permissible and convenient to perform the $\bar{T}$-integration *before* the θ-integrals. The equivalent of Eq. (91) becomes

$$\begin{aligned}\mathcal{L}^{(2)}_{\rm spin}[F] = \ & {\rm tr}(F^2)\frac{e^4}{(4\pi)^D}\int_0^\infty\frac{dT}{T}e^{-m^2T}T^{4-D}\int_0^1 du_a\int_0^1 du_b\\ &\times\Bigl\{\frac{D}{6}{G_{Bab}}^{-D/2}+\Bigl[\frac{D}{6}-4+\frac{8}{3D}+\frac{8}{3(D-2)}\Bigr]{G_{Bab}}^{1-D/2}\\ &+\Bigl[\frac{28}{3D}-\frac{8}{D-2}\Bigr]{G_{Bab}}^{2-D/2}\Bigr\}\,.\end{aligned} \tag{99}$$

This yields

$$\mathcal{L}^{(2)}_{\rm spin}[F] \sim -\frac{3}{\varepsilon}\frac{\alpha^2}{(4\pi)^2}F_{\mu\nu}F^{\mu\nu}+O(\varepsilon^0)\,. \tag{100}$$

Mass renormalization contributes a [39]

$$\Delta\mathcal{L}^{(2)}_{\rm spin}[F] = \delta m^{(1)}\frac{\partial}{\partial m}\mathcal{L}^{(1)}_{\rm spin}[F] \sim \frac{4}{\varepsilon}\frac{\alpha^2}{(4\pi)^2}F_{\mu\nu}F^{\mu\nu}\,, \tag{101}$$

yielding a total of

$$\mathcal{L}^{(2)}_{\rm spin}[F]+\Delta\mathcal{L}^{(2)}_{\rm spin}[F] \sim \frac{1}{\varepsilon}\frac{\alpha^2}{(4\pi)^2}F_{\mu\nu}F^{\mu\nu}\,. \tag{102}$$

The β-function coefficient becomes

$$\beta^{(2)}_{\rm spin}(\alpha) = \frac{\alpha^3}{2\pi^2} \quad , \tag{103}$$

which is the classical Jost–Luttinger result [57].

I have chosen to show the most straightforward, though not the most efficient versions of this calculation. Both in the scalar and the spinor QED case the 2-loop β-function calculation can be further trivialized by suitable partial integrations in τ_a. The absence of a subdivergence for the Maxwell term then becomes manifest, regularization is only needed for the trivial global T-integration, and all τ_a, τ_b-dependence disappears from the integrand before the τ_a, τ_b -integration. This is discussed elsewhere [36, 44].

2.4. The three-loop scalar master integral

Finally, let us have a look at the simplest example of a three-loop parameter integral calculation in this formalism. This is the one where the integrand consists just of the bosonic three-loop determinant factor, Eq. (54) with $m = 2$. In dimensional regularization, it reads

$$\Gamma^{(3)}_{\rm vac}(D) = \int\limits_0^\infty \frac{dT}{T} e^{-m^2 T} T^{6-\frac{3}{2}D} I(D), \tag{104}$$

$$I(D) = \int\limits_0^\infty d\hat{T}_1\, d\hat{T}_2 \int\limits_0^1 da\, db\, dc\, dd \left[(\hat{T}_1 + G_{Bab})(\hat{T}_2 + G_{Bcd}) - \frac{C^2}{4}\right]^{-D/2} . \tag{105}$$

Here $\hat{T}_{1,2} = \frac{T_{1,2}}{T}$ denote the proper-time lengths of the two inserted propagators in units of T, and $C \equiv G_{Bac} - G_{Bad} - G_{Bbc} + G_{Bbd}$. This is the most basic integral appearing if one applies this formalism to the calculation of three-loop renormalization group functions in abelian field theories, *e.g.* in QED [43] or the Yukawa Model. As far as the quenched QED β-function is concerned, the general parameter integral appearing at the three-loop level still has a very similar structure: It has the same universal denominator, possibly with a different power, and a numerator, which is again expressible in terms of the functions G_{Bab}, G_{Bcd}, C and their derivatives.

In writing Eq. (104) we have already rescaled to the unit circle, and separated off the electron proper-time integral. This integral decouples, and just yields an overall factor of

$$\int\limits_0^\infty \frac{dT}{T} \mathrm{e}^{-m^2 T} T^{6-3/2D} = \Gamma(6 - \tfrac{3}{2}D)\, m^{3D-12} \sim -\frac{2}{3\varepsilon}\,.$$

The nontrivial integrations are $\int_0^1 da\,db\,dc\,dd \equiv \int_{abcd}$, representing the four propagator end points moving around the loop (Fig. 10). This fourfold integral decomposes into 24 ordered sectors, of which 16 constitute the planar (P) (Fig. 10a) and 8 the nonplanar (NP) sector (Fig. 10b). Due to the symmetry properties of the integrand, all sectors of the same topology give an equal contribution. The integrand has a trivial invariance under the operator $\frac{\partial}{\partial a} + \frac{\partial}{\partial b} + \frac{\partial}{\partial c} + \frac{\partial}{\partial d}$, which just shifts the location of the zero on the loop.

