SIMPLE MATHEMATICAL TOOL FOR STATISTICAL DESCRIPTION OF DYNAMICAL SYSTEMS UNDER RANDOM ACTIONS. I*

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In the lectures, the differentiation formulae (DF) for statistical averages is introduced in a compact form. The first part of these lectures is devoted to a general description of the DF method. The procedure of obtaining exact and closed equations for mean values and probability distributions of linear and nonlinear macroscopic systems driven by coloured noise is illustrated. As models of coloured random perturbations, both random jump processes (Kubo–Anderson and kongaroo processes) and diffusion processes (Ornstein–Uhlenbeck, Rayleigh and Pearson processes) are considered.

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1. Introduction

The development of science of two last decades demonstrates more and more wider interest and application of probability theory methods for the description of processes occurring in nature. The simplest formulation of a statistical description problem is to study some macroscopic system *in a given stochastic field.* This field simulates surroundings ("whole other world"), in which the considered system is "immersed". Probability characteristics of surroundings can be imposed directly or with the help of dynamical equations or kinetic (master) equations.

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One of the well-known physical system of this type is a Langevin model of Brownian movement of particles in liquid or gas. It is described by a linear stochastic equation in which the interaction between a Brownian particle and molecules of gas or liquid is modeled by Gaussian white noise. The most "working" models of fluctuations are the white noise models with Gaussian and Poisson statistics. There exists developed mathematical theory for them allowing to make statistical description, *i.e.* to obtain exact and closed equations for different averages, including probability distributions [1–5]. Most of real processes in physics, chemistry and biology are driven not by white (non-correlated) noises but rather by correlated (coloured) ones and therefore a more adequate description is desired. Such a description can be realized in the frame of nonlinear models with a multiplicative (or parametrical) type of stochastic forces with non-Markovian statistics and non-zero correlation time. Among models of coloured noise, the Gaussian Markovian (Ornstein–Uhlenbeck) and Markovian dichotomous (D-noise) processes play an important role. These "working horses " of statistical simulation have been used for testing various approximate methods of calculation of averages and for studying physical phenomena generated by noises of non-zero correlation time or non-zero correlation radius (for example, noise induced phase transitions) [6, 7]. It has allowed to reveal the fact of a fundamental role of noise correlations (like behaviour of order parameters in regions of structural instabilities of nonlinear dynamical systems subjected to stochastic correlated perturbations).

The subject of studying are dynamical systems, parameters of which can randomly vary with imposed probabilistic characteristics. These systems include the huge diversity of objects and processes occurring in nature and evolution of which is described by systems of ordinary or partial differential equations, or integro-differential equations. Such systems could be devided into three groups:

1. The dynamical systems of evolutionary type (*i.e.*, described by stochastic ordinary differential equations);
2. The dynamical systems with boundary conditions;
3. Spatially extended systems.

Each of listed group has its specific features, including mathematical methods used for their description.

The paper does not present all methods for treating all listed classes of macroscopic systems driven by stochastic forces and fields. For statistical description of such systems, a large numbers of concepts and methods have been developed [7–13]. This review deals with the application of the method of the differentiation formulae to linear and nonlinear macroscopic systems driven by coloured noises of different statistics. It allows for statistical analysis of these systems directly and systematically. The work should be

considered as an introduction to this method. The reader is referred to the books [14] and [15] where details on the description of the method and its applications to a number of problems (including physical and chemical kinetics) can be found. Related problems are also presented in [16] and [17].

2. Stochastic system of evolution type

Let us introduce the notation $x(t) = (x_1(t), x_2(t), ..., x_n(t))$ for a vector function. The set of variables $x_i(t), i = 1, 2, ..., n$ characterizes the main features of the system of interest. Let the dynamics of this object be described by a set of nonlinear differential equations of the type:

$$\dot{x} = f(t, x), \tag{1}$$

where the point over x denotes a differentiation with respect to time t, $f(t, x) = (f_1(t, x), f_2(t, x), ..., f_n(t, x))$ is a nonrandom vector function. Evolution of the state vector x of the system starts at time $t = 0$ from some fixed point $x_0 = x|_{t=0}$.

The systems like (1) describe the wide class of phenomena in mechanics, physics, chemistry, biology, *etc.* Simultaneously with the action of deterministic fields, stochastic forces drive the system. Their characteristics vary by a random way. The values of stochastic forces belong to the space of random events Ω. To take into account the influence of random forces on the system, additional terms in the equations of motion occur, namely,

$$\dot{x} = f(t, x) + g(t, x, \alpha(t)), \tag{2}$$

where g is a vector function depending on random variables, which are components of the random vector function $\alpha(t) = (\alpha_1(t), \alpha_2(t), ..., \alpha_m(t))$. For $\alpha(t)$ is random therefore components of the vector $x(t)$ are random as well and now the process $x(t)$ becomes a stochastic process.

The problem of statistical description of the system (2) is the calculation of various statistical characteristics of the dynamical variable $x(t)$ in dependence on statistics of the processes $\alpha_i(t)$, where $i = 1, 2, ..., m$.

According to Eq. (2), the solution $x(t)$ at $t > 0$ is a retarded functional of the process $\alpha(t)$. It means that the macroscopic variable $x(t)$ depends on values of the process $\alpha(\tau)$ at $\tau < t$ and it will be denoted by the symbol $x(t) \equiv x_t^{(\mathrm{ret})}[\alpha(\tau)]$ or simply $x_t[\alpha]$.

To illustrate this, consider two simple physical models:

(i) Brownian motion (an *additive* type of random perturbations)

$$\dot{x} = -\lambda x + \alpha(t),$$

where $x \equiv V$ is a particle velocity, λ is a friction constant, $\alpha(t)$ is the Gaussian white noise with characteristics $\langle\alpha\rangle = 0$, $\langle\alpha(t+\tau)\alpha(t)\rangle = 2D\delta(\tau)$. The solution of this equation is

$$x(t) = e^{-\lambda t}x_0 + \int_0^t e^{-\lambda(t-\tau)}\alpha(\tau)d\tau\,.$$

It is seen that the solution depends on the values of α by a retarded way.

(ii) The Kubo oscillator (a *multiplicate* type of random perturbations)

$$\dot{x} = i\alpha(t)x\,,$$

where *e.g.* x is a spin projection of some given atom, $\alpha(t)$ characterizes a stochastic field generated by other atoms and acting on a given atom. The solution of this equation is as follows

$$x(t) = \exp\left(i\int_0^t \alpha(\tau)d\tau\right)x_0\,.$$

The solution $x(t)$ depends on the values of α by a retarded way.

In a general case, the problem of calculation of average values

$$\left\langle F(t,\alpha)\Phi_t^{(\mathrm{ret})}[\alpha]\right\rangle$$

occurs. Here, $F(t,\alpha)$ is a nonrandom function of t and α, $\Phi_t^{(\mathrm{ret})}[\alpha(\tau)]$ is a function of t and a functional of retarded type of random process $\alpha(\tau)$, where $\tau < t$. The symbol $\langle\ldots\rangle$ means an operation of averaging over all realizations of the random processes considered. It has been shown that

$$\frac{d}{dt}\left\langle F(t,\alpha)\Phi_t^{(\mathrm{ret})}[\alpha]\right\rangle = \left\langle\frac{\partial(F\Phi_t^{(\mathrm{ret})})}{\partial t}\right\rangle + \left\langle\left[\hat{L}^+F\right]\Phi_t^{(\mathrm{ret})}\right\rangle\,, \qquad (3)$$

where $\hat{L}$ is an infinitesimal generator of the process $\alpha(t)$ and the symbol " $+$ " means here the operation of conjugation. If for example $\hat{L}$ is the forward Kolmogorov (or Fokker–Planck) operator then the operator $\hat{L}^+$ is backward Kolmogorov (or Fokker–Planck) operator.

As we see below, the formula (3) named as *differentiation formula* (DF for short) is a convenient mathematical tool for transformation of the averages $\langle F(t,\alpha)\Phi_t^{(\mathrm{ret})}[\alpha]\rangle$.

Let $(\alpha_1, \alpha_2, ..., \alpha_k) \equiv \Sigma$ be a value of the random process α at time $(t_1, t_2, ..., t_k) \equiv T$ respectivelly. Let the function $Q(\alpha, t|\Sigma, T)$ be a conditional probability density, *i.e.*, $Q(\alpha, t|\Sigma, T)d\alpha$ is the probability that the random process $\alpha(t)$ at time t be found in the interval $(\alpha, \alpha + d\alpha)$ under the condition that at times $t_1, t_2, ..., t_k$ the process α takes values $\alpha_1, \alpha_2, ..., \alpha_k$, respectively. Supposing that $t \notin T$, then following [18], one can write the master equation for $\alpha(t)$ in the form

$$\frac{\partial Q(\alpha, t|\Sigma, T)}{\partial t} = \hat{L} Q(\alpha, t|\Sigma, T)\,, \tag{4}$$

where the infinitesimal generator of the process $\alpha(t)$ is presented in the differential form (for simplicity we quote $\hat{L}$ for a one-dimensional case)

$$\hat{L} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \alpha^k} \Lambda_k(\alpha, t|\Sigma, T)\,. \tag{5}$$

The kinetic coefficients $\Lambda_k(\alpha, t|\Sigma, T)$ are conditional averages and read

$$\Lambda_k(\alpha, t|\Sigma, T) = \lim_{\tau \to 0} \frac{1}{\tau} \int (\alpha(t) - \alpha(t-\tau))^k Q(\alpha, t|\eta, t-\tau; \Sigma, T) d\eta\,. \tag{6}$$

The representation (4), (5) can be considered as a master equation in a generalized Kramers–Moyal reprezentation for random processes with memory.

In particular case, when $\alpha(t)$ is assumed to be a diffusion process with memory, only two first kinetic coefficients $\Lambda_1(\alpha, t|\Sigma, T)$ and $\Lambda_2(\alpha, t|\Sigma, T)$ do not equal to zero. In this case the master equation (4) is a generalized Fokker–Planck equation. Considering a random processes with memory, we can use in the differentiation formulae the conjugate operator,

$$\hat{L}^+ = \sum_{k=1}^{\infty} \frac{1}{k!} \Lambda_k(\alpha, t|\Sigma, T) \frac{\partial^k}{\partial \alpha^k}\,, \tag{7}$$

or its generalizations for multi-dimensional cases.

It should be noted that the DF (3) is the differentiation formula of first order with respect to time variable. One can introduce formulae of differentiation of the averages $\left\langle F(t, \alpha) \Phi_t^{(\mathrm{ret})}[\alpha] \right\rangle$ of higher order with respect to time. The structure of these formulae is as follows [15]

$$\frac{d^n}{dt^n} \left\langle F(t, \alpha) \Phi_t^{(\mathrm{ret})}[\alpha] \right\rangle = \left\langle \frac{\partial^n (F \Phi_t^{(\mathrm{ret})})}{\partial t^n} \right\rangle + \left\langle \left[\hat{L}_n^+ F \right] \Phi_t^{(\mathrm{ret})} \right\rangle\,. \tag{8}$$

The explicite form of $\hat{L}_n^+$ is not presented here since we do not use it in the paper. We refer the reader to paper [19] containing a discussion on the existence of master equations of higher order with respect to time.

In the simplest case when $\alpha(t)$ is a random process of a Markovian type the dependence of kinetic coefficients Λ_k on prehistoric (Σ, T) is lost and $\Lambda_k(\alpha, t|\Sigma, T) \equiv \Lambda_k(\alpha, t)$. Below, Markovian random perturbations with finite correlation time decay imposed on a macroscopic system will be investigated.

The formulae of differentiation for selected and widely used models of Markovian processes are offerred in [20–22]. In these papers, the application of DF to analysis of dynamical systems with fluctuating parameters was given. General structures of DF has been established in [17] by use of a semigroup method and for a class of Markovian random perturbations in [23] in the framework of a functional approach. The unification of the DF methods is given in [13, 14]. The generalization of formulae of differentiation for the case of stochastic boundary problem is carried out in [24].

The paper is organized as follows. In Sections 2.1–2.3 we apply DF to linear and nonlinear macroscopic systems driven by coloured noise of a telegraphic type (Kubo-Anderson noise, kongaroo processes *etc*). In Section 2.4 we give application of DF method for statistical description of macroscopic systems driven by random processes of diffusion type. Examples of random perturbations are Ornstein–Uhlenbeck, Rayleigh and Pearson coloured noises. Section 2.5 is devoted to the application of DF method to a dynamical system subjected to exponentially correlated Poisson noise. In this section we investigate the relationship between DF and Ito approach as well. Finally, in Section 2.6 we give the application of DF in cumulant representation.

