ON QUANTUM L_p- SPACE TECHNIQUE*

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We show that using recently introduced quantum L_p-spaces one can give an explicit constructions of spin-flip and diffusive type quantum dynamics.

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1. Generalities

The purpose of statistical mechanics is the description of the mechanics of large or even infinite systems. We recall that infinite (extended) systems in statistical mechanics arose as a result of thermodynamic limit. The basic aim of this procedure is to give an unambiguous meaning of such concepts as temperature, pressure, phase transitions, *etc.* (*cf.* [13]). This makes clear that the existence of thermodynamic limit is one of the standard assumptions of statistical mechanics (*cf.* [14], vol. I). Consequently, the idealization saying that the matter fills all space with finite density is put on. Furthermore, the basic states of interest are equilibrium states. To discuss the concept of states, we note that because the number of components in a system is very large (or even infinite) we cannot hope to have a complete description in the sense of classical mechanics. Let us recall that

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such complete description of a system at any instant is given by the phase point

$$x = (q_1, \ldots, q_n, p_1, \ldots, p_n)(\in X). \tag{1}$$

It is sometimes called a pure state. An incomplete description is called a statistical description or mixed state and it is given by a measure μ on the phase space X.

The time evolution of the considered system can be expressed as a one parameter family of transformations $T_t : X \to X$ of the phase space (*e.g.* a hamiltonian flow), or as a one parameter family of transformations defined on the set of measures on X (*e.g.* by Liouville equation, Markov semigroups), or as a one parameter family of transformations on the set $C(X)$ of all bounded continuous functions on X. Thus, the following triple

$$(X, T_t, \mu) \tag{2}$$

is considered as the basis of any considerations of (classical) statistical mechanics.

Having a defined (commutative) dynamical system (2) it is natural to ask a question about its fundamental properties. Among those properties of T_t, the ergodic ones are usually considered as the most important (*cf.* [14] vol. I., [1]).

However it is an easy observation that the mere measurable structure of the phase space is too weak tool for a study of such ergodic questions as an existence and uniqueness of time invariant measure, return to equilibrium *etc.* Even to formulate in a proper way a description of Markov (classical) processes or semigroups some additional structure is necessary. This is the point where L_p-spaces come in. Namely, with the triple (X, T_t, μ) one can associate $L_p(X, \mu)$ spaces. Then, using the functional analysis one can study the time evolution as a family of transformations on $L_p(X, \mu)$. In particular, for the case $p = 2$, one can use the structure of Hilbert space and its basic tool the spectral analysis. The idea of using Hilbert space method to study classical mechanical systems is very old and goes to Koopman. For a discussion of ergodic theorems in Banach space setting (which includes L_p-technique) we refer to [7].

2. Elementary quantum mechanics

To get a hint for a generalization of the L_p-structure of classical statistical mechanics let us consider a description of dynamical systems within the framework of elementary quantum mechanics, *i.e.* let us consider a quantum system of certain (finite) number of particles. In this framework a quantum system is specified by a triple (*cf.* [3])

$$(B(\mathcal{H}), \mathcal{H}, \varrho), \tag{3}$$

where $\mathcal{H}$ is a separable Hilbert space, $B(\mathcal{H})$ the set of all linear bounded operators on $\mathcal{H}$ and ϱ is a density matrix. Let us assume that ϱ is an invertible operator, *i.e.* $\omega(\cdot) = \mathrm{Tr}\{\varrho\cdot\}$ is a faithful state on $B(\mathcal{H})$. It is an easy observation that ω can be considered as a quantum counterpart of a measure. In physical terms, ϱ can describe for example a Gibbs state at a temperature β.

When we use the Heisenberg picture, time evolution of the system can be given by a (one-parameter) family of maps on $B(\mathcal{H})$.

