CELLULAR AUTOMATA WITH VOTING RULE*,**

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The chosen local interaction- the voting (majority) rule applied to the square lattice is known to cause the non ergodic cellular automata behaviour. Presented computer simulation results verify two cases of non ergodicity. The first one is implicated by the noise introduced to the local interactions and the second one follows properties of the initial lattice configuration selected at random. For the simplified voting rule- non symmetric voting, the critical behaviour has been explained rigorously.

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1. Introduction

The cellular automaton systems are discrete dynamical systems. Space, time and states of the system are discrete. Each point in a regular spatial lattice, called a *cell* or *site*, can have one of finite number of states. The state of a cell is updated according to some local rule. All cells on the lattice are updated synchronously. Thus the state of the entire lattice, called *configuration*, changes in discrete time steps. A simply described local interaction, operating iteratively from a simple initial lattice state, often gives rise to variety of complex configurations at large time. These features make cellular automata particularly attractive systems for modelling physical complex systems [1, 2]. Therefore, there exists a huge number of computer observations.

In general, the main interest in cellular automata can be divided into two main parts:

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— *studies of cellular automata global characteristics*
This part contains, for example, problems with classification of cellular automata [3–7], problems with the construction of, so-called *natural measure*, [8], — the measure which is reached be cellular automata in the long-time evolution [5] ([9, 10] with the presence of some noise), and other very first questions when one considers a new object. The twenty main questions about cellular automata have been enumerated by Wolfram in [4] in 1985. However, it has appeared that the world of cellular automaton phenomenon is more difficult, or at least computationally more involved than the naive, first look, treatment. The mentioned twenty questions has been lately repeated by Wolfram [11] since they are still waiting for being answered. It is not surprising that this little progress in last decade has caused the little decrease in activity of scientists in this field. However, there is shared a hope, informally expressed by McIntosh [11], that maybe the time has come to reconsider the basic topics of cellular automata. The last papers of Gutowitz or Wuensche [12] as well as the Wolfram wish to come to cellular automaton subjects (see *A Note from the Publisher* in [1]) look like signs of this wider and wiser second look.

— *transferring (expressing) problems of other scientific activity into cellular automata systems*
It is a constantly increasing number of fields where the cellular automata ideas have been successfully applied. Let us mention, as examples, some very popular cellular automata applications like: lattice gases [13] which involves kind of hydrodynamics [14] and reaction-diffusion chemical systems [15], models of social [16] or biological and ecological systems [17] and finally to encrypt messages [18] (see [11] for more bibliography). All together makes the cellular automata field still focusing much attention, as one can easily verify by joining any of discussion lists present on the computer network. This part of cellular automata interest includes also the *Ising problem* transferred to cellular automata by Domany [19] — the problem of ferromagnetic interactions. Much effort is made for finding the local rules which applied to the d-dimensional lattice restore all global properties known from the theoretical predictions of d-dimensional Ising model [20]

This paper concentrates on such cellular automata, so-called *Toom model* [9], which can provide the solution to some three dimensional Ising problem. It is because, there is a natural way [19, 10] to rewrite d-dimensional time development of cellular automata layers into the $d+1$ dimensional equilibrium statistical model. However, the direct move from cellular automata to equilibrium statistical mechanics is possible if only cellular automata local dynamics is performed with some errors (so-called *probabilis-*

tic cellular automata [9]). In particular, the discrete-time dynamics of the Toom model perturbed by some noise on the square lattice is equivalent to the equilibrium statistical mechanics of the Ising model on *hcp* [19] or *fcc* [10] three dimensional triangular lattice. Therefore, it is interesting to examine the properties of Toom model, specially because the stationary measure for neither probabilistic nor deterministic models is known [21].

Thus in the following paper we will concentrate on two-level systems (called: *spins*, denoted $\sigma \in \{-1, 1\}$ or *down* and *up*, respectively) placed on the two-dimensional lattice. An every spin has a finite set of neighbours whose states determine its next time step state. The system is homogeneous — a local interaction does not depend on a lattice index. The updating spin state process is performed at discrete time steps and independently of other lattice nodes.