2.1. Dynamical systems perturbed by processes of telegraphic type

2.1.1. Kubo-Anderson noises [25–27]

The most elementary representation of random telegraphic processes is Markovian dichotomous noise. It is a jump random function of time which takes the constant values $+\sigma$ and $-\sigma$; the jumps from one value to other happen in random time and independently from one another with an average frequency ν. The values $\pm\sigma$ are equiprobable. The Kubo–Anderson processes are a more general class of telegraphic-type random processes (KA processes for short). They represent random step functions $\alpha(t)$ taking values from some set $c_1, c_2, ..., c_n$ at random time. The jumps from one value to other are independent and distributed uniformly over time with

density ν (*i.e.* on average, νdt jumps happen on interval dt). The density ν does not depend on from or to which value the jump proceeds. The probability of occurrence N jumps on intervals Δt is given by the Poisson distribution

$$P(N,t) = \frac{(\nu\Delta t)^N}{N!}\exp[-\nu\Delta t]\,.$$

The probability density $p(c)$ of amplitudes c_i of the Kubo–Anderson process $\alpha(t)$ is equal to:

$$p(c) = \sum_{k=1}^{n} p_k \delta(c - c_k)\,,$$

where p_k is the probability of values c_k. The set $\{c_k\}$ can be both discrete and continue or represents its mixture. KA processes model a stochastic motion in microscopic many particle systems, in particular, relaxation processes. The KA processes are a special class of Markovian processes that are described by the Kolmogorov–Feller master equation. The kinetic operator $\hat{L}$ for the KA processes has the form

$$\hat{L}f(t,\alpha) = -\nu f(t,\alpha) + \nu p(\alpha)\int f(t,\alpha')d\alpha'\,.$$

The formula of differentiation for KA processes has the form

$$\frac{d}{dt}\langle F(t,\alpha)\Phi_t[\alpha]\rangle = \left\langle \frac{\partial}{\partial t}(F(t,\alpha)\Phi_t[\alpha])\right\rangle - \nu\langle F\Phi_t\rangle + \nu\langle F\rangle\langle\Phi_t\rangle\,. \qquad (9)$$

In that specific case, when $F = (\alpha(t))^k$ $(k = 1,2,...)$, the result (9) reduces into the simple formula

$$\frac{d}{dt}\left\langle \alpha^k\Phi_t[\alpha]\right\rangle = -\nu\left\langle \alpha^k\Phi_t\right\rangle + \nu\left\langle \alpha^k\right\rangle\langle\Phi_t\rangle + \left\langle \alpha^k\dot{\Phi}_t\right\rangle\,, \qquad (10)$$

where $\dot{\Phi}_t[\alpha] = \partial\Phi_t[\alpha]/\partial t$.

When $\alpha(t)$ is D-noise then from (10) we obtain the simplest formula of differentiation (because the remarkable property of D-noise -$\alpha^2(t) \equiv \sigma^2 =$ *const* holds)

$$\frac{d}{dt}\langle\alpha\Phi_t[\alpha]\rangle = -\nu\langle\alpha\Phi_t\rangle + \left\langle\alpha\dot{\Phi}_t[\alpha]\right\rangle\,. \qquad (11)$$

2.1.2. Averaging of linear dynamical systems

We begin the consideration with linear systems. Let $\alpha(t)$ be KA processes and a macroscopic variable x is a solution of the following stochastic equation

$$\dot{x} = Ax + B\alpha x + f, \tag{12}$$

where A and B are matrices $n \times n$. The vector function $x^T = (x_1(t), x_2(t), ..., x_n(t))$ and the function $f^T(t) = (f_1(t), f_2(t), ..., f_n(t))$ is a deterministic or stochastic process with given characteristics. The symbol $(...)^T$ means here the transposition of a vector.

We consider the case of a scalar process $\alpha(t)$. First of all, we derive an equation for average $\langle x(t) \rangle$ for $\alpha(t)$ being the D-noise. After averaging over the statistics of α, one gets

$$\langle \dot{x} \rangle = A\langle x \rangle + B\langle \alpha x \rangle + \langle f \rangle . \tag{13}$$

The obtained equation for average $\langle x \rangle$ contains known averages $\langle x \rangle$ and $\langle f \rangle$ and new unknown average $\langle \alpha x \rangle$, in which dynamical variable $x(t)$ is a retarded functional of the process α. Assuming $\Phi_t[\alpha] \equiv x(t)$ and applying to average $\langle \alpha x \rangle$ the formula of differentiation (11) yields

$$\frac{d}{dt} \langle \alpha x \rangle = -\nu \langle \alpha x \rangle + \langle \alpha \dot{x} \rangle .$$

Now taking $\dot{x}$ from (12) and inserting it into above equation and using the property of D-noise $\alpha^2(t) \equiv \sigma^2 = const$, we find a closed system for the average $\langle x \rangle$ and $x_1 = \langle \alpha x \rangle$:

$$\begin{aligned} \langle \dot{x} \rangle &= A\langle x \rangle + Bx_1 + \langle f \rangle , \\ \dot{x}_1 &= (A - \nu)x_1 + \sigma^2 B\langle x \rangle + \langle \alpha f \rangle , \end{aligned} \tag{14}$$

with initial conditions $\langle x(t) \rangle|_{t=0} = \langle x_0 \rangle$ and $x_1(t=0) = \langle \alpha(0) x_0 \rangle$. If x_0 is a nonrandom variable then $x_1(t=0) = 0$ because $\langle \alpha(0) \rangle = 0$. The additional averaging over the stochastic process $f(t)$ does not change the structure of the system obtained. The averages $\langle f(t) \rangle$ and $\langle \alpha(t) f(t) \rangle$ are given by the relations

$$\langle f(t) \rangle = \int f P(f,t) df ,$$

$$\langle \alpha(t) f(t) \rangle = \int \int \alpha f P(\alpha, f, t) df d\alpha ,$$

where $P(\alpha, f, t)$ is a joint probability density for the processes $\alpha(t)$ and $f(t)$, $P(f,t)$ is the probability density for $f(t)$.

Let averages $\langle f \rangle$ and $\langle \alpha f \rangle$ do not depend on time. Then from (14) it follows that in the stationary state:

$$\langle x \rangle_{\rm st} = \left[A - \sigma^2 B (A - \nu)^{-1} B \right]^{-1} \left[B (A - \nu)^{-1} \langle \alpha f \rangle - \langle f \rangle \right] , \qquad (15)$$

where $\langle x \rangle_{\rm st} = \lim\limits_{t \to \infty} \langle x(t) \rangle$ and the symbol $(...)^{-1}$ denotes an inverse matrix. Further we shall use this solution to calculate the stationary mean-square of $x(t)$ for a linear oscillator driven parametrically by D-noise (the oscillator with fluctuating frequency).

Let us use the result (14) to derive an equation for the characteristic functional $\chi_t[v(\tau)]$ of Markovian D-noise. The characteristic functional of a random process $\alpha(t)$ is determined by the formula

$$\chi_t[v(\tau)] \equiv \langle \widetilde{\chi}_t[v(\tau)] \rangle = \left\langle \exp i \int\limits_0^t v(\tau) \alpha(\tau) d\tau \right\rangle , \qquad (16)$$

where $v(t)$ is a nonrandom function possessing necessary properties for existance of the average (16). The stochastic variable $\widetilde{\chi}_t[v(\tau)]$ is a solution of the following stochastic equation

$$\frac{d\widetilde{\chi}_t}{dt} = i v(t) \alpha(t) \widetilde{\chi}_t , \qquad (17)$$

with initial condition $\widetilde{\chi}_0 = 1$. This equation is a particular case of the system (12). So, we can utilize the general formula (14) for the case when $x(t) = \widetilde{\chi}_t$, $B = iv(t)$ and $A, f = 0$. The result is [23]

$$\frac{d^2 \chi_t}{dt^2} + \left(\nu - \frac{d \ln v(t)}{dt} \right) \frac{d\chi_t}{dt} + \sigma^2 v^2(t) \chi_t = 0 , \qquad (18)$$

with initial conditions: $\chi_0 = 1$, $\dot{\chi}_t|_{t=0} = 0$. It has the similar form as the equation for a linear oscillator with time-varying damping and frequency. For an arbitrary function $v(t)$, an analytical solution of this equation is not known.

Let now $\alpha(t)$ be the KA process of a general kind. Then for dynamical variables $x_k(t) = \langle \alpha^k(t) x \rangle$ $(k = 1, 2, ...)$, with the help of the formula of differentiation (10) and the stochastic equation (12), one can find the following chain of equations

$$\begin{aligned} \langle \dot{x} \rangle &= A \langle x \rangle + B x_1 + \langle f \rangle , \\ \dot{x}_k &= (A - \nu) x_k + B x_{k+1} + \langle \alpha^k f \rangle + \nu \langle \alpha^k \rangle \langle x \rangle . \end{aligned} \qquad (19)$$

It is convenient to assume that $x(0) = 0$ and the dependence on initial conditions can be transfered on vector function $f(t)$. It is always possible by performing a linear change of variables. Then $x_k(0) = 0$.

2.1.3. Transformation of the chains

Transform (19) to the form

$$\begin{aligned} \langle \dot{x} \rangle &= A\langle x \rangle + Bx_1 + \langle f \rangle , \\ x_k &= \hat{l}(Bx_{k+1} + \langle \alpha^k f \rangle + \nu \langle \alpha^k \rangle \langle x \rangle) . \end{aligned} \tag{20}$$

Here the operator $\hat{l}$ is defined by

$$\hat{l}g = \left(\frac{d}{dt} + \nu - A \right)^{-1} g = \int_0^t G(t,t')g(t')dt' ,$$

where $G(t,t')$ is the Green matrix function representing the solution of the equation

$$\left(\frac{d}{dt} + \nu - A \right) G(t,t') = 0, G(t,t) = I ,$$

(I is the unit matrix). Making the iteration in the second equation of the system (20), we get a closed equation for the average $\langle x \rangle$

$$\begin{aligned} \langle \dot{x} \rangle = A\langle x \rangle &+ \nu(\langle \alpha \rangle B\hat{l} + \langle \alpha^2 \rangle (B\hat{l})^2 + ...)\langle x \rangle \\ &+ (\langle f \rangle + B\hat{l}\langle \alpha f \rangle + (B\hat{l})^2 \langle \alpha^2 f \rangle + ...) . \end{aligned}$$

Because $(1 - cB\hat{l})^{-1}$ is equal to

$$\frac{1}{1 - cB\hat{l}} = 1 + cB\hat{l} + (cB\hat{l})^2 + (cB\hat{l})^3 + ...$$

Instead of (20), one can write

$$\langle \dot{x} \rangle = (A + \hat{A}_{\text{ef}})\langle x \rangle + f_{\text{ef}} , \tag{21}$$

where

$$\begin{aligned} \hat{A}_{\text{ef}} &= \nu \left\langle \frac{cB\hat{l}}{1 - cB\hat{l}} \right\rangle_c = \nu B \int \frac{cp(c)dc}{\frac{d}{dt} + \nu - A - cB} , \\ f_{\text{ef}} &= \left\langle \left\langle \frac{1}{1 - cB\hat{l}} f \right\rangle_f \right\rangle_c = \left(\frac{d}{dt} + \nu - A \right) \int \int \frac{1}{\frac{d}{dt} + \nu - A - cB} \\ &\quad \times fP(t,f|c)p(c)dcdf . \end{aligned}$$

The matrix $\hat{A}_{\mathrm{ef}}$ and the function f_{ef} characterize the renormalization of the system dynamics driven by random perturbations. This result is exact. For constant A and B, Eq. (21) can be solved by the Laplace transformation method.

We consider now a more difficult case, when the random parameter enters by a nonlinear way. Let dynamical variable $x(t)$ be a solution of the equation

$$\dot{x} = A(\alpha)x + f\,, \tag{22}$$

where $A(\alpha)$ is the matrix $n \times n$, other notations and conditions on the variables are the same as previously. The aim is to derive an equation for the average $\langle x(t)\rangle$ when $\alpha(t)$ is the KA process. It is convenient to introduce the variable $x_k = \langle A^k(\alpha)x\rangle$. As a result of the application of the differentiation formula (9) for KA processes, we obtain the following chain

$$\begin{aligned} \langle \dot{x}\rangle &= x_1 + \langle f\rangle\,, \\ \dot{x}_k &= -\nu x_k + x_{k+1} + \langle A^k(\alpha)f\rangle + \nu\langle A^k(\alpha)\rangle\langle x\rangle\,, \end{aligned} \tag{23}$$

with zero initial conditions, $x_k(0) = 0$, $(k = 0, 1, 2, ...)$. The structure of the derived equations is identical as considered above (see (20)). Using the same procedure, we have

$$\left[1 - \nu\left\langle \frac{1}{\frac{d}{dt} + \nu - A(c)}\right\rangle_c\right]\langle x\rangle = \left\langle \frac{1}{\frac{d}{dt} + \nu - A(c)} f\right\rangle . \tag{24}$$

Average of the right hand side is equal to

$$\begin{aligned} \varphi &\equiv \left\langle \frac{1}{\frac{d}{dt} + \nu - A(c)} f\right\rangle \\ &= \int\int\int_0^t e^{(A(c)-\nu)(t-t')} f P(t', f|c)p(c)dt'\,df\,dc\,. \end{aligned}$$

Applying the Laplace transformation to (24) one obtains

$$X(s) = \left[1 - \nu\left\langle \frac{1}{s + \nu - A(c)}\right\rangle_c\right]^{-1}\varphi(s)\,, \tag{25}$$

where

$$X(s) = \int_0^\infty \exp(-st)\langle x(t)\rangle dt, \varphi(s) = \int_0^\infty \exp(-st)\varphi(t)dt\,.$$