To generalize the classical L_p-scheme it is necessary to ask for "quantum" L_p-spaces. The traditional approach relies on the observation that the set of all Hilbert-Schmidt operators $\mathcal{H}_{\mathrm{H-S}}$ has the Hilbert space structure with the inner product given by $((a,b)) = \mathrm{Tr}\{a^*b\}$, $a,b \in \mathcal{H}_{\mathrm{H-S}}$. Furthermore, the set of all density matrices $\mathcal{F}_T$ ($\mathcal{H}_{\mathrm{H-S}}$, $B(\mathcal{H})$ respectively) can be considered as "quantum" L_1 (L_2, L_∞) spaces (see [4]). More generally, the p-th Schatten class (see [11]) can be considered as quantum counterpart of L_p-space for elementary quantum mechanics (see [12]). This leads to a conclusion that the general properties of L_p-spaces can be used for a description of quantum dynamical systems. In fact, such approach to problems of statistical physics is frequently used and sometimes is called quantum Liouville space technique (see [4]). For a slightly different definition of L_2-spaces associated with a quantum state and their applications to "probabilistic" description of quantum systems see Chapter II in [6].

Consequently, L_p-technique can be generalized for the framework of elementary quantum mechanics.

3. General quantum systems

Statistical mechanics, as it was mentioned in Section 1, deals with large (infinite) systems. Moreover, classical statistical mechanics incorporates the locality into its scheme from the very beginning (*cf.* (1)). Therefore, these features should be taken into account in any attempt of quantization of the "classical" scheme.

A natural formulation of a quantum schema which includes the locality property can be given within the algebraic formalism of quantum mechanics (*cf.* [5]). Namely, in quantum physics, (similar as in the classical theory), one can use a concept of observables (or "fields") to implement the principle of locality. Specifically[1], to each (open) bounded region $\mathcal{O}$ in the space we can associate the C^*-algebra $\mathcal{A}(\mathcal{O})$ representing physical operations performable within $\mathcal{O}$. Thus, the net of algebras, *i.e.* the correspondence

$$\mathcal{O} \longmapsto \mathcal{A}(\mathcal{O}) \tag{4}$$

[1] In this lecture we are interested in non-relativistic description only.

is the base of the mathematical description of the theory. Consequently, $\mathcal{A} = \overline{\cup\mathcal{A}(\mathcal{O})}$, where the bar denotes an appropriate strong closure, represents all quasi-local observables of the quantum extended system. It is natural to view $\mathcal{A}$ as a noncommutative analogue of the space of bounded continuous functions. Having the quantum counterpart of the set of bounded continuous functions on the phase space, as a next step, it is necessary to give a recipe for the time evolution of the system. This can be done as follows. Let $\{\mathcal{O}_n\}$ be a subnet consisting of an increasing sequence of open finite subsets of $\boldsymbol{R}^\nu$ or $\boldsymbol{Z}^\nu$ such that for every open finite subset $\mathcal{O}$ there is some finite positive integer $n(\mathcal{O})$ with the property: $\mathcal{O} \subset \mathcal{O}_k$ for all $k \geq n(\mathcal{O})$. Furthermore, suppose that the time evolution $\alpha_t^n(f) \equiv \mathrm{e}^{iH_n t} f \mathrm{e}^{-iH_n t}$ is well defined in each $\mathcal{O}_n$, where H_n is the Hamiltonian associated to the region $\mathcal{O}_n$ and $f \in \mathcal{A}_{\mathcal{O}_n}$. Let us remark that we have used the Heisenberg picture again. The choice of H_n amounts to specifying the class of interactions $\Phi(\mathcal{O}_n)$ for the region $\mathcal{O}_n$.