We will test the uniqueness of the cellular automata state in the time limit $t \to \infty$. This is the roughest, qualitative aspect of predicting the long time behaviour of complex systems. Systems which have the unique limit behaviour and converge to it from any initial conditions may be said to forget everything, when time tends to infinity. In such cases the statistical analysis of time series, by dealing with time averages, defines an unique ergodic measure. We call the system *ergodic* with respect to this measure [8]. In case of cellular automata such a measure is expected to be concentrated on an invariant set of Lebesgue measure zero [9]. However, there are systems which remember for ever some of its initial properties and its limit behaviour is not predictable. Such systems are called non-ergodic [9, 10, 21].

It is obvious that any voting system has at least two invariant configurations: all spins *up* (notation :(+)) and all spins *down* (notation: (−)). Therefore, the ergodic problem means here looking for conditions under which these invariant states are the stationary configurations and whether there are any other non trivial stationary configurations.

It is known [9], that the class of deterministic non-ergodic cellular automata systems is related to so-called *eroders*. It is said that the deterministic cellular automata system is the eroder if any finite island of one spin state is turned into the surrounding sea state, after finite steps of time. It appears that the evolution of cellular automata which are not eroders can easily be stopped by reaching some stationary configuration different from states (−) or (+). It usually happens that finite islands quickly grow and change and finally reach the shape in which the sea cannot influence them. However, if the system makes errors, then for any $\varepsilon > 0$ getting out of these traps is possible at the finite time, so then the cellular automata state relaxes to the pure sea state [9].

The examination of ergodicity in case of cellular automata systems can go two ways:

— *by adding ε- error to the deterministic dynamics*
It means that after applying the voting rule, the state of each spin makes an error with the probability ε to flip. This constitutes the probabilistic cellular automata and therefore, thanks to the correspondence of probabilistic cellular automata to equilibrium mechanics models, any high-temperature theory for the Gibbs state will find its equivalent as a high-noise theory for probabilistic cellular automata. It implies, for example, that the stationary state is Gibbsian [10]. However, the properties of the stationary state changes critically when the noise is low.

— *by varying the initial lattice state*
To examine the correlations arising from the Toom interactions itself the dependence on initial configurations is checked. Usually, the choice of an initial lattice state is restricted to, so-called, *typical initial states* [8]. These are configurations randomly prepared and their evolution does not exhibit any extravagance. For example, invariant initial states or states being the finite perturbation of them are not typical initial states. The typical initial states model thermodynamic physical systems with the highest disorder property. The time evolution leads cellular automata to highly ordered lattice configurations. Generally, there are two main attractors for Toom dynamics: all spins *up* and all spins *down*. The critical behaviour, ergodic versus non ergodic, is related to the parameter with respect to which the cellular automata change the attractor.

Computer results are the natural prerequisite for theoretical work. This paper is to establish them — Subsection 2.1. Most of them provide, at least, some qualitative suggestions for the possible explanation. Some rigorous calculus for a simplified voting rule is presented in the next Subsection 2.2. The main conclusions of the article are repeated in the last section.

2. Cellular automata with voting (Toom model)

The cellular automata system with the voting local rule is a typical example of a deterministic cellular automata system for which the observation of the critical behaviour can be examined. The local interactions, roughly speaking, go as follows: at each lattice site and at each time step there is made a vote — a spin takes the state of the majority of its nearest neighbours.

The simplest version of voting on a square lattice Z^2 is cellular automata with *North-East-Centre* majority vote. The local interactions follows the rule:

$$\sigma_{i,j}(t+1) = \mathrm{sgn}[\sigma_{i,j}(t) + \sigma_{i,j+1}(t) + \sigma_{i-1,j}(t)]\,, \qquad (1)$$

where $\sigma_{i,j}(t)$, $\sigma_{i,j+1}(t)$, $\sigma_{i-1,j}(t)$ is *Centre, East, North* nearest neighbour state, respectively.

One can consider the equivalent presentation of the model on the triangular lattice. In this case each site is directly above the centre of a triangle on the previous time layer (if the time arrow is directed up). Then the deterministic Toom rule assigns to a spin at i site the majority of the spins at the corners of the triangle directly above (*fcc* lattice).

Depending of the choice of interacting neighbours the voting rule can make cellular automata eroder or not. For example, the cellular automata with NEC rule is the eroder while the cellular automata with NEWSC rule (majority of North, East, South, West and Centre neighbours) is not the eroder. For example, the following island will exist for ever on the cellular automata configuration when NEWSC rule means the local interaction:

$$\begin{matrix} 0000 \\ 0110 \\ 0110 \\ 0000 \end{matrix}\quad . \qquad (2)$$

Generally, to have the eroder all the vectors of voting neighbours must be mutually non-coplanar [9].