2.1.4. Averaging of nonlinear systems

Let the behaviour of some macroscopic system is described by the equation

$$\dot{x} = u(x,t) + \alpha(t)v(x,t)\,, \tag{26}$$

where $x(t)$ is a dynamical variable, $u(x,t)$ and $v(x,t)$ are nonrandom vector functions of variable x and t, $\alpha(t)$ is a scalar random process of telegraphic type. We suppose that the initial conditions are fixed at $t = 0$. For nonlinearity of initial dynamic equations, the problem of finding of various average from x is usually formulated in terms of probability distributions for the dynamical variables. The base of consideration here is the Liouville stochastic equation in space of variables of the dynamic system (26). In particular, for the stochastic density $\widetilde{P}(x,t)$ we have

$$\frac{\partial \widetilde{P}(x,t)}{\partial t} + div[(u + \alpha v)\widetilde{P}(x,t)] = 0\,. \tag{27}$$

The average of the function $\widetilde{P}(x,t)$ over ensemble of realizations of the process $\alpha(t)$ presents simply a one-point density of probability $P(x,t)$, that is $P(x,t)dx = \langle \widetilde{P}(x,t)\rangle dx$ is the probability that at time t the values of a macroscopic variable x are located on an interval $(x, x + dx)$. For (27) one should impose an initial condition at $t = 0$. After averaging both hand sides of the Liouville equation over statistics α, one obtains

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}u\right) P + \frac{\partial}{\partial x}v\left\langle \alpha\widetilde{P}\right\rangle = 0\,.$$

Here and below we use the notation

$$\frac{\partial}{\partial x}(...) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(...)_i\,.$$

Let $\alpha(t)$ be D-noise. Then the application of the differentiation formula (11) leads to the equation

$$\frac{\partial}{\partial t}\langle \alpha(t)\widetilde{P}(x,t)\rangle = -\nu\langle \alpha(t)\widetilde{P}(x,t)\rangle + \left\langle \alpha(t)\frac{\partial \widetilde{P}}{\partial t}\right\rangle .$$

Using instead of $\partial\widetilde{P}(x,t)/\partial t$ the right part of the Eq. (27) and taking into account that $\alpha^2(t) \equiv \sigma^2 = \text{const}$ we get the exact set of equations for distributions $P(x,t)$ and auxillary average $P_1(x,t)=\langle \alpha(t)\widetilde{P}(x,t)\rangle$:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}u\right) P + \frac{\partial}{\partial x}vP_1 = 0\,,$$

$$\left(\frac{\partial}{\partial t} + \nu + \frac{\partial}{\partial x}u\right) P_1 + \sigma^2\frac{\partial}{\partial x}vP = 0\,. \tag{28}$$

At $t = 0$ we have

$$P(x,0) = \langle \widetilde{P}(x,0) \rangle, P_1(x,0) = \langle \alpha(0)\widetilde{P}(x,0) \rangle .$$

The obtained system is exact and closed.

In a one dimension case and time independent u and v the system (28) can easily be solved at the stationary limit. A steady-state probability distribution $P_{\rm st}(x)$ has the form

$$P_{\rm st}(x) = N \frac{|v(x)|}{|\sigma^2 v^2 - u^2|} \exp\left[\nu \int \frac{u(x)dx}{\sigma^2 v^2 - u^2}\right], \tag{29}$$

where N is the normalization constant.

Let now $\alpha(t)$ be the Kubo–Anderson process. Introducing additional averages $P_k(x,t) = \langle \alpha^k \widetilde{P}(x,t) \rangle$, $k = 0, 1, ...$ and using the formula (10), one obtains the chain

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} u\right) P + \frac{\partial}{\partial x} v P_1 = 0 ,$$

$$\left(\frac{\partial}{\partial t} + \nu + \frac{\partial}{\partial x} u\right) P_k + \frac{\partial}{\partial x} v P_{k+1} = \nu \langle \alpha^k \rangle P ,$$

with initial conditions $P_k(x,0) = \langle \alpha^k(0)\widetilde{P}(x,0) \rangle$. Similarly as it was done for the linear case (12), from the above chain one gets

$$\left[\frac{\partial}{\partial t} + \hat{u} + \nu \hat{v} \left\langle \frac{c}{\frac{\partial}{\partial t} + \nu + \hat{u} + c\hat{v}} \right\rangle_c \right] P = 0 , \tag{30}$$

where $\hat{u}$,$\hat{v}$ are the operators the action of which on an arbitrary function $g(t,x)$ is determined as

$$\hat{u}g = \frac{\partial}{\partial x}(ug) , \hat{v}g = \frac{\partial}{\partial x}(vg) .$$

The action of the operator inside $\langle ... \rangle_c$ can be calculated through the Green function method of an appropriate Cauchy problem.

We give now the results for a one-point probability density $P(x,t)$ for a dynamic system of the form

$$\dot{x} = U(x, \alpha(t)) , \tag{31}$$

where $U(x, \alpha(t))$ is a nonrandom vector function, $\alpha(t)$ is the KA process. The stochastic Liouville equation for the considered nonlinear dynamic system is of the form

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} U\right) \widetilde{P} = 0 . \tag{32}$$

We introduce the notations

$$P_k = \left\langle \hat{U}^k \widetilde{P} \right\rangle, \hat{U}g = \frac{\partial}{\partial x}(Ug), k = 0, 1, 2, \ldots .$$

Applying to averages $P_k(x,t)$ the formulae of differentiation (9) and using initial equation (32) gives the chain of equations

$$\frac{\partial P}{\partial t} + P_1 = 0 ,$$

$$\left(\frac{\partial}{\partial t} + \nu\right) P_k + P_{k+1} = \nu \left\langle \hat{U}^k \right\rangle P , \tag{33}$$

$(k = 1, 2, \ldots)$. Since the process $\alpha(t)$ is a stationary random process then we can use the Laplace transformation method. For the Laplace transform $P(x,s) = \int_0^\infty P(x,t) \exp(-st) dt$, we have

$$P(x,s) = \left[s + \nu \left\langle \frac{\hat{U}}{s + \nu + \hat{U}} \right\rangle_c \right]^{-1} \left\langle \frac{s+\nu}{s+\nu+\hat{U}} \widetilde{P}(x,0) \right\rangle_c .$$

2.2. Kongaroo type noises

The kangaroo processes are generalization of a class of Kubo-Anderson telegraphic processes in the sense that frequency of jumps from a state to another depends on the state from which the jump proceeds, *i.e.* parameter ν becomes function of the α, $\nu = \nu(\alpha)$. This feature makes this type of models more adequate in physical reality. The property allows to apply widely the kongaroo processes in spectroscopy, for example for the description of the Stark broading of spectral lines [28,29], of interaction of spin systems with thermostat [30] *etc.*

The considered class of random processes referes to Markovian processes controlled by the Kolmogorov-Feller master equation [27]

$$\frac{\partial Q}{\partial t} = -\nu(\alpha) Q + \frac{\nu(\alpha) p(\alpha)}{\langle \nu \rangle} \int \nu(\alpha') Q(\alpha, t | \alpha', 0) d\alpha' ,$$

where $Q(\alpha, t|\alpha', 0)d\alpha$ is the probability that the process $\alpha(t)$ at time t takes the value from interval $(\alpha, \alpha + d\alpha)$ under the condition that at initial time $t = 0$ the value of the process is equal to α'. Various kongaroo processes are characterized by various correlation functions,

$$K(t) = \langle (\alpha(t) - \langle \alpha \rangle)(\alpha(0) - \langle \alpha \rangle) \rangle$$

$$= \int_\alpha (\alpha - \langle \alpha \rangle)^2 e^{-\nu(\alpha)|t|} p(\alpha) d\alpha .$$

The distribution $p(\alpha)$ is defined in Section 2.1. When $\nu(\alpha)$ does not depend on α then $K(t)$ is an exponential function of time. Otherwise, $K(t)$ depends on the form of $\nu(\alpha)$ and $p(\alpha)$.

The formula of differentiation for the kongaroo processes has the form

$$\frac{d}{dt}\langle F(t,\alpha)\Phi_t[\alpha]\rangle = -\langle \nu F\Phi_t\rangle + \frac{\langle \nu F\rangle}{\langle \nu\rangle}\langle \nu\Phi_t\rangle + \left\langle \frac{\partial}{\partial t}(F\Phi_t)\right\rangle . \tag{34}$$

If the parameter ν is independent of α then the formula of differentiation (34) reduces to the formula of differentiation (9) for the KA processes.

2.2.1. Equation for averages

We begin the consideration with a simple linear stochastic equation

$$\dot{x} = A(\alpha)x + f\,, \tag{35}$$

where $x(t)$ and $f(t)$ are n-dimensional vector functions, $A(\alpha) = \| A_{ij}(\alpha)\|_1^n$ is a matrix depending on the random process α. The initial conditions can be included to the function f. So, we suppose $x(0) = 0$.

We derive the equation for average $\langle x(t)\rangle$. It is convenient to use the averages $x_{km} = \langle \nu^m A^k x\rangle$. The average we are interested in is $\langle\, x\,\rangle = x_{00}$. Putting $F(t,\alpha) = \nu^m A^k$, $\Phi_t = x$ in the formula of differentiation (34) for averages x_{km} one obtains the hierarchy of the equations

$$\dot{x}_{km} = A_{km}x_{01} + f_{km} + x_{k+1,m} - x_{k,m+1}\,, \tag{36}$$

where $k, m = 0, 1, 2, ...,$

$$A_{km} = \frac{\langle \nu^{m+1}A^k\rangle}{\langle \nu\rangle}, f_{km} = \langle \nu^m A^k f\rangle\,.$$

The initial conditions for all $x_{km}(0)$ are obviously also zero. Rewritting (36) in the form

$$x_{km} = \hat{I}(A_{km}x_{01} + f_{km} + x_{k+1,m} - x_{k,m+1})\,,$$

where

$$\hat{I}g = \int_0^t g(t')dt'$$

and iterating the terms $x_{k+1,m} - x_{k,m+1}$, we get

$$x_{km} = \hat{I}(A_{km}x_{01} + f_{km} + \hat{I}(A_{k+1,m}x_{01} - A_{k,m+1}x_{01} + f_{k+1,m} - f_{k,m+1} + \hat{I}(...)...) = \hat{I}\Big\langle [1 + \hat{I}(A-\nu) + \hat{I}^2(A-\nu)^2 + ...] \times \left(\frac{\nu^{m+1}A^k}{\langle\nu\rangle}x_{01} + \nu^m A^k f\right)\Big\rangle . \tag{37}$$

After some calculations we obtain the closed set of two equations for average $\langle x(t)\rangle = x_{00}$ and an auxiliary average x_{01}

$$\langle x\rangle = \frac{1}{\langle\nu\rangle}\left\langle \frac{\nu}{\frac{d}{dt}+\nu-A}\right\rangle_c x_{01} + \left\langle \frac{1}{\frac{d}{dt}+\nu-A}f\right\rangle_c ,$$
$$x_{01} = \frac{1}{\langle\nu\rangle}\left\langle \frac{\nu^2}{\frac{d}{dt}+\nu-A}\right\rangle_c x_{01} + \left\langle \frac{\nu}{\frac{d}{dt}+\nu-A}f\right\rangle_c . \tag{38}$$

The additional averaging over statistics of the process $f(t)$ (if $f(t)$ is a random) does not change the structure of the equations obtained. With the help of the Laplace transformation, the set can be reduced to algebraic equations

$$X(s) = \frac{1}{\langle\nu\rangle}\left\langle \frac{\nu}{s+\nu-A}\right\rangle_c X_{01}(s) + \left\langle \frac{1}{s+\nu-A}F(s)\right\rangle_c ,$$
$$X_{01}(s) = \frac{1}{\langle\nu\rangle}\left\langle \frac{\nu^2}{s+\nu-A}\right\rangle_c X_{01}(s) + \left\langle \frac{\nu}{s+\nu-A}F(s)\right\rangle_c , \tag{39}$$

where

$$\langle ...F(s)\rangle_c = \int\int ...fP(s,f|c)p(c)dcdf .$$

2.2.2. Nonlinear systems

Let us consider an example of a system described by a nonlinear equation depending on random process $\alpha(t)$

$$\dot{x} = U(x,\alpha(t)) , \tag{40}$$

where $U(x,\alpha(t))$ is a nonrandom vector function of argument x and the scalar random function $\alpha(t)$ belongs to a class of kongaroo processes.

We derive the equation for probability density $P(x,t) = \langle \widetilde{P}(x,t) \rangle$. The stochastic probability density $\widetilde{P}(x,t)$ obeys the stochastic Liouville equation

$$\frac{\partial \widetilde{P}}{\partial t} + \hat{U}\widetilde{P} = 0\,, \tag{41}$$

where

$$\hat{U}g = \frac{\partial}{\partial x}(Ug) = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(U_k g)\,.$$

It is convenient to introduce a variable $z(x,t) = \widetilde{P}(x,t) - \widetilde{P}(x,0)$ and averages $z_{km} = \langle \nu^m \hat{U}^k z \rangle$. The application of the formula of differentiation (34) to averages z_{km} leads to a chain of the equations:

$$\frac{\partial z_{km}}{\partial t} = \hat{U}_{km} z_{01} - F_{km} - z_{k+1,m} - z_{k,m+1}\,,$$

$(k, m = 0, 1, 2, ...)$. Here

$$\hat{U}_{km} = \frac{\langle \nu^{m+1} \hat{U}^k \rangle}{\langle \nu \rangle}, F_{km} = \langle \nu^m \hat{U}^{k+1} \widetilde{P}(x,0) \rangle\,.$$

The reduction procedure leads to the following closed set for averages

$$\begin{aligned} P(x,s) &= \frac{1}{\langle \nu \rangle} \left\langle \frac{\nu}{s + \nu + \hat{U}} \right\rangle_c Z_{01}(x,s) + \left\langle \frac{s+\nu}{s+\nu+\hat{U}} \widetilde{P}(x,0) \right\rangle_c\,, \\ Z_{01}(x,s) &= \frac{1}{\langle \nu \rangle} \left\langle \frac{\nu^2}{s + \nu + \hat{U}} \right\rangle_c Z_{01}(x,s) - \left\langle \frac{\nu \hat{U}}{s+\nu+\hat{U}} \widetilde{P}(x,0) \right\rangle_c\,, \end{aligned} \tag{42}$$

where $P(x,s)$ and $Z_{01}(x,s)$ are the Laplace transforms of probability distribution $P(x,t)$ and of function $Z_{01}(x,t)$, respectivelly.