Then we should study the limit of $\alpha_t^n(f)$ as $n \to \infty$ where f is taken in $\mathcal{A}(\mathcal{O})$. If it exists, in the appropriate topology, and possess some necessary properties, then the time evolution of the system can be defined,

$$\alpha_t = \lim_{n\to\infty} \alpha_t^n . \tag{5a}$$

In a similar way, one can define an equilibrium (Gibbs) state ω for an extended system. We define

$$\omega = \lim_n \omega_n , \tag{5b}$$

where $\omega_n(\cdot) = Z_n^{-1} \operatorname{Tr}\{\mathrm{e}^{-\beta H_n} \cdot\}$, with β denoting the inverse temperature and Z_n the corresponding partition function. The above program evidently involves some convergence questions and works perfectly under some additional conditions, *e.g.* for some class of interactions on a quantum spin lattice.

Let us discuss briefly the sort of physical systems which can be described by lattice systems. The typical example is provided by the Heisenberg model of a ferromagnetic material (*cf.* [2], vol II.) Namely, the ferromagnetic material can be considered as made up of atoms arranged in a regular lattice. This picture can be perfectly justified if the temperature is low enough, so that lattice vibrations can be neglected. Moreover, it is assumed that a state of each atom is described by a spin state. This assumptions means that with each site $\{x_i\}$ of the lattice we associate a set M_n of all $n \times n$ matrices. Finally it is assumed that the interactions between atoms of lattice sites, say x_i and x_j, may be approximated by the product of corresponding spins, *i.e.* by some selfadjoint element in $M_{\{x_i\}} \otimes M_{\{x_j\}} \equiv M_n \otimes M_n$. That shows that the solid body problems provide concrete examples of physical systems which can be studied within the framework described above.

Therefore we can conclude this section with the corollary that again a triple

$$(\mathcal{A}, \alpha_t, \omega) \tag{6}$$

constitutes a basis for a description of many examples of infinite quantum systems.

4. Quantum L_p-spaces

Recently we have proved that it is possible to associate with (6) a quantum counterpart of interpolating family of L_p-spaces (see [8, 9]). It is important to note that the Schatten classes can not be used here as in general the state ω in (6) has not the form $\omega(\cdot) = \mathrm{Tr}\{\varrho\cdot\}$ for some density matrix ϱ. Furthermore, even in equilibrium representation there are algebras without a (faithful) trace. Hence, our construction, (reflecting the general Haagerup's theory of noncommutative integration), was based on the notion of thermodynamic limit. In particular, we have proved

Theorem 1:
To every Gibbs state ω_β on $\mathcal{A}$ we can associate an interpolating family of Banach spaces $\{L_p(\omega_\beta, s)\}_{p\in[0,\infty),s\in[0,1]}$ in such a way that for any local observable $f \in \mathcal{A}(\mathcal{O})$ we have

$$||f||^p_{L_p(\omega_\beta,s)} = \lim_n ||f||^p_{L_p(\omega^n,s)}, \tag{7a}$$

where the local $||\cdot||^p_{L_p(\omega^n,s)}$-norm is given by

$$||f||^p_{L_p(\omega^n,s)} = \mathrm{Tr}\,|(\varrho^{(n)})^{s/p}\cdot f\cdot(\varrho^{(n)})^{1-s/p}|^p \tag{7b}$$

with $\varrho^{(n)}$ being the density matrix of ω restricted to $\mathcal{A}(\mathcal{O}_n)$.

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As it was already mentioned, the interpolating family of L_p-spaces plays an essential role in the construction and examination of classical Markov evolution. On the other hand, except for a few models, essentially no result has been obtained before in a constructive and rigorous approach to quantum Markov evolution of infinite systems. By a constructive approach we mean one in which existence of the evolution of extended system is not postulated, as in pure semigroup approach, but constructed on the basis of the local character of evolution in the bounded regions $\mathcal{O}_n$, *i.e.* the evolution should be constructed as the thermodynamic limit of the corresponding finite volume dynamics with an appropriate control of the convergence. In other words, we adopt the view that the dynamics of a extended quantum system has to be derived from intrinsic local properties of the system.

Let us remind at this point that the efforts in the construction and study of quantum Markov semigroups are as old as the equilibrium description of quantum systems. Thus to get a progress it seems natural to use the new approach — quantum Liouville space technique based on quantum L_p-spaces.