2.1. Computer results

The first class of computer experiments involves deterministic voting rule perturbed by some error ε made at each time step and each lattice site. Cellular automata evolution is supposed to generate correlations between values at different sites. The very simplest measure of these correlations is the two-point correlation function.

Starting with the stationary for deterministic Toom interactions configuration of all spins *up*, we perform the Toom voting, with some error. The ε describes the probability of errors in a local dynamics. We assume that after some initial time steps (1000 steps) the system reaches its new stationary configuration. We count the events when two spins separated at some fixed distance are in the same spin state. This way, for each value of ε we obtain the dependence of probability to have two spins in the same state on their distance. The examined dependence is related to the standard two-point correlation function $c(0,i)$ as follows:

$$c(0,i) = 2\,\mathrm{Prob}\{(+,\cdots_{(i-1)\mathrm{sites}}\cdots,+) \cup (-,\cdots_{(i-1)\mathrm{sites}}\cdots,-)\} - C\,,$$

where $C = 1 - 2p(1-p)$ and p is the probability to find spin at *up* state on the stationary configuration.

By examining this function we expect to get hints about:

— the character of dependencies between spins separated far enough (long distance influence);

— the decay of correlations for close spins.

The results for NEC and NESWC voting rules averaged over 250 000 time steps for different values of ε are presented in Figs 1ab.

In simulations we found that the stationary configuration is disordered, however at the larger scale than the single lattice site, in large interval of ε values and the Bernoulli parameter p characterizing the random state depends on ε if ε is small enough ($p = 0.94,\ 0.92,\ 0.90,\ 0.86$ in case of $\varepsilon = 0.05,\ 0.06,\ 0.07,\ 0.08$, respectively) and becomes constant $p = {}^1\!/_2$ independently of *epsilon* when ε crosses some limit value. We can say like that because the correlations between spins in these ε intervals are dumped quickly — in less than 5–6 lattice units (Fig. 1c) and finally take the value of the probability to find pairs of spins: $++$ or $--$ on the lattice state characterized by the corresponding Bernoulli measure. In Fig. 1c, there is plotted the distance between spins for which the two-point correlation function value differs by 0.02 from its stationary value. One can see the rapid change of these two properties at $\varepsilon \approx 0.09$ in case of NEC rule, and if NESWC rule is considered at $\varepsilon \approx 0.14$. The long range dependencies between spins are present. So the lattice state for the $\varepsilon \in (0.08, 0.12)$- NEC dynamics and $\varepsilon \in (0.14, 0.18)$- NEWSC dynamics, is highly correlated. The rate of the decay of two spin dependencies was estimated by using *Mathematica* packet. The best fit, in our opinion — the exponential one, is presented in Fig. 1d. It suggests a very complicated structure of the stationary configuration.

The second class of computer experiments was made with random initial states. As was mentioned before, the NEC dynamics has many invariant states. However, only two of them: all $(-)$ and all $(+)$ seems to be accessible from the typical initial conditions. To verify this suggestion we perform experiments in which we examined the probability to find a spin in $\sigma = +$ state on the stationary lattice state when different initial random states were taken. To determine the influence of the periodic boundary conditions we examine cellular automata with different lattice sizes. We simulate cellular automata with $L = 48,\ 96,\ 192,\ 384$. Figs 2 and 3 collect averages of results of evolution when initial states are described by the probability p to find a spin in $+$ state in the initial random lattice state. Fig. 2a presents the probability to find a spin in $+$ state on the final lattice state versus p value. It is usually assumed that the computer results are satisfactory when