2.3. Sum of telegraphic processes

2.3.1. Differentiation formula

Let us consider a random process $\alpha(t)$ which is a sum of statistically independent telegraphic type random processes $\alpha_k(t)$,

$$\alpha(t) = \sum_{k=1}^{N} \alpha_k(t)\,. \tag{43}$$

The statistical independence of the processes $\alpha_k(t)$ means that a one-point probability distribution $P(\alpha,t)$ and a conditional density $Q(\alpha,t|\alpha',0)$ for the joint process $\alpha(t)$ factorize into products

$$P(\alpha,t) = P^{(1)}(\alpha_1,t)P^{(2)}(\alpha_2,t)...P^{(N)}(\alpha_N,t)\,,$$
$$Q(\alpha,t|\alpha',0) = Q^{(1)}(\alpha_1,t|\alpha_1',0)\ldots Q^{(N)}(\alpha_N,t|\alpha_N',0)\,,$$

where the functions $P^{(k)}(\alpha_k,t)$ and $Q^{(k)}(\alpha_k,t|\alpha_k',0)$ are a one-point probability distribution and a conditional probability distribution of the random process $\alpha_k(t)$. For the sum of statistically independent processes $\alpha_k(t)$ the master equation reads

$$\frac{\partial P(\alpha,t)}{\partial t} = \hat{L}P = (\hat{L}_1 + \hat{L}_2 + ... + \hat{L}_N)P(\alpha,t)\,,$$

where $\hat{L}_k$ is a kinetic operator of the random process $\alpha_k(t)$. The master equation for the conditional probability denstity Q has the similar form. Now one can write down the differentiation formula for the case under consideration

$$\frac{d}{dt}\langle F(t,\alpha)\Phi_t[\alpha]\rangle = \langle\frac{\partial}{\partial t}(F\Phi_t)\rangle + \left\langle\left[\sum_{k=1}^{N}\hat{L}_k^+F\right]\Phi_t\right\rangle\,. \tag{44}$$

Let us consider two simple examples of application of this formula.

2.3.2. Central limit theorem

Let $\alpha_k(t)$ are Kubo-Anderson random jump processes with the same ν and $p(\alpha)$. Using explicit form of the kinetic operator for KA processes, one reads

$$\begin{aligned}
&\frac{d}{dt}\langle F(t,\alpha)\Phi_t[\alpha]\rangle - \langle\frac{\partial}{\partial t}(F\Phi_t)\rangle \\
&= \nu\sum_{k=1}^{N}\left\langle\left[\int F(t,\widetilde{\alpha}_k+c)p(c)dc - F\right]\Phi_t\right\rangle \\
&= \nu\sum_{k=1}^{N}\left\langle\left[\int F(t,\widetilde{\alpha}_k+c)[p(c)-\delta(\alpha_k-c)]dc\right]\Phi_t\right\rangle \\
&= \nu\sum_{n=1}^{\infty}\frac{1}{n!}\left\langle\frac{\partial^n F(t,\alpha)}{\partial\alpha^n}\Phi_t\sum_{k=1}^{N}\int(c-\alpha_k)^n p(c)dc\right\rangle\,,
\end{aligned} \tag{45}$$

where $\widetilde{\alpha}_k = \widetilde{\alpha}_k(t) = \alpha(t) - \alpha_k(t)$. We assume for simplicity $\langle \alpha_k(t) \rangle = 0$. Consider further a limit case when $N \to \infty$ and $\alpha_k \to 0$ in such a way that

$$|\alpha_k| \leq \frac{A}{\sqrt{N}}, \tag{46}$$

where A is a positive constant. Under this condition the dispersion of the process $\alpha(t)$ is bounded,

$$\sigma^2 = \langle \alpha^2 \rangle = \lim_{N \to \infty} \sum_{k=1}^{N} \langle \alpha_k^2 \rangle \leq A^2 . \tag{47}$$

The moments of higher order tend to zero in the limit $N \to \infty$,

$$\lim_{N \to \infty} \sum_{k=1}^{N} |\alpha_k^i| \leq \lim_{N \to \infty} \left(\frac{A}{\sqrt{N}} \right)^i N = 0, (i > 2) .$$

Under these conditions the sum in (45) over n contains only two terms with $n = 1, 2$ different from zero.

From the expression (45) we obtain

$$\frac{d}{dt} \langle F(t, \alpha) \Phi_t[\alpha] \rangle - \left\langle \frac{\partial}{\partial t} (F \Phi_t) \right\rangle = -\nu \left\langle \alpha \frac{\partial F}{\partial \alpha} \Phi_t \right\rangle + \lim_{N \to \infty} \sum_{k=1}^{N} \frac{\nu}{2} \left\langle \left(\alpha_k^2 + \frac{\sigma^2}{N} \right) \frac{\partial^2 F}{\partial \alpha^2} \Phi_t \right\rangle . \tag{48}$$

In particular case, when $\alpha_k(t)$ are Markovian Dichotomous noises we have $\alpha_k^2(t) \equiv \sigma^2/N$ and (48) yealds

$$\frac{d}{dt} \langle F \Phi_t[\alpha] \rangle = \left\langle \frac{\partial}{\partial t} (F \Phi_t) \right\rangle - \nu \left\langle \alpha \frac{\partial F}{\partial \alpha} \Phi_t \right\rangle + \nu \sigma^2 \left\langle \frac{\partial^2 F}{\partial \alpha^2} \Phi_t \right\rangle . \tag{49}$$

The obtained formula of differentiation is DF for Ornstein-Uhlenbeck noise (which is a Gaussian process). It can be shown that the convergence of the formula (48) to a Gaussian form like (49) holds generally. In the case when noises $\alpha_k(t)$ are of different correlation times $\tau_c^k \sim 1/\nu_k$ then the corresponding proceess tends to a Gaussian process but of non-Markovian type. Thus, we have illustrated the proving of the central limit theorem for a sum of independent KA processes by differentiation formulae and in a formulae differentiation representation.

2.3.3. Equation for means

Let us consider the application of the DF method to statistical analysis of a macroscopic system driven by a sum of random telegraphic type processes. For illustration we shell treat a linear system of the form (12), where $\alpha(t)$ is a sum (43) of identical Dichotomous noises $\alpha_k(t)$ with characterictics $\sigma_k^2 = \sigma^2/N$ and $\nu_k^{-1} = \nu^{-1}$.

The purpose is to derive an exact equation for average $\langle x \rangle$. For treating of the system under consideration we need a formulae of differentiation for averages $\langle \alpha_1(t)\alpha_2(t)...\alpha_k(t)\Phi_t \rangle$. From (11) and (44) one has

$$\left(\frac{d}{dt} + k\nu\right) \langle \alpha_1\alpha_2 \ldots \alpha_k\Phi_t \rangle = \left\langle \alpha_1\alpha_2 \ldots \alpha_k \frac{\partial \Phi_t}{\partial t} \right\rangle . \tag{50}$$

Denoting here $x_k = \langle \alpha_1\alpha_2 \ldots \alpha_k x(t) \rangle$ and using (50) and the equation for x variable, one gets [23]

$$\begin{aligned} \dot{x}_k &= (A - k\nu)x_k + B\langle \alpha_1\alpha_2...\alpha_k(\alpha_1 + \alpha_2 + ... + \alpha_N)x \rangle \\ +\langle \alpha_1\alpha_2 \ldots \alpha_k f \rangle &= (A - k\nu)x_k + k\sigma^2 B x_{k-1} + (N - k)B x_{k+1} + f_k , \end{aligned} \tag{51}$$

$k = 0, 1, 2, ..., N$ and $f_k = \langle \alpha_1\alpha_2...\alpha_k f \rangle$. The additional averaging over statistics f does not change the structure of the set obtained. The system (51) is a system of a finite number of equations. Under the condition that A and B are constants, the set (51) can be analytically solved.

2.4. Dynamical systems driven by diffusion processes

2.4.1. Formulae of differentiation

In many physical situations the character of random perturbations is essentially different from, considered in previous sections, the perturbations of a telegraphic type. We refer to a class of diffusion processes. Examples are the well known processes: Gaussian Markov, Rayleigh and Pearson processes. We restrict our consideration to Markovian processes and to a one dimension case. We will show that the method of formulas of differentiation of statistical averages is effective for this type of random perturbation, too.

For description of Markovian processes it is enough to know an initial one-point probability density $P(\alpha, 0)$ and conditional probability density $Q(\alpha, t|\alpha', t')$. All other distributions of the process $\alpha(t)$ can be expressed by this pair. The master equation for diffusion type processes is the Kolmogorov–Fokker–Plank equation

$$\frac{\partial P}{\partial t} = \left[-\frac{\partial}{\partial \alpha} a(\alpha, t) + \frac{1}{2}\frac{\partial^2}{\partial \alpha^2} b(\alpha, t) \right] P . \tag{52}$$

In frameworks of statistical description of process $\alpha(t)$ it is necessary to know the functions $a(\alpha,t), b(\alpha,t)$, initial conditions and the boundary conditions. In dependence on the latter the statistics of the process can essential be different. The detail discussion about boundary conditions for diffusion type processes can be found in [8, 31, 32]. Eq. (52) is the continuity equation,

$$\frac{\partial P}{\partial t} + div_\alpha S = 0\,,$$

where

$$S = aP - \frac{1}{2}\frac{\partial bP}{\partial \alpha}$$

is the probability current of the process α. It is a sum of systematic

$$S_{dr} = aP$$

and diffusion

$$S_{dif} = -\frac{1}{2}\frac{\partial bP}{\partial \alpha}$$

flows, respectively. The function a is a drift velocity of systematic movement and b is a diffusion coefficient (it should be $b \geq 0$).

We present a general formula of differentiation for a class of diffusion Markovian processes. It has the form

$$\frac{d}{dt}\langle F(t,\alpha)\Phi_t[\alpha]\rangle = \left\langle \frac{\partial}{\partial t}(F\Phi_t)\right\rangle + \left\langle a\frac{\partial F}{\partial \alpha}\Phi_t\right\rangle + \frac{1}{2}\left\langle b\frac{\partial^2 F}{\partial \alpha^2}\Phi_t\right\rangle . \qquad (53)$$

2.4.2. Models of processes

We consider some examples.

Ornstein–Uhlebeck process. For such process (at $\langle\alpha\rangle = 0$)

$$a(\alpha,t) = -\nu\alpha, b(\alpha,t) = 2\nu\sigma^2, (-\infty < \alpha < \infty)\,,$$

where $\sigma^2 = \langle\alpha^2\rangle$. The formula of differentiation of statistical average for OU noise has the form

$$\frac{d}{dt}\langle F(t,\alpha)\Phi_t[\alpha]\rangle = \left\langle \frac{\partial}{\partial t}(F\Phi_t)\right\rangle - \nu\left\langle \alpha\frac{\partial F}{\partial \alpha}\Phi_t\right\rangle + \nu\sigma^2\left\langle \frac{\partial^2 F}{\partial \alpha^2}\Phi_t\right\rangle . \qquad (54)$$

In applications one frequently uses the formula with $F(t,\alpha) = \alpha^k$, $k = 1, 2, \ldots$

$$\frac{d}{dt}\left\langle \alpha^k\Phi_t[\alpha]\right\rangle = \left\langle \alpha^k\dot{\Phi}_t\right\rangle - \nu k\left\langle \alpha^k\Phi_t\right\rangle + \nu\sigma^2 k(k-1)\left\langle \alpha^{k-2}\Phi_t\right\rangle . \qquad (55)$$

Here $\dot{\Phi}_t = \frac{\partial \Phi_t}{\partial t}$.

It is seen that, in contrast to the case of KA perturbations $\alpha(t)$, the average $\langle \alpha^k \Phi_t \rangle$ has more complicated character of relationships for different k. The average $\langle \alpha^k \Phi_t \rangle$ for OU noise is related not to $\langle \Phi_t \rangle$ but to more complicated structure $\langle \alpha^{k-2} \Phi_t \rangle$.

Such a character of relationships leads (see below) to other exact representation of an equation for the average $\langle x \rangle$.