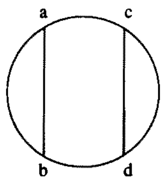

Fig.10a

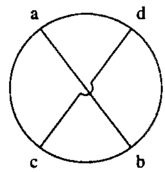

Fig.10b

As a first step in the calculation of $I(D)$, it is useful to add and subtract the same integral with $C = 0$ and rewrite

$$I(D) = I_{\text{sing}}(D) + I_{\text{reg}}(D), \tag{106}$$

$$I_{\text{sing}}(D) = \int_0^\infty d\hat{T}_1\, d\hat{T}_2 \int_{abcd} \Big[(\hat{T}_1 + G_{Bab})(\hat{T}_2 + G_{Bcd})\Big]^{-D/2}. \tag{107}$$

$I_{\text{sing}}(D)$ factorizes into two identical three-parameter integrals, which are elementary:

$$I_{\text{sing}}(D) = \left\{ \int_0^\infty dT \int_0^1 du \Big[T + u(1-u)\Big]^{-D/2} \right\}^2 = \left[\frac{2B(2-\frac{D}{2}, 2-\frac{D}{2})}{D-2} \right]^2.$$

The point of this split is that the remainder $I_{\text{reg}}(D)$ is finite. To see this, set $D = 4$, expand the original integrand in $\frac{C^2}{G_{Bab}G_{Bcd}}$, and note that for all terms but the first one the zeroes of G_{Bab} (G_{Bcd}) at $a \sim b$ ($c \sim d$) are offset by zeroes of C^2. Only the $\frac{1}{\varepsilon}$ – pole is required in renormalization group function calculations, so that one can set $D = 4$ for the calculation of $I_{\mathrm{reg}}(D)$. The integrations over $\hat{T}_1, \hat{T}_2$ are then elementary, and we are left with

$$I_{\mathrm{reg}}(4) = \int\limits_{abcd} \left[-\frac{4}{C^2} \ln\Big(1 - \frac{C^2}{4G_{Bab}G_{Bcd}}\Big) - \frac{1}{G_{Bab}G_{Bcd}} \right]. \tag{108}$$

For the calculation of this integral, observe the following simple behaviour of the function C under the operation $D_{ab} \equiv \frac{\partial}{\partial \tau_a} + \frac{\partial}{\partial \tau_b}$:

$$D_{ab}C = \pm 2\chi_{NP}, \; D^2_{ab}C = 2\Big(\delta_{ac} - \delta_{ad} - \delta_{bc} + \delta_{bd}\Big), \tag{109}$$

where χ_{NP} denotes the characteristic function of the nonplanar sector. From these identities and the symmetry properties one can easily derive the following projection identities, which effectively integrate out the variable C:

$$\begin{aligned} \int\limits_P f(C, G_{Bab}, G_{Bcd}) &= 4\int\limits_0^1 da \int\limits_0^a dc(a-c) f\Big(-2c(1-a), a - a^2, c - c^2\Big), \\ \int\limits_{NP} f(C, G_{Bab}, G_{Bcd}) &= -4\int\limits_0^1 da \int\limits_0^a dc \int\limits_0^{-2c(1-a)} dC f\Big(C, a - a^2, c - c^2\Big). \end{aligned} \tag{110}$$

Here f is an arbitrary function in the variables G, G_{Bab}, G_{Bcd}, and $\int\limits_0^C dCf$ denotes the integral of this function in the variable C, with the other variables fixed. The integrals on the left hand side are restricted to the sectors indicated. For f the integrand of our formula Eq. (108), we have

$$\begin{aligned} \int\limits_0^C dCf = &- \frac{C}{G_{Bab}G_{Bcd}} + \frac{4}{C} \ln\Big(1 - \frac{C^2}{4G_{Bab}G_{Bcd}}\Big) \\ &+ \frac{4}{\sqrt{G_{Bab}G_{Bcd}}} \operatorname{arctanh}\Big(\frac{1}{2}\frac{C}{\sqrt{G_{Bab}G_{Bcd}}}\Big). \end{aligned} \tag{111}$$

Inserted in the second equation of (110) this leaves us with three two-parameter integrals, of which the first one is elementary. Applying the substitution

$$y = \frac{c(1-a)}{a(1-c)}$$

to the second integral, and

$$y^2 = \frac{c(1-a)}{a(1-c)}$$

to the third integral, those are transformed into known standard integrals, tabulated for instance in [58]. The result is

$$\int_{NP} f = 12\zeta(3) - 8\zeta(2)\,.$$

The calculation in the planar sector is elementary, and we just give the result,

$$\int_{P} f = 4\zeta(2) - 4\,.$$

Putting the pieces together, we have, up to terms of order $O(\varepsilon^0)$,

$$\Gamma^{(3)}_{\mathrm{vac}}(D) = m^{3D-12}\Gamma(6-\frac{3}{2}D)\left\{\left[\frac{2B(2-\frac{D}{2},2-\frac{D}{2})}{D-2}\right]^2 + 12\zeta(3) - 4\zeta(2) - 4\right\}.$$

I thank M. Jeżabek and the organizers of the school for their hospitality and the opportunity to deliver these lectures. I am also grateful to M.G. Schmidt for discussions, and to D. Fliegner for computer support.

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