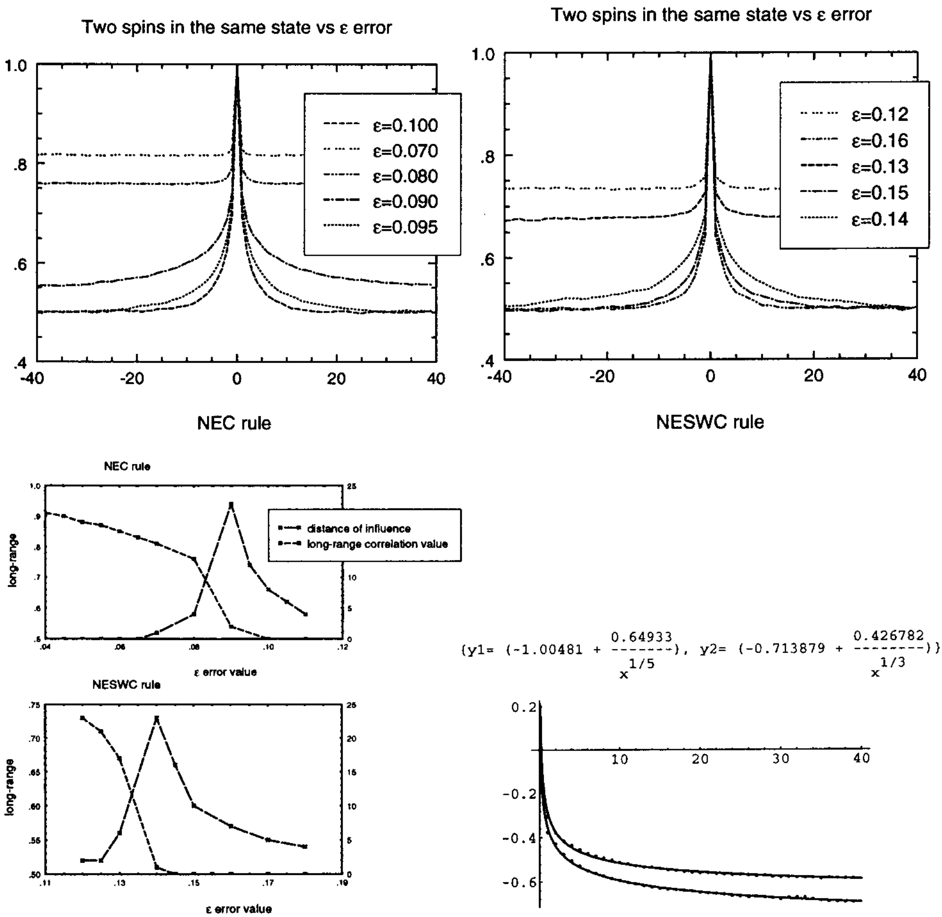

Fig. 1. The probability to find two spins in the same spin state is plotted at different errors ε for a) NEC rule, b) NESWC rule. Figure c) redraws the a) and b) results with respect to the error. Figure d) suggests the exponential fit for the decay of the correlations at the limit values of ε with y_1 and y_2 possible exponents for NEC and NESWC rules, respectively.

they describe effects which are obtained in time shorter than a lattice size. Therefore, Fig. 2b shows the probability to find a spin in + state but at given moments of time: $t = 48$, 96, 192, 384 for each lattice size whenever possible. The independence of corresponding values of the lattice size is evident. The other characteristic of final stationary states can be made when one compares their geometrical properties. We record the following types of final configurations: $(+)$, $(-)$ and there are two separated areas — parallelograms, of all spins in *up* state and all spins in *down* state. These