Rayleigh's processes [8, 31]. The drift and diffusion coefficients have the form

$$a(\alpha, t) = -\frac{\nu}{2}\left(\alpha - \frac{\chi^2}{\alpha}\right), b(\alpha, t) = \nu\chi^2, (0 \leq \alpha < \infty),$$

where the parameter χ is connected with average $\langle \alpha \rangle$ by expression $\chi = \sqrt{\frac{2}{\pi}}\langle \alpha \rangle$. The correlation function of the process α can be expressed as a sum of exponential decreasing functions with decrements proporional to frequency ν. The formula of differentiation for process the Rayleigh has the form

$$\frac{d}{dt}\langle F\Phi_t \rangle = \left\langle \frac{\partial}{\partial t}(F\Phi_t) \right\rangle - \frac{\nu}{2}\left\langle \left(\alpha - \frac{\chi^2}{\alpha}\right)\frac{\partial F}{\partial \alpha}\Phi_t \right\rangle + \frac{\nu\chi^2}{2}\left\langle \frac{\partial^2 F}{\partial \alpha^2}\Phi_t \right\rangle . \quad (56)$$

Processes with Pearson distributions. Pearson's processes are widely used. There are several types of Pearson processes [33] (see also [31]). The drift and diffusion coefficients are polynomial functions of first and second degree

$$a = a_0 + a_1\alpha, b = b_0 + b_1\alpha + b_2\alpha^2 .$$

The classification of Pearson processes is based on classification of solutions of the stationary kinetic equation (52) with given drift and diffusion coefficients. Consider two types of Pearson's processes:

The beta Pearson process (a symmetrical case). For this case

$$a = -\nu\alpha, b = \frac{\nu}{1+r}(d^2 - \alpha^2), r > -1, (|\alpha| \leq d).$$

The gamma Pearson process.

$$a = -\nu(\alpha - (1+s)g), b = 2\nu g\alpha, g > 0, s > -1, (0 \leq \alpha < \infty).$$

2.4.3. Averaging linear stochastic system

Let the behaviour of a macroscopic dynamical system be described by the system (12). We suppose now that the noise model is one of the listed

diffusion process. The problem is to find a closed equation for average $\langle x(t)\rangle$. Introduce the averages $x_k = \langle \alpha^k x\rangle$ and suppose that in the differentiation formulae $F = \alpha^k$ and the retarded functional $\Phi_t[\alpha]$ is the dynamical variable x. Then we obtain the following chain of equations:

— for the OU process

$$\dot{x}_k = (A - k\nu)x_k + Bx_{k+1} + k(k-1)\nu\sigma^2 x_{k-2} + \langle \alpha^k f\rangle\,, \tag{57}$$

— for the Rayleigh processes

$$\dot{x}_k = (A - \frac{1}{2}k\nu)x_k + Bx_{k+1} + \frac{1}{2}k^2\nu\chi^2 x_{k-2} + \langle \alpha^k f\rangle\,, \tag{58}$$

— for the Pearson process with symmetrical beta distribution

$$\dot{x}_k = (A - \nu(k + e_k))x_k + Bx_{k+1} + \nu d^2 e_k x_{k-2} + \langle \alpha^k f\rangle\,, \tag{59}$$

where $e_k = k(k-1)/2(1+r)$,

— for the Pearson process with gamma distribution

$$\dot{x}_k = (A - k\nu)x_k + Bx_{k+1} + k(k+s)g\nu x_{k-1} + \langle \alpha^k f\rangle\,, \tag{60}$$

where $k = 0, 1, 2, \ldots$ and for each chain of equations, the initial conditions are zero. The above sets are sets of an infinite number of equations and rather not tractable. What we want to find is $x_0 = \langle x\rangle$. In the next paragraph we present a general procedure of finding an exact equation for $x_0 = \langle x\rangle$.

2.4.4. Truncation of chain and rapidly fluctuating field limit

We illustrate the procedure of reduction of equations for evearges x_k on example the chain (57) for OU noise. All others chains can be reduced similarly. It is convenient to write the chain in the form

$$\begin{aligned} \langle \dot{x}\rangle &= A\langle x\rangle + Bx_1 + f_0\,, \\ x_k &= \hat{l}_k(Bx_{k+1} + k(k-1)\nu\sigma^2 x_{k-2} + f_k), k = 1, 2, \ldots, \end{aligned} \tag{61}$$

where $\hat{l}_k$ is the integral operator

$$\hat{l}_k = \left(\frac{d}{dt} + k\nu - A\right)^{-1} \tag{62}$$

and $f_k = \langle \alpha^k(t) f\rangle$. We suppose that the chain (61) is equivalent to the following chain

$$\begin{aligned} \langle \dot{x}\rangle &= A\langle x\rangle + Bx_1 + f_0\,, \\ x_k &= \hat{L}_k x_{k+1} + \hat{D}_k\langle x\rangle + q_k, k = 1, 2, \ldots. \end{aligned} \tag{63}$$

The unknown variables can be found by comparison of the chains (61) and (63). After some simple calculation, one obtains the recurrence relations

$$\begin{aligned}
\hat{L}_k &= [1 - k(k-1)\nu\sigma^2 \hat{l}_k \hat{L}_{k-2} \hat{L}_{k-1}]^{-1} \hat{l}_k B \equiv \hat{S}_k B\,, \\
q_k &= \hat{S}_k [f_k + k(k-1)\nu\sigma^2 (q_{k-2} + \hat{L}_{k-2} q_{k-1})]\,, \\
\hat{D}_k &= k(k-1)\nu\sigma^2 \hat{S}_k (\hat{D}_{k-2} + \hat{L}_{k-2} \hat{D}_{k-1})\,.
\end{aligned} \tag{64}$$

The initial conditions for the new chain are

$$\hat{L}_0 = 0, q_0 = 0, \hat{D}_0 = 1\,.$$

Using the initial conditions and the chain (63) leads to the closed equation for average $\langle x(t) \rangle$ of the form

$$\begin{aligned}
\langle \dot{x} \rangle = &\left(A + B \sum_{k=2}^{\infty} \hat{L}_1 \hat{L}_2 ... \hat{L}_{k-1} \hat{D}_k \right) \langle x \rangle \\
&+ f_0 + B \left(\hat{l}_1 f_1 + \sum_{k=2}^{\infty} \hat{L}_1 \hat{L}_2 ... \hat{L}_{k-1} q_k \right)\,.
\end{aligned} \tag{65}$$

Thus, we obtain the exact equation for average $\langle x(t) \rangle$ in the form (21) where the effective operator $\hat{A}_{\text{ef}}$ and function f_{ef} take the form

$$\begin{aligned}
\hat{A}_{\text{ef}} &= B \sum_{k=2}^{\infty} \hat{L}_1 \hat{L}_2 ... \hat{L}_{k-1} \hat{D}_k\,, \\
f_{\text{ef}} &= f_0 + B \left(\hat{l}_1 f_1 + \sum_{k=2}^{\infty} \hat{L}_1 \hat{L}_2 ... \hat{L}_{k-1} q_k \right)\,.
\end{aligned} \tag{66}$$

For fixed numbers k, the operators $\hat{L}_k$,$\hat{D}_k$ and functions q_k are partial units of branching continued fractions. In the same way one can obtain the effective operators $\hat{A}_{\text{ef}}$ and functions f_{ef} for other noise perturbation models listed above and many others as well. Such a representation for operators $\hat{A}_{\text{ef}}$ and function f_{ef} is useful for evaluation of accuracy of approximative methods.

To illustrate this statement, consider the limit of high-frequency fluctuations. It means that noise correlation scales ν ($1/\nu$ is a characteristic correlation time) is much larger all others time scales of the problem under consideration. For dynamical system controlled by Eq. (12) with OU noise perturbation $\alpha(t)$, this limit means that there should be small parameters

$$\varepsilon = \frac{\sigma \parallel B \parallel}{\nu}\,, \qquad \eta = \frac{\omega}{\nu} \ll 1\,, \tag{67}$$

where ω is a characteristic frequency of nonperturbed motion when $\alpha = 0$, $\| \ldots \|$ means a norm of a vector or matrix. Here we use the Euclidian norm.

It can be shown that for the case when the solution $\langle\, x(t)\,\rangle$ is stable ($\| \langle x(t) \rangle \| < \infty$) the following inequalities hold

$$\| \hat{L}_1 \hat{L}_2 \ldots \hat{L}_{k-1} \hat{D}_k \langle x \rangle \| \leq \frac{\| B \|^{k-1} (k-1)!!}{\nu^{k-1}(k-1)!} \sigma^k \| \langle x \rangle \|$$

$$\approx \frac{\varepsilon^{k-1}}{(k-2)!!} \sigma \| \langle x \rangle \|,$$

$$\| \hat{L}_1 \hat{L}_2 \ldots \hat{L}_{k-1} q_k \| \leq \frac{\varepsilon^{k-1}}{(k-2)!!} \sigma \| f_0 \| .$$

Here $(k-1)!! = (k-1)(k-3)(k-5)\ldots 1$.

Thus, the series on the right hand side of the equation (65) is convergent if the conditions (67) are fulfilled. One has an estimation

$$\| \hat{A}_{\text{ef}} \langle x \rangle + f_{\text{ef}} - f_0 \|$$

$$\leq \| B \| \left(\sigma\varepsilon + \sigma \sum_{k=3}^{\infty} \frac{\varepsilon^{k-1}}{(k-2)!!} \right) (\| \langle x \rangle \| + \| f_0 \|).$$

In the white noise limit, when $\nu \to \infty$ and $\sigma \to \infty$ but $\sigma^2/\nu \to$ const, the sum with respect to k tends to zero, but the first term is equal to $\lim_{\nu \to \infty}(\sigma\varepsilon) =$ const $\| B \|$. Under the assumption (67), keeping only first k terms in the right hand side of the expression (65) means the truncation of the chain (57) on step k ($x_{k+1} = 0$) with accuracy $o(\varepsilon^{k-1})$. Another procedure is used in the approach *based on the differentiation formulae.* Consider DF (55) where $\Phi_t = x$

$$\left(\frac{d}{dt} - \frac{\partial}{\partial t} \right) \langle \alpha^{k+1} x \rangle$$

$$= -\nu(k+1) \langle \alpha^{k+1} x \rangle + \nu\sigma^2 k(k+1) \langle \alpha^{k-1} x \rangle .$$

If the parameter $\nu \to \infty$ and the function $\langle x(t) \rangle$ is a smooth function on time scale $t \sim \nu^{-1}$ then from the above equation it follows that

$$\langle \alpha^{k+1} x \rangle \approx k\sigma^2 \langle \alpha^{k-1} x \rangle . \tag{68}$$

The application of the chain truncation rule (68) to $x_2 = \langle \alpha^2 x \rangle$ leads to the approximation $x_2 = \langle \alpha^2 x \rangle \approx \sigma^2 \langle x \rangle$. Let us list rules of chain truncation followed from DF for (57)–(60):

— for OU noise

$$x_{k+1} \approx k\sigma^2 x_{k-1}\,,$$

— for Rayleigh noise

$$x_{k+1} \approx (k+1)\chi^2 x_{k-1}\,,$$

— for Beta Pearson noise

$$x_{k+1} \approx \frac{k}{2(1+r)+k} d^2 x_{k-1}\,,$$

— for Gamma Pearson noise

$$x_{k+1} \approx (k+1+s)g x_k\,.$$

2.4.5. White noise limit

Consider Gaussian white noise as a limit of OU noise when

$$\nu \to \infty, \sigma^2 \to \infty\,, \lim_{\nu\to\infty} \frac{\sigma^2}{\nu} = D = \text{const}\,.$$

Under this condition the correlation function of OU noise reduces to

$$K(\tau) = \langle \alpha(t+\tau)\alpha(t)\rangle = 2D\delta(\tau)\,.$$

Our purpose is to derive the equation for a probability distribution $P(x,t)$ of some dynamical system subjected to Gaussian white noise. Let the vector function $x(t)$ is the solution of (26) with Gaussian white noise parameter α. Using DF (55) for averages $P_k = \langle \alpha^k \widetilde{P}(x,t)\rangle$, one obtains at $k = 0, 1, 2$.

$$\frac{\partial P}{\partial t} + \frac{\partial uP}{\partial x} + \frac{\partial vP_1}{\partial x} = 0\,,$$

$$\frac{\partial P_1}{\partial t} + \nu P_1 + \frac{\partial uP_1}{\partial x} + \frac{\partial vP_2}{\partial x} = 0\,,$$

$$\frac{\partial P_2}{\partial t} + 2\nu P_2 + \frac{\partial uP_2}{\partial x} + \frac{\partial vP_3}{\partial x} - 2\nu\sigma^2 P = 0\,.$$

From last two equations we get in white noise limit

$$P_1 = -\lim_{\nu\to\infty} \frac{1}{\nu}\frac{\partial vP_2}{\partial x} = -D\frac{\partial vP}{\partial x}\,.$$

Inserting this into the first equation, one has the following kinetic equation

$$\frac{\partial P}{\partial t} + \frac{\partial uP}{\partial x} = D\frac{\partial}{\partial x} v \frac{\partial}{\partial x} vP\,.$$

Thus, the formulae gives a simple method for obtaining kinetic equations in the white noise limit.