Looking for some basic examples of explicitly constructed Markov semigroups we take the point of view that jump and diffusive processes are the basic ones in the domain of stochastic dynamics. Having this in mind in the following sections we describe a construction of corresponding Markov generators for quantum lattice systems.

5. Spin-flip dynamics

In this section we introduce a family of Markov generators and semigroups for a quantum spin system on a lattice, which are similar to a block spin flip stochastic dynamics of classical spin systems . We start with a definition of finite volume dynamics. For $\mathcal{O}$, a finite subset of the lattice, let $E_{\mathcal{O},\Lambda} : \mathcal{A} \longrightarrow \mathcal{A}$ be a map defined as follows

$$E_{\mathcal{O},\Lambda}(f) = \mathrm{Tr}_{\mathcal{O}} \left(\gamma^*_{\mathcal{O},\Lambda} f \gamma_{\mathcal{O},\Lambda} \right) \tag{8}$$

with

$$\gamma_{\mathcal{O},\Lambda} \equiv \rho_\Lambda^{1/2} \left(\mathrm{Tr}_{\mathcal{O}}\, \rho_\Lambda \right)^{-1/2} , \tag{9}$$

where $\mathrm{Tr}_{\mathcal{O}}$ is the partial trace and ρ_Λ the density matrix of a finite volume Gibbs state ω_Λ and Λ is another finite subset of the lattice (see [9] for details).

Let $\boldsymbol{L}_{\mathcal{O},\Lambda}$ be an operator on $\mathcal{A}$ defined by

$$\boldsymbol{L}_{\mathcal{O},\Lambda} f \equiv E_{\mathcal{O},\Lambda}(f) - f \,. \tag{10}$$

We define

$$\boldsymbol{L}^\Lambda f = \sum_{j\in\Lambda} \boldsymbol{L}_{\mathcal{O}+j,\Lambda} f \tag{11}$$

and the corresponding semigroup by $\mathcal{P}_t^\Lambda$, *i.e.*

$$\mathcal{P}_t^\Lambda = \exp\left(t\boldsymbol{L}^\Lambda \right) .$$

It has the following properties.

Theorem 2:

(i) Positivity preserving: For any $f \in \mathcal{A}^+$

$$\mathcal{P}_t^\Lambda f \geq 0 \tag{12}$$

(ii) Unit preserving

$$\mathcal{P}_t^\Lambda \mathbf{1} = \mathbf{1} \tag{13}$$

(iii) L_2 - Symmetry

$$\langle \mathcal{P}_t^\Lambda(f), g\rangle_{\omega_\Lambda} = \langle f, \mathcal{P}_t^\Lambda(g)\rangle_{\omega_\Lambda} \tag{14}$$

(iv)

$$||\mathcal{P}_t^\Lambda||_{L_2(\omega_\Lambda)\to L_2(\omega_\Lambda)} \leq 1 \tag{15}$$

$$\langle L^{X,\Lambda}(f), f\rangle_{\omega_\Lambda} \leq 0 \tag{16}$$

(v) Invariance: For any $f \in \mathcal{A}$

$$\omega_\Lambda\left(\mathcal{P}_t^\Lambda(f)\right) = \omega_\Lambda(f) \tag{17}$$

where $\langle f, g\rangle_{\omega_\Lambda}$ stands for the inner product of L_2 ($\equiv L_2(\omega_\Lambda, 1/2)$.