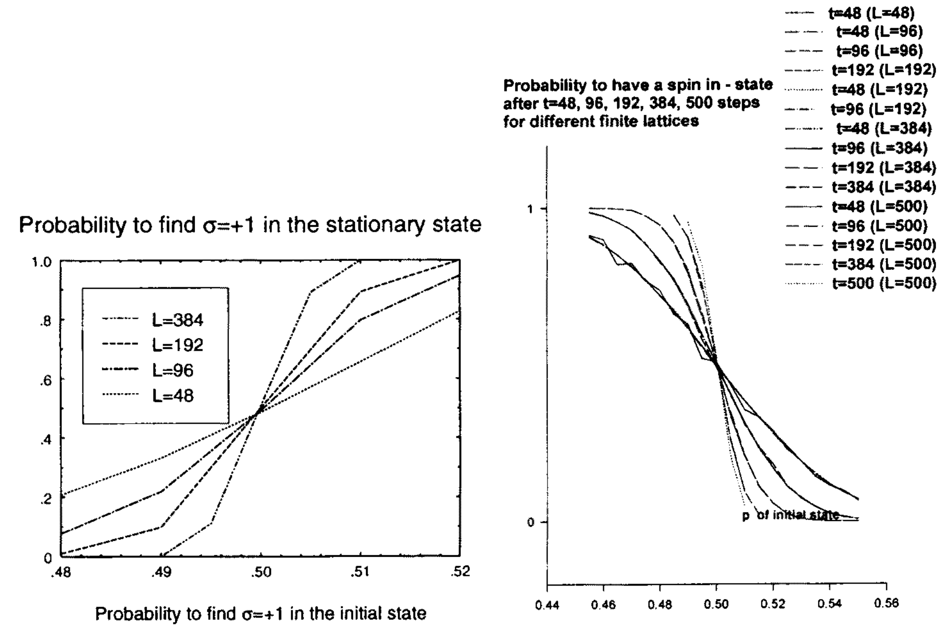

Fig. 2. The configuration characterization via contents of *ones*: a) in the stationary configurations, b) at the given moments $t = 48, 96, 192, 384$

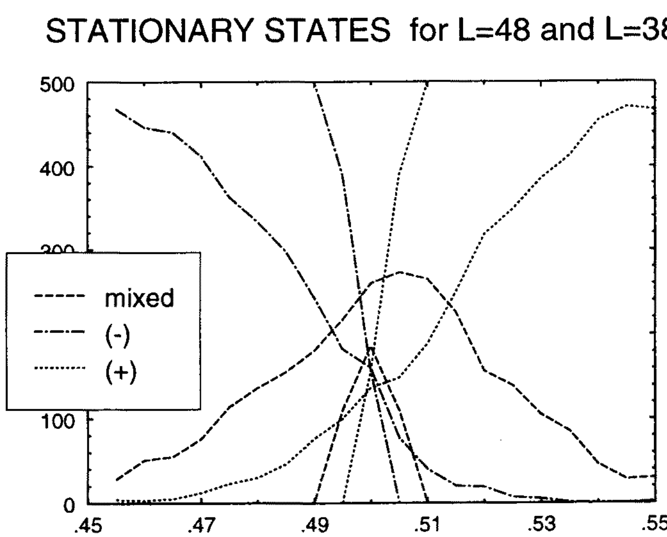

Fig. 3. The distribution into the three classes of stationary configurations for two lattice sizes.

geometrical properties of the stationary states of the cellular automata with periodic boundary condition can be expressed by the following:

Proposition [7]

If σ is any typical initial state then the set of measures of stationary states $\mathcal{M}_s$ consists of the following measures:

$$\mu \in \mathcal{M}_s \Leftrightarrow \bigwedge_{\substack{W\subset Z^2\times\{0,1,2,\ldots\}\\ |W|<\infty}} \text{either } \mu(\{\sigma_{((i,j),t)} \neq \sigma_{((i,j),t-1)} \bigwedge_{((i,j),t)\in W}\}) = 0$$

$$\text{or } \mu(\{\sigma_{((i,j),t)} \neq \sigma_{((i-1,j),t-1)} \bigwedge_{((i,j),t)\in W}\}) = 0$$

$$\text{or } \mu(\{\sigma_{((i,j),t)} \neq \sigma_{((i,j+1),t-1)} \bigwedge_{((i,j),t)\in W}\}) = 0\,. \tag{3}$$

Due to (3) the final pattern either does not change at all, or moves up, or moves right by one lattice unit at each time step if only the evolution starts from the typical initial state to exclude extraordinary behaviour. One can easily see that both lattice states $(+)$ and $(-)$ satisfy all three conditions while states with parallelograms fulfil only one of the above demands. In Fig. 3 the distribution of final states among these three classes is presented for different lattice sizes $L = 48, 384$, and with respect to the initial value of p. One can notice that the mixed states appear as stationary states only if an initial state is characterized by p close to $p = {}^1/_2$. For each lattice size one can find the p value at which these states occur. This observation indicates that the interval around $p = {}^1/_2$ contains points of the transition in the cellular automata behaviour. Therefore, the transition in case of cellular automata with periodic boundary conditions means the final stabilization of the lattice state at mixed patterns. Looking again at Fig. 2b one can see that the periodic boundaries break the free development of randomly scattered homogeneous islands built from one spin state. Earlier in time the stop is made (the lattice size shorter), the bigger number of islands are present on the lattice state. The periodic boundary condition makes the non-zero probability for an island to become a stationary parallelogram. Furthermore, because with the increase of the lattice size, the transition interval shrinks, then one can expect that finally single point $p = {}^1/_2$ is the critical point for the unbounded lattice. It indicates that the set of stationary measures $\mathcal{M}_s$ consists of two elements: one concentrated on $(-)$ and one concentrated on $(+)$.