2.5. Poisson processes, formulae of differentiation, relation with Ito approach

2.5.1. Exponentially correlated Poissonian process

Let a stationary random process $\alpha(t)$ be a sequence of impulses of exponential shape

$$\alpha(t) = \sum_k z_k \Theta(t - t_k) e^{-\nu(t-t_k)}, \tag{69}$$

where $\Theta(t)$ is the unit step function

$$\Theta(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}.$$

The amplitudes z_k and moments t_k are assumed to be random. The moments t_k are uniformly distributed over time and they are statistically independent of each other and independent of amplitudes z_k. Impulses are distributed according to the Poisson statistics, *i.e.* the probability $P(n, \Delta t)$ of n impulses on the interval Δt is

$$P(n, \Delta t) = \frac{(\mu \Delta t)^n}{n!} \exp(-\mu \Delta t).$$

The parameter μ is a mean number of impulses per unit time. We assume that random amplitudes z_k are statistically independent and distributed according to the probability density $p(z)$. Let $\langle z_k \rangle = \int z p(z) dz = 0$. Then $\langle \alpha(t) \rangle = 0$. The correlation function of the process (69) is

$$\langle \alpha(t_1) \alpha(t_2) \rangle = \sigma^2 e^{-\nu |t_1 - t_2|}, \tag{70}$$

where σ^2 is a dispersion of the α process,

$$\sigma^2 = \langle \alpha^2 \rangle = \frac{\mu}{2\nu} \int z^2 p(z) dz. \tag{71}$$

The process $\alpha(t)$ is a Poisson process belonging to a class of Markovian jump processes obeying the Kolmogorov– Feller type master equation

$$\frac{\partial Q(\alpha, t|\alpha', 0)}{\partial t} = \nu \frac{\partial}{\partial \alpha}(\alpha Q) + \mu \int [Q(\alpha + z, t|\alpha', 0) - Q(\alpha, t|\alpha', 0)] p(z) dz. \tag{72}$$

The conjugate operator $\hat{L}^+$ reads

$$\hat{L}^+ f = -\nu\alpha\frac{\partial f}{\partial\alpha} + \mu\int[f(\alpha+z) - f(\alpha)]p(z)dz. \tag{73}$$

By use of the represenation (73) for the operator $\hat{L}^+$ one can present the differentiation formula for the exponentially correlated Poisson process. The result reads

$$\frac{d}{dt}\langle F(t,\alpha)\Phi_t\rangle = \left\langle\frac{\partial}{\partial t}(F\Phi_t)\right\rangle - \nu\left\langle\alpha\frac{\partial F}{\partial\alpha}\Phi_t\right\rangle + \mu\int\langle[F(t,\alpha(t)+z) - F(t,\alpha(t))]\Phi_t\rangle\, p(z)dz. \tag{74}$$

We can introduce another representation for this formula using power series expansion of integrand (74). One gets

$$\frac{d}{dt}\langle F(t,\alpha)\Phi_t\rangle = \left\langle\frac{\partial}{\partial t}(F\Phi_t)\right\rangle - \nu\left\langle\alpha\frac{\partial F}{\partial\alpha}\Phi_t\right\rangle + \mu\sum_{n=1}^{\infty}\frac{\langle z^n\rangle}{n!}\left\langle\frac{\partial^n F}{\partial\alpha^n}\Phi_t\right\rangle. \tag{75}$$

Here $\langle z^n\rangle = \int z^n p(z)dz$. In applications one frequently uses this formula with $F = \alpha^k$, $k = 0, 1, 2, \ldots$

$$\frac{d}{dt}\langle\alpha^k\Phi_t\rangle = -k\nu\langle\alpha^k\Phi_t\rangle + \mu\sum_{n=1}^{k}C_k^n\langle z^n\rangle\langle\alpha^{k-n}\Phi_t\rangle + \langle\alpha^k\dot{\Phi}_t\rangle, \tag{76}$$

where $C_k^n = \frac{k!}{n!(k-n)!}$.

In the limit $\mu\to\infty$, $\langle z^2\rangle\to 0$ and $\mu\langle z^n\rangle\to 0$ at $n>2$ but the product $\mu\langle z^2\rangle$ is finite, Eq. (75) reduces to (54) for Ornstein–Uhlenbeck noise.

2.5.2. Equation for mean and reduction of the chain

Let us consider a linear dynamical system (12) driven by the Poisson noise (69). The application of the DF (76) to Eq. (12) leads to the following hierarchy of equations for averages $x_k = \langle\alpha^k x\rangle$:

$$\dot{x}_k = (A - k\nu)x_k + Bx_{k+1} + \mu\sum_{n=1}^{k}C_k^n\langle z^n\rangle x_{k-n} + \langle\alpha^k f\rangle. \tag{77}$$

We consider the zero initial condition for x variable. Therefore $x_k(0) = 0$ for all k. It is seen that the average $x_k(t)$ for given k depends on a set of all averages $x_n(t)$ with $n \leq k$. In paragraph 2.4.4 we have discussed the procedure of exact truncation of similar hierarchies to an effective closed equation (65) for average $\langle x(t) \rangle$. For the case under consideration the recurrence relations for operators $\hat{L}_k$, $\hat{D}_k$ and function q_k are

$$\hat{L}_k = \left[1 - \sum_{i=1}^{k-1} \beta_{ki} \hat{l}_k \hat{L}_i \hat{L}_{i+1} \ldots \hat{L}_{k-1}\right]^{-1} \hat{l}_k B \equiv \hat{S}_k B\,,$$

$$\hat{D}_k = \hat{S}_k(\mu\langle z^k \rangle + \sum_{i=1}^{k-1} \beta_{ki} \hat{D}_i + \sum_{i=1}^{k-2} \beta_{ki} \hat{L}_i \hat{D}_{i+1}$$

$$+ \sum_{i=1}^{k-3} \beta_{ki} \hat{L}_i \hat{L}_{i+1} \hat{D}_{i+2} + \ldots + \beta_{k,k-1} \hat{L}_1 \hat{L}_2 \ldots \hat{L}_{k-2} \hat{D}_{k-1})\,,$$

$$q_k = \hat{S}_k(f_k + \sum_{i=1}^{k-1} \beta_{ki} q_i + \sum_{i=1}^{k-2} \beta_{ki} \hat{L}_i q_{i+1}$$

$$+ \sum_{i=1}^{k-3} \beta_{ki} \hat{L}_i \hat{L}_{i+1} q_{i+2} + \ldots + \beta_{k,k-1} \hat{L}_1 \hat{L}_2 \ldots \hat{L}_{k-2} q_{k-1})\,,$$

where $\beta_{ki} = \mu C_k^i \langle z^{k-i} \rangle$. The initial conditions for these recurrence relations read

$$\hat{L}_1 = \hat{l}_1 B, q_1 = \hat{l}_1 f_1, \hat{D}_1 = \mu\langle z \rangle \hat{l}_1\,.$$

By the similar way one can construct a closed operator equation for nonlinear dynamical systems driven by the exponential Poisson noise.

2.5.3. White noise limit

Let us consider a nonlinear dynamical system

$$\dot{x} = u(x,t) + \alpha(t) v(x,t) \tag{78}$$

driven by the Poissonian white noise $\alpha(t)$. By the Poissonian white noise we understand here a limit of the Poissonian Markov process

$$\alpha(t) = \sum_k z_k g(t - t_k) = \sum_k z_k \nu \Theta(t - t_k) \exp[-\nu(t - t_k)]\,, \tag{79}$$

when $\nu \to \infty$. This definition differs from (69) by a coefficient ν. When $\nu \to \infty$ then the function $g(t - t_k)$ tends into Dirac δ-function $\delta(t - t_k)$.

Thus, the sequence (79) is a sequence of delta impulses. It is assumed that the product $\mu\langle z^n\rangle$ is bounded for any integer $n > 0$ when $\nu \to \infty$ as well.

Taking into account the definition (79), the formula of differentiation (76) can be rewritten as

$$\frac{d}{dt}\langle \alpha^k \Phi_t\rangle = -k\nu\langle \alpha^k \Phi_t\rangle + \langle \alpha^k \dot{\Phi}_t\rangle + \mu \sum_{n=1}^{k} C_k^n \nu^n \langle z^n\rangle \langle \alpha^{k-n} \Phi_t\rangle . \tag{80}$$

Using a stochastic Liouville equation for dynamical system (78) and applying the DF (80), one gets a set of equations for averages $P_k(\alpha, t) = \langle\, \alpha^k \widetilde{P}(\alpha, t)\rangle$,

$$\frac{\partial P}{\partial t} + \frac{\partial uP}{\partial x} + \frac{\partial vP_1}{\partial x} = 0,$$
$$\frac{\partial P_k}{\partial t} + k\nu P_k + \frac{\partial uP_k}{\partial x} + \frac{\partial vP_{k+1}}{\partial x} = \mu\nu^k\langle z^k\rangle P + \sum_{i=1}^{k-1} \nu^i \beta_{ki} P_{k-i}. \tag{81}$$

In white noise limit the chain (81) transforms to

$$\frac{\partial vP_{k+1}}{\partial x} = -k\nu P_k + \mu\nu^k\langle z^k\rangle P. \tag{82}$$

The chain (82) with the first equation of the system (81) can easily be reduced to the closed Kolmogorov–Feller-type kinetic equation

$$\frac{\partial P}{\partial t} + \frac{\partial uP}{\partial x} = \left[\mu \int p(z)\left(e^{-z\frac{\partial}{\partial x}v} - 1\right) dz\right] P. \tag{83}$$

In the simplest case when function v is independent of x, *i.e.* $v = v(t)$, the operator $\exp(-z\frac{\partial}{\partial x}v)$ is a shift operator with respect to the variable x. Under this asssumption we get from (83)

$$\frac{\partial P}{\partial t} + \frac{\partial uP}{\partial x} = \mu \int p(z)\,(P(x - zv, t) - P(x, t))\, dz. \tag{84}$$

Thus, it is possible to analyse a dynamical system with the Poisson white noise perturbations in the frame of DF approach.

2.5.4. Relation with Ito approach

The Ito formalism is developed and widely used approach to analyse stochastic differential equations [34] (see also [1, 8, 31, 32, 35]). The Ito theory is based on two simplest models of random processes with independent increments. They are the Wiener and Poisson processes. The Wiener process $w(t)$ having Gaussian statistics has been used for modeling of continuous components of a macroscopic variable $x(t)$. The Poisson process is responsible for random jumps of a macroscopic variable x. The time evolution of a macroscopic variable $x(t)$ is described by the following stochastic integral equation [35]

$$x(t) = x(0) + \int_0^t a(x(\tau),\tau)d\tau + \int_0^t \sigma(x(\tau),\tau)dw(\tau) + \int_0^t \Theta(x(\tau),d\tau)\,, \quad (85)$$

where $x(t) = (x_1(t),...,x_n(t))$, $a(x,t)$ is a n-dimensional vector- function, $\sigma(x,t)$ is a positive definite matrix function $n \times m$, $dw(t)$ is the increment of the Wiener process $w(t)$ corresponding to a small time interval dt. For arbitrary t_1 and t_2 the increments $w_i(t_1) - w_j(t_2)$, $i,j = \overline{1,m}$ are Gaussian variables with

$$\langle w_i(t_1) - w_j(t_2)\rangle = 0\,,$$

and dispersion

$$\langle (w_i(t_1) - w_j(t_2))^2\rangle = D_{ij}|t_1 - t_2|\,.$$

Here D_{ij} is a positive definite matrix.

The last term on the right hand side of the Eq. (85) describes a jump component of the process $x(t)$. It can be presented in the form

$$\int_0^t \Theta(x(\tau),d\tau) = \int_o^t \int g(x,z,\tau)\rho(dz,d\tau)\,, \quad (86)$$

where $g(x,z,\tau)$ is an amplitude of jumps of the random process x at time t. It depends on x and on the parameter z; $\rho(z,t) = \widetilde{\rho}(z,t) - \langle\, \widetilde{\rho}(z,t)\rangle$. The function $\widetilde{\rho}(z,t)$ is a random integer function of z and t such that $\widetilde{\rho}(dz,dt)$ is the number of x-process jumps in time interval $(t,t+dt)$ when z values be found in the interval $(z,z+dz)$. The statistical dependence ρ on z variable is arbitrary and on time variable t is Poissonian. It means that on non-overcrossing intervals of time the function ρ has independent increments and on average μdt jumps occur on the interval $(t,t+dt)$. The probability that the increment $\rho(z,t+dt) - \rho(z,t) = m$ is given by the Poisson low

$$P(\rho(z,t+dt) - \rho(z,t) = m) = \frac{(\mu dt)^m}{m!}e^{(-\mu dt)}\,.$$

For processes under consideration the generalized Ito formula reads [35]

$$G(x(t),t) = G(x(0),0) + \int_0^t \left(\frac{\partial}{\partial\tau} + \hat{L}_d^+ + \hat{L}_p^+\right) G(x(\tau),\tau)d\tau + \int_0^t \sigma_{ij}(x(\tau),\tau)\frac{\partial G(x(\tau),\tau)}{\partial x_j}dw_i(\tau) + \int_0^t \int [G(x(\tau)+g(x(\tau),z,\tau),\tau) - G(x(\tau),\tau)]\rho(dz,d\tau)\,, \tag{87}$$

where $G(x,t)$ is a vector function of x and t variables and operators $\hat{L}_d^+$ and $\hat{L}_p^+$ read

$$\hat{L}_d^+ G(x(t),t) = a_i(x,t)\frac{\partial G}{\partial x_i} + \frac{1}{2}\sigma_{il}(x,t)\sigma_{lj}(x,t)\frac{\partial^2 G}{\partial x_i x_j}\,, \tag{88}$$

$$\hat{L}_p^+ G(x(t),t) = \int [G(x(t)+g(x(t),z,t),t) - G(x(t),t) + g_i(x(t),z,t)\frac{\partial G(x(t),t)}{\partial x_i}]\pi(t,dz)\,. \tag{89}$$

Here $\pi(t,dz)$ is the probability density for amplitudes of the Poisson process.