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It follows that $\mathcal{P}_t^\Lambda$ is a well defined Markov semigroup. Such semigroups of block-spin flip type can be generalized for a thermodynamic limit of local Gibbs states on an infinite lattice. In other words we are looking for a limit of $\gamma_{\mathcal{O},\Lambda}$ as $\Lambda \to \mathbf{Z}^\nu$. We proved (see [10]) that such limit, γ_0, exists. However in general, γ_0 is an element of the von Neumann algebra $\mathcal{M} \equiv \pi_\omega(\mathcal{A})''$ obtained in the GNS construction with ω being a Gibbs state on the entire lattice. Thus we can define

$$\mathcal{L}_{\mathcal{O}} f = \mathcal{E}_{\mathcal{O}}(f) - f \tag{18}$$

with

$$\mathcal{E}_{\mathcal{O}}(f) = \mathrm{Tr}_{\mathcal{O}}\{\gamma_0^* f \gamma_0\}\,, \tag{19}$$

where $f \in \mathcal{M}$.

Theorem 3:

Suppose $|\beta| \leq \beta_0$, with β_0 a critical temperature (so high temperature region) or in one dimension that the interaction Φ is of finite range. Then $\mathcal{E}_{\mathcal{O}}$ is a well defined positivity and unit preserving map on $\mathcal{M}$ into $\mathcal{M}$ which extends to a symmetric contraction on the quantum $L_2(\mathcal{M}, 1/2)$-space.

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Corollary:

The formula (18) gives a well defined Markov generator of flip-spin type dynamics on infinite quantum system on a lattice.

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6. Diffusion type dynamics

To describe this type of dynamics let us begin with the introduction of "noncommutative" derivation ∇_x by the following formula

$$\nabla_x(f) \equiv i[x, f]$$

for x and f in $\mathcal{A}$. We need to assume that the following condition is true.

We say that the system $(\mathcal{A}, \alpha_t)$ possess the Strong Asymptotic Abelianess property iff

$$\int_0^\infty ||\nabla_{\alpha_s(x)}(f)||ds < \infty \tag{20}$$

for f in a dense subalgebra $\tilde{\mathcal{A}}$ in $\mathcal{A}$ and x is taken from some specified subset of $\mathcal{A}$ (see [9] for details). Let $\boldsymbol{K}(\cdot)$ be a positive defined function belonging to $L_1(\boldsymbol{R}, ds)$ and suppose that the condition (20) is satisfied. Then *an elementary Dirichlet form* $\mathcal{E}_x(\cdot\,,\,\cdot)$ can be introduced in the following way. Define on the domain $\mathcal{D} \equiv \mathcal{D}(\mathcal{E}_x) = \tilde{\mathcal{A}}$ the form $\mathcal{E}_x(\cdot,\cdot)$ as follows

$$\mathcal{E}_x(f, g) = \int dr\, ds\, \boldsymbol{K}(r-s)\langle \nabla_{\alpha_r(x)}(f), \nabla_{\alpha_s(x)}(g)\rangle_\omega \tag{21}$$

with $\langle\cdot,\cdot\rangle_\omega$ denoting a scalar product in a quantum L_2-space associated to a state ω. The subscript x denotes that the noncommutative derivation is performed in "x"-direction.

Suppose additionally that the function $\boldsymbol{K}$ is analytic in a strip Im $z \in [0, \beta]$ and satisfies the following conditions

$$\boldsymbol{K}(s-r) = \boldsymbol{K}(r-s+i\beta) \tag{22}$$

Then, under suitable technical conditions on the kernel $\boldsymbol{K}$, one can prove the following result, (see [9]).

Theorem 4:

The quantum Dirichlet form (22) is well defined. It defines the operator $\mathcal{L}_x; \tilde{\mathcal{A}} \to L_2(\omega)$ such that its closure (on the quantum L_2-space) is a Markov generator.

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7. Conclusions

We would conclude this lecture with the following corollary. *It is possible to define quantum L_p-spaces as well to give some concrete and explicit examples of Markov generators of stochastic dynamics for extended quantum systems.* In other words, Liouville space technique can be generalized to general quantum systems. Moreover, we have found (*cf.* [9, 10]) that this approach gives a possibility to study important ergodic properties (*e.g.* a convergence to equilibrium) of the considered examples of dynamics. Consequently, the proposed framework of quantum L_p-spaces is an interesting and useful tool for study of quantum stochastic dynamics.

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