The other type of computer experiment within this class of computer simulation is based on so-called coupled lattice method and it investigates the damage spreading [21]. In this experiment one records the results of simultaneous simulations of two cellular automata, which differs initially at one spin state. Brief studies of other lattices with the Ising type of interactions [21] allow to expect that such a consideration provide the estimation for the value of the dynamical critical exponent z which characterizes the critical slowing down at the second-order phase transition.

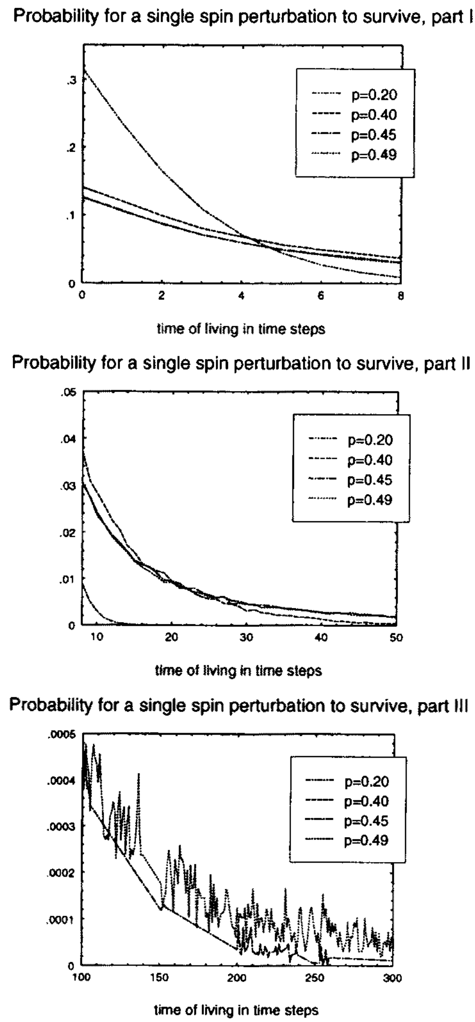

Fig. 4. The probability of propagation of a single state perturbation in three time regions: a) very initial development $0\ldots8$ first time steps, b) in $8\ldots50$ steps c) long living perturbations $100\ldots300$.

In Figs 4abc we present the probability to decay the single spin perturbation. The data come from experiments with the large lattice $L = 500$ and for typical initial states with $p = 0.20,\ 0.40,\ 0.45,\ 0.49$. The statistic was made over 182 000, 81 000, 168 000, 96 500, respectively, experiments. One can see that the relaxation to the equilibrium stationary configuration goes very fast in case of a small p (the maximum time observed for $p < 0.40$ was 23 steps). The complete decay of the discrepancies was also observed for the next two values: $p = 0.40$ and 0.45. However with the increase of p the number of perturbations long living on the lattice is greater. Results of experiments when we start at $p = 0.45$ and $p = 0.49$ do not differ much until about the 50th time step. If one starts the evolution on the random configuration at $p = {}^1/_2$, then there is non zero probability that the perturbation will survive for ever. In case of finite lattices, it denotes that the stationary states reached by coupled lattices were different.

2.2. Non-symmetric voting — rigorous results

Let us follow the simplified NEC rule — let only spins in *down* state take part in voting while spins in *up* do not vote, although, they are flipped. It means that the only situation when a spin state is changed is the following one: a Centre spin is in *up* state while both its nearest neighbours North and East are in *down* state. Let us examine thoroughly the propagation of the flip of a single spin. Because of this flip, states of the other spins will be adjusted at the next time step. The area of maximal influence of one spin flip is shown in Fig. 5a. It forms an expanding with time triangle. The place where the first flip occurs is the firm vertex of this triangle. The bond of the triangle opposite the vertex (lying along the diagonal of the lattice) moves one lattice unit per one time step in both directions West and South. The maximal propagation of one spin flip occurs only if there exist a special border. This special border consists of spins in *down* state

Remarks:

i: the maximal infection area never covers the entire lattice state. Moreover, any finite set of flipped spins cannot change the entire lattice state.

ii: the area of maximal infection grows with time as $1 + 2 + \ldots + t = {}^1/_2 t(t+1)$.