Let in the generalized Ito formula (87) $x(t) = \alpha(t)$ and $G(x(t),t) = F(\alpha(t),t)\Phi_t[\alpha]$, where $\Phi_t[\alpha(\tau)]$ is a function of t and a retarded functional of $\alpha(\tau)$ with $\tau < t$. Note that in (87) the derivative of F with respect to t is calculated over $\alpha(t)$-dependence only, but not $\alpha(\tau)$-dependence where $\tau < t$. Taking into account this and after averaging of (87) over statistics of the Wiener and Poisson processes, we obtain the following differentiation formula

$$\frac{d}{dt}\langle F(\alpha(t),t)\Phi_t[\alpha]\rangle = \left\langle \frac{\partial}{\partial t}(F\Phi_t)\right\rangle + \left\langle \left[\left(\hat{L}_d^+ + \hat{L}_p^+\right)F\right]\Phi_t\right\rangle\,. \tag{90}$$

The action of the operators $\hat{L}_d^+$ and $\hat{L}_p^+$ on an arbitrary function $F(\alpha,t)$ is defined by the formulas (88), (89).

Thus, by using generalized Ito formula and special choice of G one can derive a differentiation formula for a general class of the Markovian random processes generated by the operators $\hat{L}_d$ and $\hat{L}_p$.

2.6. Differentiation formulae of cumulant functions and their applications

2.6.1. A general notations

In the previous section we have considered the application of DF for moment functions. In many cases it is more convenient to use another representation of DF, namely the cumulant representation. The cumulants of a random process $\alpha(t)$ appear as coefficients of a functional Taylor series of the logarithm of the characteristic functional $\chi_t[v] = \left\langle \exp i \int_0^t v(\tau)\alpha(\tau)d\tau \right\rangle$:

$$\ln \chi_t[v] = \sum_{k=1}^{\infty} \frac{i^k}{k!} \int_0^t \dots \int_0^t \varphi(v_1, v_2, ..., v_k)\langle\alpha(t_1), \alpha(t_2), ..., \alpha(t_k)\rangle dt_1 dt_2 ... dt_k .$$

Here $\varphi(v_1, v_2, ..., v_k)$ is a symmetric function of the variables $v_1, v_2, ..., v_k$ and equal to

$$\begin{aligned}\varphi(v_1, v_2, ..., v_k) &= \varphi(v(t_1), v(t_2), ..., v(t_k) \\ &= \frac{\delta^k \ln \chi_t[v]}{\delta\alpha(t_1)...\delta\alpha(t_k)}\Big|_{\alpha(t_i)=0, \forall i=\overline{1,k}} .\end{aligned}$$

where the symbol $\frac{\delta}{\delta\alpha(t_i)}$ denotes the variation derivative. The cumulants of a random process $\alpha(t)$ $\langle\alpha(t_1), \alpha(t_2), ..., \alpha(t_k)\rangle \equiv \langle\alpha_1, \alpha_2, ..., \alpha_k\rangle$ [1] are connected with moment functions $\langle\alpha(t_1)\alpha(t_2)...\alpha(t_k)\rangle$. The relations between cumulants and moments and some related questions are contained in the books [5, 8, 9, 36]. The cumulant of the first order (it can be denoted as $\langle, \alpha\rangle$ or $\langle\alpha, \rangle$) equals the mean $\langle\alpha\rangle$. The cumulant of the second order $\langle\alpha_1, \alpha_2\rangle$ is equal to the correlation function of the random process

$$\langle\alpha_1, \alpha_2\rangle = \langle\alpha_1\alpha_2\rangle - \langle\alpha_1\rangle\langle\alpha_2\rangle .$$

For cumulant of the third and fourth orders one has

$$\begin{aligned}\langle\alpha_1, \alpha_2, \alpha_3\rangle =& \langle\alpha_1\alpha_2\alpha_3\rangle - \langle\alpha_1\alpha_2\rangle\langle\alpha_3\rangle - \langle\alpha_1\rangle\langle\alpha_2\alpha_3\rangle \\ &- \langle\alpha_1\alpha_3\rangle\langle\alpha_2\rangle + 2\langle\alpha_1\rangle\langle\alpha_2\rangle\langle\alpha_3\rangle , \\ \langle\alpha_1, \alpha_2, \alpha_3, \alpha_4\rangle =& \langle\alpha_1\alpha_2\alpha_3\alpha_4\rangle - \{\langle\alpha_1\rangle\langle\alpha_2\alpha_3\alpha_4\rangle\}_4 \\ &- \{\langle\alpha_1\alpha_2\rangle\langle\alpha_3\alpha_4\rangle\}_3 + 2\{\langle\alpha_1\rangle\langle\alpha_2\rangle\langle\alpha_3\alpha_4\rangle\}_6 \\ &- 6\langle\alpha_1\rangle\langle\alpha_2\rangle\langle\alpha_3\rangle\langle\alpha_4\rangle .\end{aligned}$$

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[1] We use here and further the notation accepted in the book [36].

where $\{...\}_k$ means a sum of different terms with all kinds of index transposition. The number of such terms is equal to k. For example

$$\{\langle\alpha_1\rangle\langle\alpha_2\alpha_3\alpha_4\rangle\}_4 = \langle\alpha_1\rangle\langle\alpha_2\alpha_3\alpha_4\rangle\langle\alpha_2\rangle\langle\alpha_1\alpha_3\alpha_4\rangle+$$

$$+\langle\alpha_3\rangle\langle\alpha_1\alpha_2\alpha_4\rangle + \langle\alpha_4\rangle\langle\alpha_1\alpha_2\alpha_3\rangle\,.$$

The cumulants of third and fourth order are usually used to characterize a symmetry property and shape of the probability distributions. For example, the cumulant of third order at the same times is called an asymmetry coefficient γ_1 or "skewness" [37],

$$\gamma_1 = \frac{\langle\alpha^{[3]}\rangle}{\langle\alpha^{[2]}\rangle^{3/2}}\,. \tag{91}$$

It characterizes symmetry of a one-point probability distribution $P(\alpha,t)$. Here and further the notation $\langle\alpha^{[k]}\rangle$ means $\langle\alpha^{[k]}\rangle = \underbrace{\langle\alpha,\alpha,...,\alpha\rangle}_{k}$. The equality $\gamma_1 = 0$ indicates that the function $P(\alpha,t)$ is an even function of α variable. The positive sign of γ_1 indicate that the distribution has longer right tail then left. In opposite case $\gamma_1 < 0$ the probability distribution P has longer left tail.

Using cumulant $\langle\alpha^{[4]}\rangle$ one can create a coefficient γ_2 named as kurtosis [37]

$$\gamma_2 = \frac{\langle\alpha^{[4]}\rangle}{\langle\alpha^{[2]}\rangle^{2}}\,. \tag{92}$$

The sign of the coefficient γ_2 characterizes a shape of probability distribution $P(\alpha,t)$ near a maximum point in comparison to a shape of the Gaussian distribution. When $\gamma_2 > 0$ the probability distribution $P(t,\alpha)$ has sharper peak then the Gaussian distribution. This case corresponds to *leptokurtic* shape of the distribution. The negative sign of the kurtosis indicates the *platykurtic* shape of the probability distribution $P(t,\alpha)$ near a maximum point. In this case the shape of the distribution is more flat then Gaussian.

Let us put $\langle\alpha,\rangle = M(t)$, $\langle\alpha_1,\alpha_2\rangle = K(t_1,t_2)$ and $\langle\alpha_1,...,\alpha_k\rangle = 0$ at $k > 2$. Then, from series expansion of $\ln\chi_t[v]$, one easy obtains

$$\chi_t[v] = \exp\left\{ i\int_0^t M(\tau)v(\tau)d\tau - \frac{1}{2}\int_0^t\int_0^t v(\tau_1)v(\tau_2)K(\tau_1,\tau_2)d\tau_1 d\tau_2 \right\}.$$

The characteristic functional is well-known for a Gaussian random process. In this case the set of two first cumulant functions characterize completely

the statistical properties of the Gaussian process. The simplicity of Gaussian random processes in cumulant representation and universality have created approximate methods for description of probability distributions by means of a series with respect to deviation from the Gaussian distribution. The measure of such deviations is smallness of higher order cumulants. As an example we refer to an Edgeworth series [38].

The discussion about apllications of cumulant technics for analysis of dynamical systems under influence of random perturbations can be found in [9, 12, 13, 39-42].

Now we shall use the cumulant representation to derive formulae of differentiation of cumulant functions and apply them to a macroscopic system driven by random perturbations. We need some properties of cumulants.

2.6.2. Some useful properties of cumulants

1. Cumulants are invariant to any transposition of arguments, for example

$$\langle \alpha_1, \alpha_2, ..., \alpha_k \rangle = \langle \alpha_2, \alpha_1, ..., \alpha_k \rangle .$$

2. If c_k is a deterministic variable or an operator acting on the variable α_k only, then

$$\left\langle \sum_{k=1}^{m} c_k \alpha_k, \alpha_{m+1}, ..., \alpha_n \right\rangle = \sum_{k=1}^{m} c_k \langle \alpha_k, \alpha_{m+1}, ..., \alpha_n \rangle . \tag{93}$$

3. A joint cumulant of a set $\alpha_1, \alpha_2, ..., \alpha_n, \beta$ of random variables is equal to zero if the set contains at list one variable statistically independent of other random variables

$$\langle \alpha_1, \alpha_2, ..., \alpha_n, \beta \rangle = 0 . \tag{94}$$

The expression (94) holds in a particular case when β is a nonrandom variable. Let the set c_k be a set of nonrandom parameters. Then as a consequence of Eqs. (93), (94) one has

$$\langle \alpha_1 + c_1, \alpha_2 + c_1, ..., \alpha_n + c_1 \rangle = \langle \alpha_1, \alpha_2, ..., \alpha_n \rangle . \tag{95}$$

4. We shall use the following important formula [36]:

$$\langle \alpha_1, \alpha_2, ..., \alpha_k, xy \rangle = \langle \alpha_1, \alpha_2, ..., \alpha_k, x, y \rangle + \sum_{l=1}^{k} \{ \langle \alpha_1, \alpha_2, ..., \alpha_{k-l}, x \rangle \langle \alpha_{k-l+1}, ..., \alpha_k, y \rangle \}_{C_k^l} . \tag{96}$$

where the symbol $\{...\}_{C_k^l}$ denotes a sum of different terms with all kinds of index transposition. The number of such terms is equal to $C_k^l = \frac{k!}{l!(k-l)!}$. In the particular case $\alpha_1 = \alpha_2 = ... = \alpha_k = \alpha$ one has

$$\langle \alpha^{[k]}, xy \rangle = \langle \alpha^{[k]}, x, y \rangle + \sum_{l=1}^{k} C_k^l \left\langle \alpha^{[k-l]}, x \right\rangle \left\langle \alpha^{[l]}, y \right\rangle . \tag{97}$$

2.6.3. Differentiation formulae of cumulant functions

Let us consider a cumulant of the special structure

$$\langle \alpha_1(t), \alpha_2(t), ..., \alpha_m(t), \Phi_t[\alpha] \rangle , \tag{98}$$

where the variable $\Phi_t[\alpha]$ is a retarded functional of the random process $\alpha(t)=(\alpha_1(t), \alpha_2(t), ..., \alpha_m(t))$ The application of general DF (3) to average $\langle \alpha_1 \alpha_2 ... \alpha_m \Phi_t[\alpha] \rangle$ gives the following result:

$$\frac{d}{dt} \langle \alpha_1 \alpha_2 ... \alpha_m \Phi_t \rangle = \left\langle \alpha_1 \alpha_2 ... \alpha_m \dot{\Phi}_t \right\rangle + \left\langle \left[\hat{L}^+ (\alpha_1 \alpha_2 ... \alpha_m) \right] \Phi_t \right\rangle . \tag{99}$$

We use the representation (7) for operator $\hat{L}^+$. If $\alpha_1 = \alpha_2 = ... = \alpha_m = \alpha$ then from (99) and (7), one obtains

$$\frac{d}{dt} \langle \alpha^m \Phi_t \rangle = \left\langle \alpha^m \dot{\Phi}_t \right\rangle + \sum_{i=1}^{m} C_m^i \left\langle \Lambda_i \alpha^{m-i} \Phi_t \right\rangle , (m = 1, 2, ...) . \tag{100}$$

What is the analog of these formulae in cumulant representation? Using property 2 from the previous paragraph one obtains, analogically to (99), the form

$$\begin{aligned} \frac{d}{dt} \langle \alpha_1, \alpha_2, ..., \alpha_m, \Phi_t \rangle &= \langle \alpha_1, \alpha_2, ..., \alpha_m, \dot{\Phi}_t \rangle \\ &+ \{ \langle \Lambda_1, \alpha_2, \alpha_3, ..., \alpha_m, \Phi_t \rangle \}_m \\ &+ ... + \{ \langle \Lambda_k, \alpha_{k+1}, \alpha_{k+2}, ..., \alpha_m, \Phi_t \rangle \}_{C_m^k} + \langle \Lambda_m, \Phi_t \rangle . \end{aligned} \tag{101}$$