If the evolution starts with a random lattice configuration, then the chain of spins being in the same state- *down*, is little probable. Instead, one obtains infinitely many sources of infection which are scattered randomly all over a lattice. Each flipping spin is the origin for the infection and it owns its infection area. The individual infection area is randomly cut off from the maximal infection area. These cuts are random because of a random lattice state. However, one can formulate directives for marking the whole infection area:

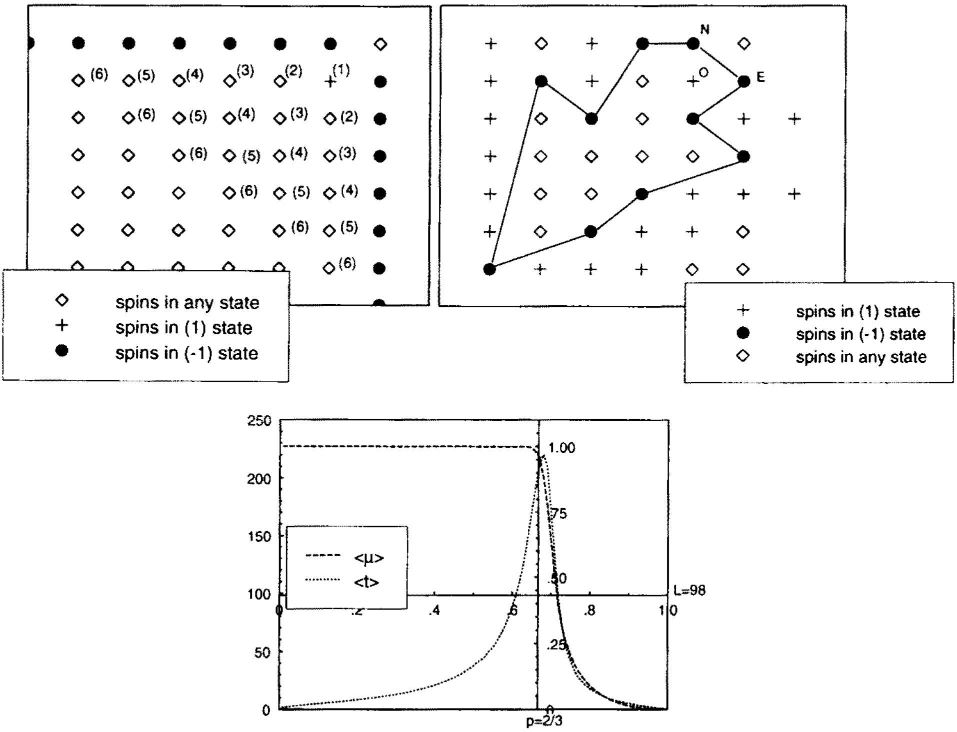

Fig. 5. a) The maximal infection area when one spin flips. b) The example of the construction of North_ and East_ Sides on some random configuration. North_ Side starts at N spin and goes West-South. East_ Side starts at E spin and moves to South- West. This is an example of the close border. The infection process stops after 8 time steps. If there is any other random infection area inside the presented one, the infection will stop earlier. c) The results of simulations for simplified voting rule: $\langle \mu \rangle$ represents the rate of spins in -1 state, $\langle t \rangle$ the average time steps to reach the stationary state.

Having an origin for the infection process — a single spin which will flip at the next time step, to determine North_ and East_Sides for the spin infection area one must go as follows (Fig. 5b):

— *the North_Side starts at the North neighbour of the origin and it moves West by one lattice unit to the next column. The next vertex of the side is chosen as the site of a spin in $(-)$ state in the following order: one lattice unit above, at the present position, one lattice unit below, two lattice units below, etc. This procedure repeats every next vertex selection.*

— *the East_Side starts at the East neighbour of the origin and it moves South by one lattice unit to the nearest row. The next vertex of the side* *is chosen as the site of a spin in (−) state in the following order: one unit to East, at the present position, one unit to West, two units to West, etc. This procedure repeats every next site selection.*

Generally, the sub-sequential points of North_ and East_Sides can be represented as the following random processes:

$$\overrightarrow{w(t)} = \overrightarrow{e_W} + \xi_t \overrightarrow{e_S}\,, \qquad \overrightarrow{s(t)} = \eta_t \overrightarrow{e_W} + \overrightarrow{e_S}\,, \tag{4}$$

where $\overrightarrow{e_W}, \overrightarrow{e_S}$ are lattice versors directed to West and South, respectively, and ξ_t and η_t are independent random variables with properties:

$$\xi_t, \eta_t = \begin{cases} -1 & \text{with probability } (1-p) \\ 0 & \text{with probability } (1-p)p \\ 1 & \text{with probability } (1-p)p^2 \\ \vdots & \\ k & \text{with probability } (1-p)p^{k+1} \end{cases} .$$

p is, as usual, the probability to meet (+) state in the initial random lattice state.
This way the North_ and East_Sides are represented by the following lattice sites:

$$\text{North\_Side} = \{(0,-1), (1,\xi_1), (2, \xi_1+\xi_2), \ldots\},$$
$$\text{East\_Side} = \{(-1,0), (\eta_1, 1), (\eta_1+\eta_2, 2), \ldots\}.$$

The condition that these two sides will cross somewhere is:
Observation:
There exist such k and m, $k, m = 1, 2, \ldots$ *that*

$$\text{for} \quad \eta_1 + \ldots + \eta_k = m \quad \text{is} \quad \xi_1 + \ldots + \xi_m \geq k \tag{5}$$

Hence, if we start with a random prepared initial configuration, then the expectation values of one step North_Side and East_Side vectors are:

$$\begin{aligned}\langle \overrightarrow{w} \rangle &= \overrightarrow{e_W} + \langle \xi_t \rangle \overrightarrow{e_S} = \overrightarrow{e_W} + (1-p)\overrightarrow{e_S}[-1 + p^2 \sum_{k=1}^{\infty} k p^{k-1}] \\ &= \overrightarrow{e_W} + \frac{2p-1}{1-p}\overrightarrow{e_S}\end{aligned}$$

and:

$$\langle \overrightarrow{s} \rangle = \langle \eta_t \rangle \overrightarrow{e_W} + \overrightarrow{e_S} = \frac{2p-1}{1-p}\overrightarrow{e_W} + \overrightarrow{e_S}.$$

Therefore, after k steps along North_Side we reach $k\overrightarrow{e_W} + k\frac{2p-1}{1-p}\overrightarrow{e_S}$ and after m steps along East_Side we get $m\overrightarrow{e_S} + m\frac{2p-1}{1-p}\overrightarrow{e_W}$. According to (5) it provides the following condition for crossing:

$$\frac{2p-1}{1-p}^2 \geq 1\,. \tag{6}$$

This condition is satisfied for all $p \in [^2/_3, 1]$. This result matches the result of simulations. The computer experiment was performed on a lattice $L = 98$ and the probability to find a spin in $(+)$ state on the initial lattice state p was increased by 0.01 from $[0, 1]$. There were 500 experiments for each p. The results by means the averages of the total final magnetization and the average number of time steps which was done to get the final stabilization are presented in Fig. 5c. The rapid change of both properties at $p = {}^2/_3$ is evident.

3. Summary

The analysis of the stationary states obtained in cellular automata evolution with local interactions described by North-East-Center nearest neighbours voting, yields the following statements:

1. the stationary state of probabilistic cellular automata, *i.e.* when the dynamics is perturbed by some noise measured by ε in case when $\varepsilon \in (0, 0.08) \cup (0.12, 1)$ is random at the scale of lattice blocks larger than 5. Moreover, the Bernoulli parameter p of that random state takes the value $p = {}^1/_2$ for all $\varepsilon > 0.12$. If $\varepsilon \in (0.08, 0.12)$ then the long-range correlations between spins, are present on the long-time configurations, and it implicates that the stationary measure posses its own complex structure. The exponential decay of two-spin dependencies suggests that the measure is of the Gaussian type.
2. The phenomenon of obtaining as stationary configurations the mixed configurations: clusters of all spins *up* and clusters of all spins *down* is the effect of the periodic boundary lattice conditions and it seems that those configurations are not reachable by unbounded systems which start at typical initial configurations.
3. The cellular automata evolution which starts at the typical initial configuration characterized by the Bernoulli parameter $p = {}^1/_2$, is chaotic in the sense that there exists non-zero probability to reach the different stationary state when the infinitesimal change to the lattice configuration is introduced.

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