The analog of the formula (100) reads

$$\frac{d}{dt} \left\langle \alpha^{[m]}, \Phi_t \right\rangle = \left\langle \alpha^{[m]}, \dot{\Phi}_t \right\rangle + \sum_{i=1}^{m} C_m^i \left\langle \Lambda_i, \alpha^{[m-i]}, \Phi_t \right\rangle . \tag{102}$$

There are two terms in sum on the right hand side of (102) ($i = 1$ and $i = 2$) for a particular case of diffision type random processes. Further

we restrict ourselves to consideration of a Gaussian Markovian process or Ornstein–Uhlenbeck noise. The differentiation formula reads

$$\frac{d}{dt}\left\langle \alpha^{[m]}, \Phi_t \right\rangle = \left\langle \alpha^{[m]}, \dot{\Phi}_t \right\rangle - m\nu \left\langle \alpha^{[m]}, \Phi_t \right\rangle . \tag{103}$$

This formula can be rewritten as

$$\left\langle \alpha^{[m]}, \dot{\Phi}_t \right\rangle = \left(\frac{d}{dt} + m\nu \right) \left\langle \alpha^{[m]}, \Phi_t \right\rangle . \tag{104}$$

Its generalization is as follows

$$\left\langle \alpha^{[m]}, \frac{\partial^k \Phi_t}{\partial t^k} \right\rangle = \left(\frac{d}{dt} + m\nu \right)^k \left\langle \alpha^{[m]}, \Phi_t \right\rangle . \tag{105}$$

For an arbitrary linear differential operator $M\left(t, \frac{d}{dt}\right)$:

$$M\left(t, \frac{d}{dt}\right) = \sum_k m_k(t) \frac{d^k}{dt^k} ,$$

with nonrandom coefficients $m_k(t)$, Eq. (105) leads to the form

$$\left\langle \alpha^{[m]}, M\left(t, \frac{d}{dt}\right) \Phi_t \right\rangle = M\left(t, \frac{d}{dt} + m\nu\right) \left\langle \alpha^{[m]}, \Phi_t \right\rangle . \tag{106}$$

2.6.4. Equation for mean

Let us apply the result (103) to the linear stochastic system (12). After averaging both sides of the system equations, one obtains

$$\begin{aligned} \langle \dot{x} \rangle &= A\langle x \rangle + B\langle \alpha x \rangle + \langle f \rangle \\ &= A\langle x \rangle + B\langle \alpha, x \rangle + \langle f \rangle . \end{aligned}$$

For the unknown cumulant $\langle \alpha, x \rangle$, the formula of differentiation (103) can be appleid. After differentiation of the cumulant $\langle \alpha, x \rangle$ with respect to time and using DF (103), we find

$$\begin{aligned} \frac{d}{dt}\langle \alpha, x \rangle &= \langle \alpha, \dot{x} \rangle - \nu\langle \alpha, x \rangle = \langle \alpha, (Ax + B\alpha x + f) \rangle \\ -\nu\langle \alpha, x \rangle &= (A - \nu)\langle \alpha, x \rangle - B\langle \alpha, \alpha x \rangle + \langle \alpha, f \rangle \\ &= (A - \nu)\langle \alpha, x \rangle + B\langle \alpha, \alpha, x \rangle + \sigma^2 B\langle x \rangle + \langle \alpha, f \rangle . \end{aligned} \tag{107}$$

We take into account here that $\langle\alpha\rangle = 0$ and $\langle\alpha,\alpha\rangle = \sigma^2$. It is seen that a new cumulant $\langle\alpha,\alpha,x\rangle$ occurs. The same procedure is applied to this cumulant,

$$\frac{d}{dt}\langle\alpha,\alpha,x\rangle = \langle\alpha,\alpha,\dot{x}\rangle - 2\nu\langle\alpha,\alpha,x\rangle = (A-2\nu)\langle\alpha,\alpha,x\rangle + B\langle\alpha,\alpha,\alpha,x\rangle + 2\sigma^2 B\langle\alpha,x\rangle + \langle\alpha,\alpha,f\rangle\,. \tag{108}$$

We use here the following general formula obtained from (97) under conditions that $\langle\alpha\rangle = 0$ and $\alpha(t)$ is a process with Gaussian statistics ($\langle\,\alpha^{[k]}\rangle = 0, k > 2$):

$$\left\langle\alpha^{[k]},\alpha x\right\rangle = \left\langle\alpha^{[k+1]},x\right\rangle + k\sigma^2\left\langle\alpha^{[k+1]},x\right\rangle\,. \tag{109}$$

Continuation of this procedure leads to the following hierarchy of equations for cumulants $x_k = \langle\alpha^{[k]},x\rangle$:

$$\dot{x}_k = (A-k\nu)x_k + k\sigma^2 Bx_{k-1} + Bx_{k+1} + f_k\,, \tag{110}$$

where $f_k = \langle\alpha^{[k]},f\rangle$, $k = 0,1,2,....$

As an example of application of obtained results, let us consider a Brownian particle motion in coloured noise being Ornstein-Uhlenbeck process. The equation of motion was written in Introduction. We are interested in mean and meansquare as $t\to\infty$. The equation for variables $x(t)$ and $x^2(t)$ can be written in the form (12) where

$$x = \begin{pmatrix} x \\ x^2 \end{pmatrix},\ A = \begin{pmatrix} -\lambda & 0 \\ 0 & -2\lambda \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},\ f = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}.$$

In the stationary limit, we have equation (110) with $\dot{x} = 0$ and $f_k = \begin{pmatrix} \langle\alpha^{[k]}\rangle \\ 0 \end{pmatrix}$. Because for a Gaussian process the cumulants $\langle\alpha^{[k+1]}\rangle = 0$ for $k+1 > 2$, that is for $k > 1$, then for $k = 0$ one gets $\langle\alpha^{[1]}\rangle = \langle\alpha\rangle = 0$ and for $k = 1$, $\langle\alpha^{[2]}\rangle = \sigma^2$. Using explicit form for matrices A and B we have

$$\langle x\rangle_{\rm st} = 0, \langle x^2\rangle_{\rm st} = \frac{\sigma^2}{\lambda(\lambda+\nu)}\,.$$

2.6.5. Application to equation of higher order

Formulae of differentiation for cumulants are rather convenient when using Gaussian Markovian random perturbations. They can be easily applied to stochastic equations of higher order. Let us consider a macroscopic system of the form

$$N\left(t,\frac{d}{dt}\right)x + \alpha(t)M\left(t,\frac{d}{dt}\right)x = f\,, \tag{111}$$

where

$$N\left(t,\frac{d}{dt}\right)=\sum_{k=0}^{n}n_k(t)\frac{d^k}{dt^k},M\left(t,\frac{d}{dt}\right)=\sum_{k=0}^{m}m_k(t)\frac{d^k}{dt^k},$$

and $m<n$. The coefficients $n_k(t)$ and $m_k(t)$ are nonrandom functions, $\alpha(t)$ is the Ornstein–Uhlenbeck noise. The initial condition $x(t=0)=x_0$ and a function $f(t)$ on the right hand side of the equation (111) can be random and depends on α but in a non-advanced way.

The averaging both sides of the Eq. (111) with respect to the noise statistical ensemble gives

$$N\left(t,\frac{d}{dt}\right)\langle x\rangle+\langle\alpha(t)M\left(t,\frac{d}{dt}\right)x\rangle=\langle f\rangle.$$

Taking into account (106), one obtains

$$N\left(t,\frac{d}{dt}\right)\langle x\rangle+M\left(t,\frac{d}{dt}+\nu\right)\langle\alpha,x\rangle=\langle f\rangle. \tag{112}$$

Applying cumulant averaging to the Eq. (111), we can find

$$\langle\alpha,N\left(t,\frac{d}{dt}\right)x\rangle+\langle\alpha,\alpha M\left(t,\frac{d}{dt}\right)x\rangle=\langle\alpha,f\rangle,$$

.....

$$\langle\alpha^{[k]},N\left(t,\frac{d}{dt}\right)x\rangle+\langle\alpha^{[k]},\alpha M\left(t,\frac{d}{dt}\right)x\rangle=\langle\alpha^{[k]},f\rangle.$$

Utilizing (109) to the averages $\langle\alpha^{[k]},\alpha M_0x\rangle$ where $M_0=M(t,\frac{d}{dt})$ a new averages of the type $\langle\alpha^{[k\pm1]},M_0x\rangle$ appear. The application of the formula (106) allows to write a hierarchy of equations for averages $x_k=\langle\alpha^{[k]},x\rangle$ $(k=0,1,2,...)$

$$N_kx_k+M_{k+1}x_{k+1}+k\sigma^2M_{k-1}x_{k-1}=f_k, \tag{113}$$

where

$$N_k=N\left(t,\frac{d}{dt}+k\nu\right),M_k=M\left(t,\frac{d}{dt}+k\nu\right),f_k=\langle\alpha^{[k]},f\rangle. \tag{114}$$

The chain obtained is a chain of an equation for a scalar macroscopic variables x_k. Let us consider here a simplest case when the functions n_k and

m_k are independent of time t. In this case we can use the Laplace transformation. Denoting

$$x_k(s) = \int_0^\infty e^{-st} x_k(t)dt, F_k(s) = \int_0^\infty e^{-st} f_k(t)dt,$$

from (103), one obtains

$$N_k(s)x_k(s) + M_{k+1}(s)x_{k+1}(s) + k\sigma^2 M_{k-1}(s)x_{k-1}(s) = F_k(s), \quad (115)$$

where

$$N_k(s) = \sum_{i=0}^{n} n_i(s + k\nu)^i, M_k(s) = \sum_{i=0}^{m} m_i(s + k\nu)^i.$$

We choose zero initial conditions for simplicity.

Let F_0 and F_1 not be equal to zero. It means that we addmit nonzero correlation $\langle \alpha, f \rangle$. Then after some calculations we get an explicit expression for desired average $x_0(s)$ expressed by the continuous fraction

$$x_0(s) = \frac{F_1(s)}{\sigma^2 M(s)} + \frac{F_0(s)M(s) - \sigma^2 N(s)}{M(s)(N(s) - Z(s))}, \quad (116)$$

where

$$Z(s) = \cfrac{u_1(s)}{N_2(s) - \cfrac{u_2(s)}{N_2(s) - \cfrac{u_3(s)}{N_3 - \ldots}}},$$

is a continuos fraction with $u_k(s) = k\sigma^2 M_k(s)M_{k-1}(s)$. The first application of the formulae of differentiation for cumulants was done in [43] for a linear oscillator with fluctuating frequency in the form of OU noise.

In this part, we considered some examples of applications of differentiation formulae to macroscopic systems driven by Gaussian OU noise. The DF for others coloured noise models leads to the hierarchy of equations for averages of more complicated structures. For example, the application to (12) gives the chain of equations where the time derivative for average $x_k = \langle \alpha^{[k]}, x \rangle$ contains a set of variables x_j with $j = 1, 2, ..., k$. Nevetheless, the cumulant DF can be considered as one of general approaches to truncation of such hierarchies, especially for random perturbations with short correlation time. The cumulant representation in analysis of system perturbed by random forces with short correlation time has been developed and used in [12, 13].

3. Summary

We have described the general framework of the DF method. Advantages of the method for obtaining exact and closed equations for mean values and probability distributions of linear and nonlinear macroscopic systems perturbed by coloured noises of different statistics have been illustrated. We have considered both Markovian jump processes (such as the Kubo–Anderson and kongaroo random processes) and diffusion processes (OU noise, Rayleigh and a set of the Pearson processes). It is shown that for listed models of Markovian jump processes the application of the DF leads to a hierarchies of equations for averages. The hierarchy obtained can be presented as a series. The important feature of this series is that the series is an operator resolvent expansion. It can be factorized to a closed form. As a result we get a closed and exact representation for equations corresponded to averages of macroscopic variables. Among this averages there are several moment functions and probability distributions as well. Utilization of the DF method does not require an additional procedure connected with consideration of conditional averages [27].

The application of the DF method to dynamical systems driven by diffusion type processes leads to hierarchy of equations for averages. The structure of these hierarchies are different from that we have for jump noise perturbations. Nevertheless, in this case one can derive an exact and closed equation for averages as well. The contribution of the noise perturbation to averaged dynamics of the system is represented as a series. Each term of this series is a unit of a branching continuous fraction. This representation is convenient when applying approximations, especially when random perturbations have small correlation time. It is important that for this case differentiation formulae contain some rules for a hierarchy truncation. The given representation of the DF for cumulants is very convenient at application when random perturbations have Gaussian statistics.

We have illustrated a close relation between the Ito approach and DF approach. Different generalizations of the Ito approach can be used for getting new formulae of differentiation. The DF method gives a regular procedure to analyse a macroscopic system driven by Gaussian and Poissonian white noise. Thus, the DF method allows to study systematically a macroscopic system perturbed by coloured random noises of different statistics.

The discussion of DF applications to macroscopic systems with boundary conditions, with retarded arguments and with many-parametric stochastic perturbations is beyond the scope of the present paper. In the second part of the paper we will consider examples of applications the DF method to selected physical problems. We will investigate a fluid particles motion in turbulent flow, a Brownian particle in stochastic layered media, a

test charged particle in random nonhomogeneous fields, a linear oscillator with periodically modulated random frequency and plane wave transmission through stochastic layer. Finally, two problems related to random switching on and of between two different dynamics will be studied.

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